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**THE MATHEMATICS**  
**OF**  
**APPLIED ELECTRICITY**

**A PRACTICAL MATHEMATICS**

**BY**

**ERNEST H. KOCH, JR.**

*Instructor in Mathematics, School of Science and Technology,  
Pratt Institute, Brooklyn, N. Y.*

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**"THE KEY TO EVERY MAN IS HIS THOUGHT"**

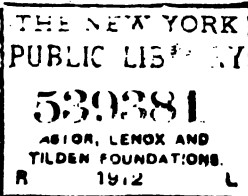
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**FIRST EDITION**

**FIRST THOUSAND**

**NEW YORK**  
**JOHN WILEY & SONS**  
**LONDON: CHAPMAN & HALL, LIMITED**

**1912**



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ERNEST H. KOCH, JR.

ROY VAN  
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ROBERT DRUMMOND AND COMPANY  
BROOKLYN, N. Y.

## PREFACE

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THIS text was developed primarily for the mathematics instruction of second year students in applied electricity courses at Pratt Institute. It is intended to follow a year of drill in the essential elements of algebra, plane geometry and plane trigonometry and also in the elementary principles of mechanics, heat and electricity.

The introductory chapter has been written to meet the requirements of other young men who are occupied in various electrical industries and who find their progress retarded in electrical matters owing to their deficiency in the knowledge of elementary practical mathematics. The text will prove a very helpful aid as a short practical course in Trade, Industrial and Technical High Schools, and in Apprenticeship courses.

The purpose of practical mathematics is to bring to the student's attention the underlying basic structure of the relations of the elements of electrical phenomena so that he may formulate them, interpret them physically and work with accuracy and facility in making numeric and graphic computation. When correlated with a practical, industrial or theoretic course in electricity, practical mathematics gives the student a better appreciation of his major subject and assures him an intellectual penetration into secrets of nature and the utilization of her powers. Instruction from the mimeographed form of this text has been conducted for a number of years and has demonstrated its

usefulness as a powerful and interesting presentation which has promoted the student's judgment, fidelity, poise, common sense and tact in his applied work and in subsequent success in foremanship, superintendence and in other executive and administrative positions. The author aims to present the text according to the modern principles of education which makes the text not only teachable but also self-instructive. Although the work is graded and arranged in logical sequence it is so written that it may be presented in any order irrespective of chapter divisions. There is considerable material of a more advanced character which may be omitted on first assignment and used later for review or in further preparation for more advanced study.

One aim has been to use simple diction, good English, and carefully phrased statements which shall assist the student in becoming familiar with technical terms and guide him in expressing his observations in his own language. Abstract mathematic theory has been avoided as non-essential since the author realizes that any so-called utilitarian or practical book cannot fail to provide the so-called cultural aspects of education. The text contains a limited but an adequate amount of information concerning electrical phenomena so as to minimize the necessary references to supplementary electrical texts. Problems and examples are both designated under the abbreviation of exercise—Ex.

For convenience the text is divided into three parts, viz. I. The Transformation and Interpretation of Formulas. Direct Current Problems. II. The Graphs of Formulas and the Formulation of Graphs. III. Vectors and Vector Diagrams, Alternating Current Problems.

I have collected problems during a number of years from many sources, making it impossible to give specific acknowledgment. I am indebted to my colleagues for many suggestions and express my thanks and appreciation to Mr. H. W. Marsh, Head of the Department of Mathematics, to Mr. S. S. Edmands, Director of the School



PREFACE

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of Science and Technology, Pratt Institute, and to Mr. A. L. Williston, Principal of Wentworth Institute, who inspired the creation of the text. I am indebted to many of my former students, more particularly to Messrs. C. H. Meeker, H. A. Ketcham and J. G. Brown, for valuable assistance in preparing plates and diagrams. It gives me great pleasure to commend the engravers, printers and publishers for their careful and conscientious labors in producing this book.

E. H. KOCH, JR.

NEWARK, N. J., May, 1912.





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# PRACTICAL MATHEMATICS

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## CHAPTER I

### FUNDAMENTAL OPERATIONS

**1. Numbers, Magnitude, Measurement.** Numbers are combinations of nine figures—1, 2, 3, 4, 5, 6, 7, 8, 9, called **digits**, the letter 0, called naught, cipher or **zero**, and the period, called the **decimal point**.

These numeric characters are written either singly or together without commas separating them. They express rank, order, greatness, vastness, extent, size, intensity, activity, strength or significance of quantities.

A **scale** is a measuring stick, instrument, or device upon which there is a series of graduated lines numbered in sequence. Any division or part of a scale may be used as a **unit of measure** and thereby extend the range of measurement.

The **magnitude** of a quantity is the number of units which it contains. The magnitude of a 5-lb. article is 5 and not 5 lbs. If the weight were expressed in ounces, its magnitude would be eighty.

**Ex. 1.** In the following measurements use the most convenient devices at hand, such as a yardstick, foot-rule, tape-measure, store scale. In as many ways as possible express the magnitude by

- (1) Measuring the length, breadth and height of your room.
- (2) Measuring the length, breadth and thickness of a door.
- (3) Measuring a table, desk, box, quart can, ash can, oil can.
- (4) Measuring the parts of a motor, generator, engine, shafting.
- (5) Measuring the sides and perimeters of regular and irregular sections of materials provided by the instructor.
- (6) Weighing a quart of water, sand, cement, loose earth, packed earth, 100' No. 18 copper wire.
- (7) Measuring the length, breadth, and thickness and also by weighing a copper bus-bar, bars of steel, wrought and cast iron.
- (8) Measuring and weighing a brick, cement building block.
- (9) Describe at least ten other scales not used in the above examples.

*The observation from the above list of examples is that magnitudes express the results of measurements or comparisons. Magnitude is a number which states how many and not what kind of units the object contains. The unit of measure must possess the same quality, characteristic, i.e., denomination of the object measured. The unit of measure, as its name implies, has a magnitude of one.*

**2. The Decimal System.** In the decimal notation, all numbers are based on a **system of ten**, making it possible to write a number with comparatively few figures. The annexing of zeros after a number multiplies by powers of ten. A number followed by **one 0** is multiplied by **10**. A number followed by **00** is multiplied by **100**. A number followed by **000** is multiplied by **1000**, and so on. We then have the following summary, assuming the original number to be 95623:

$$95623 = 95623 \times 1$$

$$956230 = 95623 \times 10$$

$$9562300 = 95623 \times 100 = 956230 \times 10$$

$$95623000 = 95623 \times 1000 = 9562300 \times 10$$

*It is observed that the multiple value of each figure depends on its position in a given number. Numbers from 1 . .*

*inclusive require one figure. To write a number ten times as great move the figure one place to the left and fill in its former position with a zero.*

Suppose the process is reversed. We begin with the number 95623000. To write a number one-tenth of this value remove one zero on the right and move each figure one place to the right. If this process be continued the number will be divided by ten in each stage. Although we may continue to move the figures to the right we shall have exhausted all the zeros in three stages, i.e., we have reached the number 95623. To write a number one-tenth of 95623, move the figures one place to the right and insert a decimal point between 2 and 3. To write a number one-tenth of the last result, move the figures one place to the right and shift the decimal point one place to the left, and so on. This is illustrated below:

$$\begin{aligned}
 95623000 \\
 9562300 &= \frac{1}{10} \text{ of } 95623000 \\
 956230 &= \frac{1}{10} \text{ of } 9562300 \\
 95623 &= \frac{1}{10} \text{ of } 956230 \\
 9562.3 &= \frac{1}{10} \text{ of } 95623 \\
 956.23 &= \frac{1}{10} \text{ of } 9562.3
 \end{aligned}$$

Fractions whose denominators are powers of ten are united into a decimal. Thus  $\frac{3}{10} + \frac{9}{100} + \frac{7}{1000} + \frac{6}{10000}$   
 $= \frac{3}{10} + \frac{9}{10^2} + \frac{7}{10^3} + \frac{6}{10^4} = .3976$ . Each numerator takes its position in the decimal answer corresponding to the power of 10 in its denominator.

The number 956.23 means 9 times 100 and 5 times 10 and 6 units and 2 tenths and 3 hundredths. In other words 956.23 means 900 and 50 and 6 and  $\frac{2}{10}$  and  $\frac{3}{100}$ , or

to 900 add 50 add 6 add  $\frac{2}{10}$  add  $\frac{3}{100}$ . The symbol for add is (+) read **plus**, so the above may be written,  $900+50+6+.2+.03$ , or, rearranging in a vertical column, we have

$$\begin{array}{r}
 900 \\
 50 \\
 6 \\
 .2 \\
 .03 \\
 \hline
 956.23
 \end{array}$$

Therefore 956.23 is a condensed form which saves space and the writing of zeros. It is read nine fifty-six point twenty-three, or nine five six point two three.

**3. Addition.** The **addition** of numbers is a process of condensation or uniting of figures in the same column, file or row. An excess over nine in any column is counted into the next column to the left.

**Ex. 2.**

	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)
	12	37	48	52	73	94	50	66
	37	64	78	32	90	33	62	98
	45	76	21	45	67	78	32	54
	68	43	64	38	23	79	22	65
	77	11	13	6	60	5	4	23
	29	46	7	8	23	54	59	31
	<hr/>	<hr/>	<hr/>	<hr/>	<hr/>	<hr/>	<hr/>	<hr/>
column	38							
sums	23							
	<hr/>	<hr/>	<hr/>	<hr/>	<hr/>	<hr/>	<hr/>	<hr/>
sum	268							

The process is identically the same when the numbers are arranged horizontally,  $12+37+45+68+77+29=268$ . In the latter case the figures are united in the order of their numeric position. The symbol (=) means **equals** or **is equal to**.

It is of primary importance for the computer to be able to recognize instantly the sum of two integers. In the following arrangement of two reversed rows of integers the student should recognize that each column sums 10.

1	2	3	4	5	6	7	8	9
9	8	7	6	5	4	3	2	1

**Ex. 3.** Write the integers in any mixed order, then state or write immediately the complementary number. Numbers are complementary when their sum is 10.

**Ex. 4.** Rearrange the digits in double rows so that their sums will give nines, eights, sevens, sixes, fives, fours, threes, twos, ones. It will be observed that some of these combinations are repetitions. There should be 45 different combinations. Why?

**Ex. 5.** What is the sum of the nine digits? What is the sum of the next ten numbers? What is the sum of the ten numbers following? What is the increase for any group of ten numbers following a set of ten numbers?

The addition of numbers should be checked by adding the columns from the top downward as well as from the bottom upward. Where the numbers contain many figures, the **sums of each column** should be written below the columns, and finally the **total sum** should be written at the bottom as shown in the following example. The marks \*, †, °, ; to the right of the figures indicate that those similarly marked were grouped in the summing.

The sum of the figures in the units column is 66, which is the same as 60 and 6. Therefore we write 6 in the units column and 6 under the tens column. The sum of the figures in the tens column is 18, which means that we have 18 times 10, or 180, or 100 and 80. Therefore we write 8 under the tens column and 1 under the hundreds column. Proceeding in like manner with the other columns, we obtain 80 for the sum of the hundreds column, which means 80 times 100, or 8000. Therefore we write 8 under the thousands column and zero or nothing under the hundreds column. The sum of the thousands column is 45, which means 45 times 1000, or 45,000, or 40,000 and 5000. Therefore we write 4 under the ten-thousands column and 5 under the thousands column. The sum of the ten-thousands column is 9. Therefore we write 9 under the ten-thousands column.

The figures set down under the columns are added. Their sum gives the sum total, which is 143,246. Setting down the

figures under the columns enables us to check the work more rapidly.

	3	1°	4	1*	6*		excess over 9 = 6	
	1	5	7	0	8†		3	
		7*	8	5	4†		6	
		3*	9	2	7		3	
	2	7†	8	1	8†		8	
	1	3†	9	0	9		4	
		4	5	4*	5*		0	
			9	0	9*		0	
			3	0	3		6	
		6	9	5*	4		6	
	1	3°	3	0	1		8	
	1	6°	6	0	2		6	
			6	6		excess over 9 = 3	56	excess over 9 = 2
the sums			1	8		0		
of each	8	0				8		
column	4	5				0		
	9					0		
sum	1	4	3	2	4	6	excess over 9 = 2	11 excess over 9 = 2
hundred-thousands								
ten-thousands								
thousands								
hundreds								
tens								
units								

The sum total is a short way of stating that we have altogether 100,000, 40,000, 3,000, 200, 40, and 6. In words we have one hundred-thousands plus four ten-thousands plus three thousands plus two hundreds plus four tens plus six units. Therefore  $100000 + 40000 + 3000 + 200 + 40 + 6 = 143246$ .

An additional check is obtained by casting out the nines. Write the excess over 9 from the sum of the digits in each row and also from the sums of excesses. The final excesses 2 are the same when the addition has been performed without error.

**The Use of Letters as Symbols of Abbreviations.** In mathematics work the alphabet serves not only to expr

our ideas in words, but also to symbolize objects by abbreviating their names by a contracted form or by a single letter. Thus amperes may be abbreviated by amp; volts by v; kilo-watts by k.w.; electromotive force by E.M.F. or by E; intensity of current by I; feet by ft.; inches by in.; miles by mi.; circular mils by C.M., etc.

The engineering alphabet is suggested because of its simplicity. The letters are constructed of thin straight lines and arcs of circles. The letters are equally broad and high with the exception of the I, M, W. The sides of the M are vertical, whereas the sides of the W are inclined. Arcs occur in the letters B, C, D, G, J, O, P, Q, R, S, U.

**Ex. 6.** Obtain a piece of paper ruled in squares and upon it prepare an alphabet of capital and also an alphabet of lower case letters. Follow this by the nine digits. Repeat for practice.

Whenever the letters of the alphabet are used for abbreviations we should be consistent in the use of either CAPS or l. c., and should not change from one style to the other unless the conditions of the problem require it.

A succession of figures or letters connected by + signs means that the figures or letters are to be united by addition. Thus  $n+5n+3n$  means that a number which is represented by the letter  $n$  is to be added to five times the same number  $n$  and in addition three times the same number  $n$  is to be added to the preceding amount. The total result or sum will be  $9n$  or  $(1+5+3) n$ , or (one+five+three) times  $n$ . Suppose  $n$  stands for 321, then  $5n=5$  times  $321=1605$ , and  $3n=3$  times  $321=963$ . Therefore instead of  $n+5n+3n$  we may write  $321+1605+963$ , which equals  $2889=9$  times  $321=9n$ .

*Observation.* The use of a letter to abbreviate a number saves time, space, and unnecessary multiplication. The number written in front of  $n$  is its multiplier and is called the coefficient of  $n$ . The coefficient expresses the number of times the attached letter is used in addition.  $5n$  is therefore

*a contraction of  $n+n+n+n+n$ , and  $3n$  is a contraction of  $n+n+n$ . In like manner  $9n$  is a double contraction, because  $9n$  is first of all a contraction of  $n+5n+3n$ , which is in turn a contraction of  $n+n+n+n+n+n+n+n+n$ . In addition the sum of the coefficients of any letter is the coefficient of the same letter in the answer.*

**Ex. 7.** Perform the following indicated additions:

- (1)  $2b+5b+7b+3b+11b=?b$  = how many  $b$ 's?
- (2)  $7h+4h+h+6h+10h=28?$  = twenty-eight of what?
- (3)  $5k+5k+5k+5k+5k=??$  = how many of what?
- (4)  $2E+3E+E+8E=?E$  = how many  $E$ 's?
- (5)  $3a+6v+5a+v+3a+7v=?a+?v$  = how many  $a$ 's plus how many  $v$ 's?

If the example contains different letters, these must be summed separately, as different letters represent different things and therefore cannot be united. This is exactly the the same kind of a statement we would make if we were asked to render an inventory of electrical apparatus in a stockroom or in a shop or laboratory. There would be one sum obtained by adding the number of lamps, another sum would be obtained by adding the number of switches, and other sums to correspond to the different types of equipment. Suppose we were given the addition of  $20s+502p+36c+57s+31p+92c+32s+173p+19c$ , the answer would be  $109s+706p+147c$ . Since  $s$ ,  $p$ , and  $c$  respectively represent different pieces of apparatus. In expressing the sum we state how many there are of each kind, i.e., how many  $c$ 's,  $p$ 's,  $s$ 's.

*Observation. Addition can be made only of quantities of the same kind or of the same denomination. Therefore in a series of additions we unite quantities or terms with like letters.*

**Ex. 8.** Perform the following indicated additions:

(1)  $3b + 2c + 5d + 2b + 4c + 3.2d + b + c + 5.7d + 2.5c$

(2)  $8.1a + x + 3x + \frac{1}{2}x + \frac{9}{10}a + \frac{3}{2}x + 2.5a + 2$

The student may find it convenient to arrange the work in columns, putting like quantities in the same column as shown below:

$3b + 2c$	$+ 5d$	$3b + 2$	$c + 5$	$d$	$b3 + 2$	$+ 1$	$6$
$2b$	$4c$	$2$	$4$	$3.2$	$c2 + 4$	$+ 1$	$+ 2.5$
$b$	$c$	$1$	$1$	$5.7$	$d5 + 3.2$	$+ 5.7$	$13.9$
$2.5c$		$2.5$			$6b + 8.5c + 13.9d$		
$6b + 8.5c + 13.9d$		$6b + 8.5c + 13.9d$			$6b + 8.5c + 13.9d$		

(3)  $312 + 6t + 523k + 2\frac{1}{2}t + 319.2k + 139$

(4)  $5mi + 6yd + 7ft + 3mi + 308ft + 25yd + 45yd + 2.5ft$

**4. Subtraction.** When we wish to subtract one quantity from another quantity, a minus sign ( $-$ ) is written preceding the **subtrahend**. Thus  $6h - 4h$  means from the **minuend**  $6h$  subtract  $4h$  the **subtrahend**. The **remainder** is  $2h$ , which is the **excess** of  $6h$  over  $4h$ .

If a series of quantities are to be united so as to involve both addition and subtraction the operations may be performed in any order.

(5)  $7a + 3a - 5a + 4a - 2a$

Example (5) means to  $7a$  add  $3a$ , then subtract  $5a$ , then add  $4a$ , and then subtract  $2a$ . Then the remainder or that which is left over or the excess is  $7a$ . The analysis of the procedure is as follows:

Since  $7a + 3a = 10a$ , then  $7a + 3a - 5a + 4a - 2a$  becomes  $10a - 5a + 4a - 2a$ .

Since  $10a - 5a = 5a$ , then  $10a - 5a + 4a - 2a$  becomes  $5a + 4a - 2a$ .

Since  $5a + 4a = 9a$ , then  $5a + 4a - 2a$ , becomes  $9a - 2a$ .

Since  $9a - 2a = 7a$ , then  $9a - 2a$  becomes  $7a$ .

Therefore

$$7a + 3a - 5a + 4a - 2a = 7a.$$

From the above it will be observed that there were a total of 14  $a$ 's added and a total of 7  $a$ 's subtracted and again  $14a - 7a$  gives the excess  $7a$ . In the following examples we have the same quantities and operations to deal with as in ex. (5), except that the order of the quantities has been changed. Show that the remainder is the same in each case. When a sign is omitted before the first term the plus sign is understood as intended.

$$(6) 7a + 3a + 4a - 5a - 2a$$

$$(9) 3a + 7a - 5a - 2a + 4a$$

$$(7) 3a + 4a - 5a + 7a - 2a$$

$$(10) 3a - 5a + 4a + 7a - 2a$$

$$(8) 3a + 4a - 2a - 5a + 7a$$

$$(11) -2a + 3a + 4a - 5a + 7a$$

$$(12) \text{ add the columns formed by examples (6) . . . (11).}$$

*Observation. Subtraction is the reverse operation to addition. When the minuend is greater than the subtrahend the excess is positive, and when the minuend is less than the subtrahend then the excess is negative and indicates a deficiency. The latter would correspond to a debt or liability in a system of accounts, whereas the former would correspond to a credit or asset. A negative amount of any quantity will always cancel a like positive amount of the same quantity. It is for this reason that the excess always takes the sign of the uncanceled amount in any series of additions and subtractions.*

In an election contest  $+A$  may represent the number of votes received for a candidate and  $-B$  the number of votes against a candidate. Suppose the favorable votes are 10568 and the unfavorable votes are 8717. Then the plurality of the candidate over his opponent is an excess of  $A$  over  $B$ , or  $A - B = 10568 - 8717 = 1851$  plurality. In other words the candidate has received 10568 (+ votes) and 8717 (- votes).

The minus sign preceding a quantity may be regarded in two ways. We may interpret it as a symbol of oper-

ation, i.e., a command to subtract or remove or take away the thing which follows it. We may also regard the thing which follows the minus sign as a deficiency or something which is stored away or passive or inactive. It would therefore have a contrary sense to a like positive quantity which might represent something being used or operative or active.

In a tug of war there are pulls in both directions. If we call those acting to the right positive pulls and those acting to the left negative pulls, then the sum total of all the effects is zero when the + and - pulls balance. There is a + or - excess depending upon the preponderance of the + or - pulls respectively.

During part of the day when the demand for power from a station is low the plant may be operated efficiently by storing its excess of energy in a bank of batteries. Suppose a 500-k.w. machine is running at full load, and only 300 k.w. of this amount is supplied to a lamp circuit, then the balance 200 k.w. is inactive and is stored in a series of storage batteries. Then,

Total power = power used in lamps + power stored in batteries.

$$(A) \quad 500 \text{ k.w.} = 300 \text{ k.w.} + 200 \text{ k.w.}$$

This may be rewritten:

Total power = useful or active power - inactive power.

$$(B) \quad 500 \text{ k.w.} = +300 \text{ k.w.} - (-200 \text{ k.w.})$$

The + sign before 300 k.w. indicates + sense or active power.

The - sign before 200 k.w. indicates - sense or inactive power.

The - sign before the parenthesis ( ) means stored or removed or subtracted. (A) and (B) are two different ways of expressing the same idea.

*Observation.* A double negative sign preceding a quantity restores the quantity to a positive sense. Therefore every example in subtraction is changed to one of addition by changing the sign of the subtrahend, and adding the quantities.

**Ex. 13.** What is the meaning of  $6x - (7x - 3x)$ ?

$6x$  is the **minuend** and  $(7x - 3x)$  is the **subtrahend**. The  $(7x - 3x)$  is preceded by a  $-$  sign and is therefore to be subtracted from  $6x$ . The parenthetical quantity alone implies the subtraction of  $3x$  from  $7x$  and reduces to  $4x$ . Therefore the  $(7x - 3x)$  may be replaced by its equal  $4x$  and the original example may be restated as  $6x - 4x$ . The result or excess is  $2x$ . The same result would have been obtained if the original expression had been altered by removing the  $( )$  and reversing the signs of all the quantities contained within it. Thus:

$$6x - (7x - 3x) = 6x - 7x + 3x = 2x.$$

These forms are also equivalent to

$$6x + (-7x + 3x) = 6x + (-4x) = 6x - 4x = 2x.$$

*Observation.* When a  $+$  sign precedes a  $( )$  there is no need to retain the  $( )$ . When a  $-$  sign precedes a  $( )$  the  $( )$  can be removed, only providing we change the signs preceding each term inside the  $( )$ .

The **terms** are the letters or numbers or combinations of both which are separated distinctly by  $+$ ,  $-$ , or  $=$  signs.

A **parenthesis**  $( )$  is a grouping, bonding or aggregating symbol. If an expression requires more than one group of terms it is customary to use different forms of bonding symbols to limit or define the extent of each group. The following bonding symbols are equivalent to the  $( )$  and may replace one another:

$[ ]$  **bracket**;  $\{ \}$  **brace**;  $\text{—}$  **vinculum**.

$$\begin{aligned} 6x - (7x - 3x) &= 6x - [7x - 3x] = 6x - \{7x - 3x\} = 6x - \overline{7x - 3x} \\ &= 6x - 4x = 2x. \end{aligned}$$

Although the vinculum is usually written over the terms which are to be bonded, it is also recognized as the **solidus** or the dividing line which is written under the terms of a numerator to show that all the terms above it are to be operated upon by a common divisor. These

grouping symbols are especially useful when it is desirable to indicate that a number of terms are to be subjected to a like operation.

$3a(5-6b)$  means that both 5 and 6  $b$  are to be multiplied by 3a, giving the result  $15a-18ba$ .

$2x(y-3a-6b)=2xy-6ax-12bx$  means that each one of the three terms on the right of the equality sign contains a common factor or part. The parenthesis indicates not only that the  $2x$  is common to all the terms, but it also shows the remaining constituent parts or factors of each term.

$$\sqrt{A+B+C} = \sqrt{A+B+C}$$

means that the sum of  $A$ ,  $B$ , and  $C$  are collectively and not singly subject to the root symbol. If  $A=6$ ,  $B=9$ , and  $C=10$ , then

$$\sqrt{A+B+C} = \sqrt{6+9+10} = \sqrt{25} = 5.$$

**Ex. 14.** What is the distinction between  $\sqrt{X^2+R^2}$  and  $\sqrt{X^2} + R$ ? Illustrate by assuming  $X=9$  and  $R=4$ .

**5. Multiplication.** Numeric calculations should be made in the easiest manner so as to economize in time, effort, and the chance of mistakes. Thus in multiplying 6305 by 1453, we may write either of these numbers as the multiplier and then use the other number as the multiplicand. In other words,  $6305 \times 1453 = 1453 \times 6305 = 9161165$ . The latter order saves one line of multiplication as follows:

6305		1453	
1453		6305	
<hr/>		<hr/>	
18915 =	$3 \times 6305$	7265 =	$5 \times 1453$
31525 =	$5 \times 6305$	0 =	$0 \times 1453$
25220 =	$4 \times 6305$	4359 =	$3 \times 1453$
6305 =	$1 \times 6305$	8718 =	$6 \times 1453$
<hr/>		<hr/>	
9161165 =	$1453 \times 6305$	9161165 =	$6305 \times 1453$

Since the figures change only on writing the partial product we need not write the multiplier and multiplicand. This method of omission is illustrated below:  $63.05 \times 14.53 = 916.1165$ :

18915	.7265
31525	43.59
25220	878.8
6305	
<hr/>	<hr/>
916.1165	916.1165

*Observation.* The multiplier times the multiplicand equals the product; also the multiplicand times the multiplier equals the product.

If we abbreviate the numbers 6305, 1453, and 9161165 by substituting the letters  $a$ ,  $b$ , and  $p$ , respectively; times by  $\times$ ; equals by  $=$ ; then we may write the product

$$6305 \times 1453 = 9161165 \quad \text{as} \quad a \times b = p,$$

$$1453 \times 6305 = 9161165 \quad \text{as} \quad b \times a = p.$$

Since letters are used to abbreviate quantities they are used in computations in exactly the same way as the numbers for which they have been substituted. The letters of a product are usually written close together, so that the multiplication symbol " $\times$ ," is omitted or contracted to a point. It will be understood that two or more letters, or letters and numbers, written close together imply multiplication. The **factors** are the makers of a product.

$$a \times b = a \cdot b = ab = \text{product} = p,$$

$$b \times a = b \cdot a = ba = \text{product} = p.$$

**Factor times Factor = Product.**

Since we obtain identical products whether we multiply  $a$  by  $b$  or multiply  $b$  by  $a$ , we write accordingly,

$$a \times b = b \times a \quad \text{or} \quad ab = ba.$$

**Observation.** *Things equal to the same thing are equal to each other. This statement is known as the Axiom of Equality and is abbreviated by  $ax=ty$ .*

An **axiom** is a truth which the mind recognizes as self-evident but which it cannot prove or demonstrate. When a fact is at the bounds of our most fundamental ideas there is no expression more simple in terms of which to present it.

**Observation.** *In multiplication the order of multiplier and multiplicand does not affect the result.*

*A product always consists of two or more parts or makers called factors.*

The computation of multiplication may be abbreviated whenever the arithmetic product is to be expressed with a few significant figures. Sometimes the multiplier is used backward, as shown below:

**Ex. 15.** Multiply 7235 by 1294, and express the answer with four significant figures.

7235	7235
1294	4912
29000	7235
65100	1447
14470	651
7235	29
93620000	9362

Therefore  $7235 \times 1294 = 9362 \times 1000$  approximately  $= 9362 \times 10^3$ .

**Ex. 16.** By ordinary and contracted methods determine the following products to four significant figures:

- |                          |                          |
|--------------------------|--------------------------|
| (a) $6305 \times 333$    | (e) $2819 \times 4567$   |
| (b) $7070 \times 212$    | (f) $2000 \times 356$    |
| (c) $8532 \times 6729$   | (g) $3.325 \times 267$   |
| (d) $4.56 \times .00653$ | (h) $.0379 \times .0045$ |

**Ex. 17.** What is the meaning of  $1000 = 10 \times 10 \times 10 = 10^3$ ? The three written above and to the right of the ten is called an **exponent**. It indicates that 1000 is composed of three equal "10"

factors. In other words, the "10" is made use of three times in producing 1000.

In the same manner the product  $aaa$  which consists of three equal " $a$ " factors is abbreviated by  $a^3$ , or  $a \times a \times a = aaa = a^3$ .

*Observation.* In multiplication we add exponents of like factors and write this sum over the like factor in the product.

A product of two equal " $b$ " factors, three equal " $z$ " factors, and four equal " $m$ " factors may be written " $bbzzzmmmm$ ," but with the use of exponents, this is written  $b^2m^4z^3$  or  $b^2z^3m^4$ . The first form gives preference to alphabetic order and the second form gives preference to numeric order.

*Observation.* An exponent serves to abbreviate multiplication.

**Ex. 18.**  $630500000 = 6.305 \times 100\,000\,000 = 6.305 \times 10^8$ . Why is  $10^8$  an abbreviation for 100 000 000?

**Ex. 19.** Explain the following forms:

(a)  $923\,500\,000 = 9235 \times 10^5$

(b)  $17,280 \times 1000 = 1728 \times 10^4$

(c)  $8753 \times 10^3 = 8.753 \times 10^6$

(d)  $783abc \times 10abd = 7.83 \times 10^3 a^2 b^2 cd$

(e)  $a^2 b^3 c^4 \times abc^2 = a^3 b^3 c^6$

(f)  $5xy \times 2yz \times 3xz^2 = 30x^2 y^2 z^3$

(g)  $6(a+b) \times 7(a+b) = 42(a+b)^2$

The following notations are often used where it is found desirable to carry the contraction still further. A divisor may be brought up into the numerator as a factor, providing the sign of the factor's exponent is changed from a positive to a negative sense, or vice versa.

(h)  $89360000 = 8936 \times 10^4 = 89360^4$ .

(i)  $0.000001236 = 0.0_5 1236 = 0.0^5 1236$ .

(j)  $78.92 = 7.892 \times 10 = \frac{7892}{100} = 7892 \times 10^{-2}$ .

(k)  $\frac{1}{10} = 10^{-1} = \frac{1}{10^1} = .1$ .      (l)  $\frac{1}{10^2} = 10^{-2} = .01$ .

Explain the following forms:

$$(m) \quad 0.937 = 9.37 \times 10^{-1} = 937 \times 10^{-3}$$

$$(n) \quad 0.00546 = 5.46 \times 10^{-3} = 546 \times 10^{-5}$$

$$(o) \quad \frac{3.14 \times 10^8}{0.04785} = ? \quad (q) \quad \frac{1}{1000} = \frac{1}{10} \times \frac{1}{10} \times \frac{1}{10} = 10^{-3}.$$

$$(p) \quad \frac{1}{10^2} \times \frac{1}{10^3} \times \frac{1}{10^5} = 10^{-2} \times 10^{-3} \times 10^{-5} = \frac{1}{10^{10}} = 10^{-10}.$$

Give four significant figures in the answers of the following:

**Examples**

20.  $7891 \times 9944.$

21.  $7529 \times 3366.$

22.  $5278 \times 3246.$

Perform the following by reversing the multiplier:

23.  $6547 \times 6531.$

24.  $8054 \times 9601.$

25.  $6665 \times 8772.$

When the numeric values of the factors of a product are given or known, they may be substituted for the letters and thereby enable the numeric value of the product to be obtained.

**Ex. 26.** Determine the numeric value of  $b^3 m^2 z^3$  by substituting the following values:  $b=2$ ;  $m=11$ ;  $z=5$ . From the data we obtain

$$b^3 = 2^3 = 2 \times 2 \times 2 = 8; \quad m^2 = 11^2 = 11 \times 11 = 121; \quad z^3 = 5^3 = 5 \times 5 \times 5 = 125$$

but  $b^3 m^2 z^3$  means  $b^3$  times  $m^2$  times  $z^3$ , therefore,

$$b^3 m^2 z^3 = 8 \times 121 \times 125 = 60500.$$

The multiplication of 4 by 121 may be performed first or the 4 may be multiplied by 125, or we may proceed by multiplying 121 by 125 first. The product so obtained is then multiplied by the remaining factor. In every case the final product is the same,

$$4 \times 121 \times 125 = 484 \times 125 = 60500$$

$$4 \times 125 \times 121 = 5000 \times 121 = 60500$$

$$121 \times 125 \times 4 = 15125 \times 4 = 60500$$

Do these facts verify any previously observed law?

**Ex. 27.** What is the value of  $m^2s^2y^2a^2$ , given  $m=2$ ;  $s=3$ ;  $y=2.5$ ;  $a=10$ ?

Every example in multiplication should suggest the best system to be used in performing the work quickly and accurately.

**Ex. 28.** Multiply 62.5 by 25. Consider  $25 = \frac{100}{4}$ , then the example may be written:

$$62.5 \times 25 = 62.5 \times \frac{100}{4} = \frac{62.5 \times 100}{4} = \frac{62.5}{4} \times 100$$

and

$$\frac{62.5}{4} = 15.625;$$

but

$$15.625 \times 100 = 1562.5;$$

$$\therefore 62.5 \times 25 = 1562.5.$$

*Observation.* When the multiplier is 25, divide the given number by 4 and multiply by 100. The multiplier 100 shifts the decimal point two places to the right.

**Ex. 29.** Multiply 5.23 by 9. Consider  $9 = (10 - 1)$ , therefore,

$$5.23 \times 9 = 5.23(10 - 1) = 5.23 \times 10 - 5.23 = 47.07.$$

**Observation.** When the multiplier is 9 shift the decimal point one place to the right and then subtract the original number.

**Ex. 30.** Multiply 78.91 by 72.36. We may write

$$78.91 = \frac{7891}{100} \quad \text{and} \quad 72.36 = \frac{7236}{100}.$$

$$\therefore 78.91 \times 72.36 = \frac{7891}{100} \times \frac{7236}{100} = \frac{7891 \times 7236}{10000} = ?$$

Express the answer as a decimal.

**Observation.** The product of several factors has as many decimal figures as the sum of the several decimal figures in the factors.

**Ex. 31.** Devise quick methods for performing the following multiplications:

(a)  $3366 \times 9.449$

(c)  $72.6 \times 19$

(b)  $8.93 \times 27$

(d)  $25 \times 24$

The **decimal point**, although the smallest symbol in computation, is nevertheless the most important.

In dividing 62.5 by 4 the quotient should not be written 156.25, because the result should not be larger than the numerator 62.5.

In dividing 7.02 by 0.351 the quotient should not be written 2, because the result should not be smaller than the numerator 7.02.

**Observation.** A quotient is smaller than the dividend or numerator when the divisor or denominator is greater than one, and is larger than the dividend when the divisor is less than one.

$$\frac{0.996}{1.002}$$

should not be written 9.75, because the numerator

of the fraction is smaller than the denominator, which means that the quotient should be less than one.

*Observation.* A proper fraction reduces to a value less than one and an improper fraction reduces to a value greater than one.

**6. Division.** Division may be performed in the usual manner, or, where the answer is required to be given with a few significant figures, then the contracted form suggested below will prove advantageous.

**Ex. 32.** Divide 6295 by 1453. This may be written in the following equivalent forms:  $6295 \div 1453$  or  $6295 : 1453$  or  $\frac{6295}{1453}$  or  $6295/1453$ :

$$\frac{\text{Dividend}}{\text{Divisor}} = \text{Quotient} = \frac{\text{Numerator}}{\text{Denominator}} = \text{Fraction} = \text{Ratio}.$$

ORDINARY METHOD	
6295	1453 Divisor
5812	4.33241 Quotient
<hr/>	
4830	
4359	
<hr/>	
4710	
4359	
<hr/>	
3510	
2906	
<hr/>	
6040	
5812	
<hr/>	
2280	
1453	
<hr/>	
827	Remainder

ROMANCE METHOD	
4.332	
6295	1453
4830	
4710	
3510	

CONTRACTED METHOD	
6295	1453 Divisor
5812	4.332 Quotient
<hr/>	
483	
436	
<hr/>	
47	
44	
<hr/>	
3	
3	
<hr/>	

DECIMAL METHOD	
	4.332 Quotient
Divisor 1453	6295.000 Dividend
	5812
<hr/>	
	483
	435 9
<hr/>	
	47 1
	43 59
<hr/>	
	3 51
	2 906
<hr/>	

**Ex. 33.** By ordinary and contracted methods, determine the value of the following fractions to four significant figures:

$$(a) \frac{9999}{1274},$$

$$(e) \frac{3533}{172.3},$$

$$(b) \frac{7831}{6103},$$

$$(f) \frac{3756}{9834},$$

$$(c) \frac{0.9999}{0.1274},$$

$$(g) \frac{9.009}{15.06},$$

$$(d) \frac{0.7831}{0.6592},$$

$$(h) \frac{0.001293}{0.04567},$$

(i) What is the value of  $\frac{a}{b}$  when  $a=7.03$  and  $b=9.52$ ?

(j) What is the value of  $\frac{wy}{z}$  when  $w=2$ ;  $y=27$ ;  $z=6$ ?

Give an approximate mental estimate of the values in the following examples and then determine the exact values by calculation:

$$(k) 327 \times 56.3,$$

$$(l) .002 \times 1.0018 \times 0.996,$$

$$(m) \frac{1.003}{0.997},$$

$$(p) \frac{1}{98},$$

$$(n) \frac{0.993}{1.005},$$

$$(q) \frac{1}{506},$$

$$(o) \frac{1.0023 \times 0.9984}{1.0016 \times 0.98},$$

$$(r) \frac{1}{1.009}.$$

**7. Properties of Simple Geometric Constructions.** The retention of more decimal figures than the work warrants is dishonest and without meaning in commercial practice. The **degree of accuracy** with which we make our measurements determines the number of significant figures which should be used.

**Ex. 34.** Use a straightedge and draw a line about 4.5 inches. For convenience in referring to this line we label it by placing a letter "L" above it. With a 12-inch scale or rule

measure "*L*" to the nearest quarter-inch division. Express the answer as a mixed number and also in terms of its decimal equivalent.

Remeasure "*L*" to the nearest eighth-inch division. Express the answer as a mixed number and also as a decimal. How many decimal figures are dependable?

Remeasure "*L*," using sixteenths of an inch as the unit of measure. In expressing the answer be careful not to use more figures than the work warrants.

**Ex. 35.** Using  $\frac{1}{8}$  of an inch, the smallest subdivision of a 12-inch rule, measure the length of a pencil. Suppose we discover that there are 7 full inches and a remainder lying between  $\frac{3}{8}$  and  $\frac{4}{8}$  scale divisions, but nearer the former. We should be correct if we write the length of the pencil =  $7\frac{3}{8}$ " =  $7\frac{1.5}{2}$ " = 115 times the unit  $\frac{1}{8}$ ". Now  $7\frac{3}{8}$  = 7.1875 = 7.19, approximately. The answer is not reliable beyond 7.18, and therefore the figures 75 should be omitted and added in as 1 to 8 in the second decimal place. This is explained by considering the fact that we cannot measure closer than one-half of the smallest scale division. Therefore the degree of accuracy equals

$$\frac{115 \text{ measured divisions}}{\frac{1}{2} \text{ division (estimated)}} = \frac{230}{1} = \text{one part in 230.}$$

Therefore the figure 8 or 9 in the second decimal place is questionable, because an error 1 too large or 1 too small in this means an accuracy of one part in 700.

Explain why the answer may have been written 7.21 and still lie within the limits of accuracy.

**Ex. 36.** Three quantities, *A*, *B*, *C*, are measured. The number of units in *A* = 8096, the number of units in *B* = 9.91, and the number of units in *C* = 0.025. Suppose an error of 1 is made in the last figure of each measurement, what is the degree of accuracy in each result?

*Observation.* The degree of accuracy is not indicated by the position of the decimal point, but by the number of significant figures.

**Ex. 37.** Rule a sheet of paper into squares as shown on page 25 by drawing horizontal and vertical lines equally spaced  $\frac{1}{4}$ " apart. Leave a blank marginal space at the four edges. This is called **squared cross-section paper** and also **squared paper**. The trade supplies such paper divided into eighths of an inch, tenths of an inch, and also into millimeters.

Fig. 4 shows the method of using a **T-square** and a **right triangle** for constructing and testing parallel and perpendicular lines. The T-square has its upper edge planed true, i.e., straight, and is therefore a **straightedge**. The T-square is held rigidly to the paper to prevent its slipping, while the triangle is slid along the straightedge. The two triangles shown in Fig. 5 are used in exactly the same way as the T-square and triangle shown in Fig. 4, excepting that either of the triangles may serve as the straightedge. Lines which are parallel are marked in a like manner by single, double, or triple strokes or crosses. The two right triangles are called  $30^{\circ}$ - $60^{\circ}$  and  $45^{\circ}$  respectively. Test the angles of the triangles and the sum of the angles of the triangles.

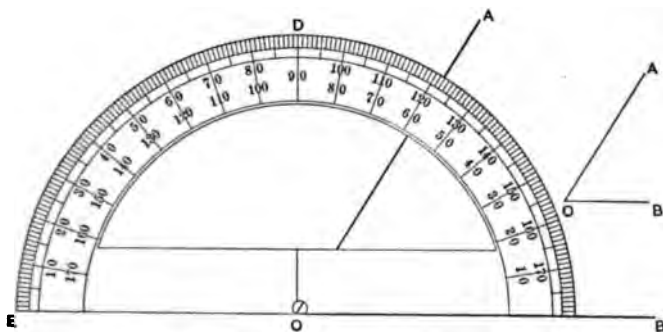


FIG. 1.—A Protractor.

**Ex. 38A.** Test the angles of your squared paper with T-square and right-angled triangles and measure the angles with a **protractor**. An **angle** has two sides. The point of intersection of the sides is called the **vertex**. To measure an angle with a protractor, place the center of the latter at the vertex of the angle with the zero mark of the graduated arc over one side of the angle, read the nearest degree of arc over the other side of the angle. In Fig. 1 the angle  $AOB$  is measured by placing the center of the protractor at the vertex  $O$  and  $0^{\circ}$  of the protractor on side  $OB$ , then  $OA$  cuts the protractor at  $58^{\circ}$ . Therefore  $AOB = 58^{\circ}$ .

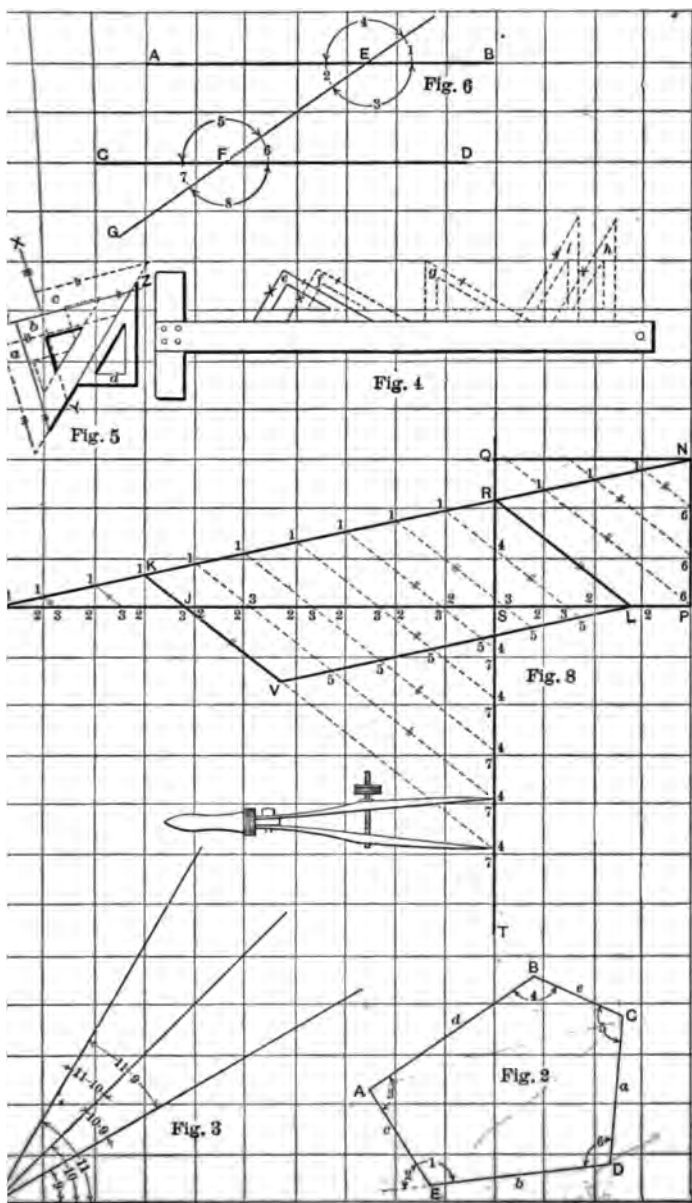
**Lines** (always refer to straight lines) are designated by two capital letters placed usually at the extremities of the line or by a single lower case letter which usually corresponds to the capital letter at the opposite vertex. When the line forms part of a figure (see Fig. 2) a small arc is often drawn at the vertex of the angle with an arbitrary number inserted in the arc instead of the number of degrees. There may be several angles having an equal degree measure, but these may be distinguished readily with accents by calling them angles 1, 1', 1'', 1'''. If they are unequal, it is better to call them more distinctly by angles 1, 2, 3, 4, etc. Arcs are usually designated by three letters (see Fig. 7). Consult the list of symbols and the list of abbreviations.

**Ex. 38B.** Through the lower left-hand corner *O* of the squared paper draw three lines making angles 9, 10, 11, respectively, with the lower ruled bounding line, see Fig. 3. As near as your eye can judge without measuring, make these angles approximate to  $30^\circ$ ,  $45^\circ$ ,  $60^\circ$ , respectively. Now use the protractor to measure angles 9, 10, 11, and also mark and measure the angle 11 minus angle 10; angle 11 minus 9; angle 10 minus 9. What is the degree of accuracy in each measurement, assuming that we can estimate to one-half of a degree?

*Observation.* In measuring quantities with the same unit of measure the greatest magnitude has the greatest degree of accuracy.

Two **parallel lines** lie in the same plane and do not meet or intersect within a finite distance even when produced or extended. Lines are either **parallel** or **non-parallel**. Name the sets of parallel and non-parallel lines on the squared paper, which you have constructed. Why are they parallel or non-parallel?

Intersecting vertical and horizontal lines are **perpendicular**. A **perpendicular** is a line which forms a **right angle**, i.e., a  $90^\circ$  angle with another line. Name the sets of perpendicular and non-perpendicular lines on the squared



paper, which you have constructed. Why are they perpendicular or non-perpendicular?

**Ex. 39.** Construct Fig. 6, which represents a portion of the squared paper, with two of the horizontal parallel lines  $AB$  and  $CD$ , accentuated. These are cut at  $E$  and  $F$  respectively by the transversal  $GH$ . It is called a transversal because it is a straight line which cuts across several other lines. These intersecting lines form four angles, 1, 2, 3, 4, at  $E$ , and four angles, 5, 6, 7, 8, at  $F$ . If these angles are measured and taken in pairs, we find they are either equal or supplementary.

Angles whose sum is  $180^\circ$  are **supplementary**. Angles whose sum is  $90^\circ$  are **complementary**.

Angles having a common side and common vertex and two sides exterior are **adjacent**.

Angles 6 and 3, 5 and 2, 8 and 1, 7 and 4, are supplementary. Verify by measurement.

Angles 1 and 4, 1 and 3, 3 and 2, 2 and 4, are **supplementary adjacent**. Why? Name all other supplementary adjacent angles in Fig. 6.

**Vertical** angles lie opposite to the intersection of two lines and are non-adjacent but equal. 1 and 2 are vertical angles. Name the other vertical angles.

**Interior** angles lie within the parallels and **exterior** angles lie outside the parallels. When angles lie on opposite sides of the transversal they are called **alternate**. When they lie on the same side they are called **unilateral** or **direct**.

Angles 2 and 6 are **alternate interior**. Name one other pair of alternate interior angles. Are they equal or unequal? Why?

Angles 1 and 7 are **alternate exterior**. Name one other pair of alternate exterior angles. Are they unequal or equal? Why?

Use a sheet of thin tracing paper and carefully copy your drawing of Fig. 6, using thumb tacks to secure the work in position. Move the copy vertically downward by following the left-hand margin. Observe the direction

relation of all corresponding, i.e., like named lines, and the magnitude relations of all correspondingly numbered angles.

*Direction axiom. Two lines are either parallel, perpendicular, or oblique.*

*Magnitude axiom. One quantity is either less than, equal to, or greater than another quantity.*

Move the copy horizontally to the right by following the lower margin. Observe the direction of all corresponding lines and the magnitude relation of all correspondingly numbered angles.

Restore the copy or tracing to its original position.

Without lifting the copy move it through  $90^\circ$ , keeping the  $F$  points matched by pivoting with a needle point. The  $AB$  line of the copy will then be at right angles to the  $AB$  line of the squared paper. Observe the direction of all corresponding lines and the relations of all correspondingly numbered angles.

**Ex. 40.** Locate a point near the center of the work sheet and mark this point " $O$ ." Through  $O$  draw a horizontal line  $1\frac{1}{2}$ " to the right. Label the right end of the line " $A$ ." Set your compass to extend from  $O$  to  $A$  and with  $O$  as a center draw a circle. What is the radius of the circle and what line can be used to represent it. Place a needle point at  $O$  and the straightedge along  $OA$ . Rotate the straightedge in a counter-clockwise direction until it is in a vertical position, and represent it by a line  $OB$ . What kind of angle and how many degrees have been described in the rotation?  $OA$  is the **initial side** of the angle, and  $OB$  is the **terminal side** sometimes referred to as the **radius vector** or moving arm. An **acute angle** is greater than  $0^\circ$  but less than  $90^\circ$ . Continue the rotation until the straightedge is horizontal and draw  $OC$  the terminal side of the straight angle thus formed.  $COA$  is a **straight angle** because its sides  $OA$  and  $OC$  are in the same straight line with the vertex  $O$ . An **obtuse angle** is greater than  $90^\circ$  but less than  $180^\circ$ .

Referring to Fig. 7,  $O$  is the **center** of the circle. The **perpendicular diameters**  $CA$  and  $BF$  divide the circle into

four quadrants I, II, III, and IV, and also indicate the four directions which are designated by the four cardinal points N, E, W, S, respectively. If we stand facing the north the right hand will extend to the east.

The boundary of the circle is called the **circumference** and any portion of the latter, such as *LDA*, is called an **arc**, i.e., an arc of the circle.

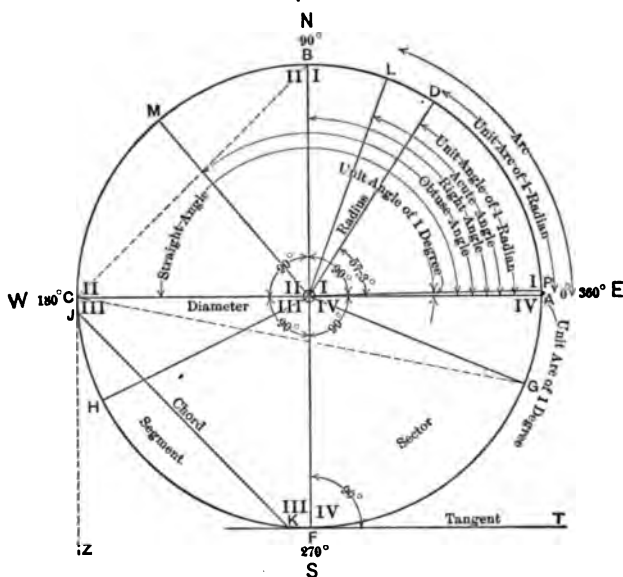


FIG. 7.

An angle such as *LOA*, whose vertex is at the center of the circle, is called a **central angle** and its sides are radii (plural of radius) of the circle.

A **radius** is a line, such as *OD*, joining the center of a circle to a point in the circumference. A **chord** is a line joining two points in the circumference. A **diameter** equals twice the radius, passes through the center, and is the longest chord.

An **inscribed angle**, such as  $ACG$ , has its center on the circumference and its sides are chords of the circle.

An arc may be described in terms of the angle which intercepts it as arc  $LD$  or  $\sim \angle LOD$ .

The circumference of the circle has  $360^\circ$  of arc, and each quadrant contains an arc of 90 arc degrees and a central angle of 90 angle degrees. The sum of the angles about the central point  $O$  equals  $360^\circ$ . A unit central angle intercepts, i.e., cuts off, a unit arc. Therefore a central **angle of one degree**, i.e., one-ninetieth of a right angle, intercepts an **arc of one degree**, i.e., one-ninetieth of a quadrant arc. A central **angle of one radian** (approximately  $57.3^\circ$ ) intercepts **one radian of arc** (a length equal to the radius). A central angle is measured by the arc which it intercepts.

An angle may be measured, therefore, by a protractor by placing the center of the protractor at the vertex of the angle and reading the number of degrees intercepted on the arc.

**Ex. 41.** Referring to Fig. 7 designate all the central angles having the initial side  $OA$ , and state also their respective intercepted arcs. These angles are also designated as angles of the I, II, III, or IV quadrants, if their terminal sides lie in these respective quadrants. Describe the angles lying in each quadrant of Fig. 7.

**Ex. 42A.** What will be the position of the terminal side of the following angles:  $30^\circ$ ,  $45^\circ$ ,  $57.3^\circ$ ,  $60^\circ$ ,  $114.6^\circ$ ,  $120^\circ$ ,  $135^\circ$ ,  $150^\circ$ ,  $210^\circ$ ,  $240^\circ$ ,  $270^\circ$ ,  $300^\circ$ ,  $360^\circ$ ?

**Ex. 42B.** Construct the angles mentioned in Ex. 42A and with a radius of 2.25" draw a **concentric circle**, i.e., a circle with the same center  $O$ . Extend the sides of the angles to the new circumference. Does this extension of the sides of the angles in any way effect the degree measure of the angles or the measure of their respective arcs on the new circumference?

**Ex. 43.** Designate the chords in Fig. 7. Designate the inscribed angles and the respective arcs which they intercept; the same arcs are also intercepted by which respective central angles? Show by measurement that each of the inscribed angles is measured by one-half of its respective arc.

A line such as  $FT$  which touches a circle in one point is called a **tangent**.

**Ex. 44.** What is the greatest value which the inscribed angle can assume when  $CA$  is kept fixed and the chord  $CG$  is rotated about  $C$ ? Is a tangent perpendicular to a radius drawn to its point of contact? Why?

**Ex. 45.** Construct a tangent to a circle and through the point of contact  $P$  draw a chord which terminates at the point  $R$ . Determine the relation of the angle formed by tangent and chord compared with the intercepted arc.

**Ex. 46.** Draw two parallel chords. Measure the arcs intercepted between the chords. What is their relation?

**Ex. 47.** Draw a tangent and a parallel chord. What is relation between all the arcs intercepted?

**Ex. 48.** From a point  $C$  draw two tangents to a circle with center  $O$ . Measure the tangents, the angle between the tangents and the central angle drawn to the point of contact. Draw  $OC$  and remeasure the central angles. By what name would you call  $OC$ ?

**Ex. 49.** From a point  $E$  draw a tangent and a prolonged chord, i.e., a secant. Measure all lines and arcs and state their relation.

**Ex. 50.** Draw two angles 1 and 2 whose sides are parallel and extend in the same direction, measure them and state their relation.

Draw an angle 3 whose sides are parallel to 1 but extending in the opposite directions, measure and state their relation.

Draw an angle 4 whose sides are parallel to 1 with one pair of sides extending in the same and the other pair in opposite directions, measure and state their relation.

**Ex. 51.** Draw two acute angles 5 and 6 with their sides perpendicular, measure and state their relation.

Draw two obtuse angles 7 and 8 whose sides are perpendicular, measure and state their relation.

Draw an obtuse angle 9 whose sides are perpendicular to 5, measure and state their relation.

**Ex. 52. The Bisection of an Angle.** On the sides of angle 1 beginning at the vertex lay off a unit distance (one inch) and at these points draw perpendiculars. The perpendiculars (two straight lines) can intersect only in one point. The line passing through the intersection and the vertex is the **bisector** of the angle. Draw the bisector and verify by measuring the **resulting angles**. Measure the perpendiculars and state their

relation. From the unit distant points as centers strike arcs of equal radius. Their intersection lies on the bisector of the angle.

**Ex. 53. The Bisection of a Line.** Draw a straight line. Label its extremities *A* and *B*, respectively. From *A* and *B* strike equal arcs which intersect above and below the line at points *C* and *D* respectively. A line passing through *C* and *D* is the perpendicular bisector of *AB*.

**Ex. 54.** The method of Ex. 53 may be used to draw a perpendicular to a line through a given point on or off the line, *AB*. Select a point *C* off the line. From *C* draw any arc intersecting the line *AB* at *E* and *F*. Use *E* and *F* as centers and strike equal arcs intersecting above and below the line at points *G* and *H* respectively. The line *GH* is the perpendicular to the given line through *C*. Repeat changing the position of *C* so that it is on the given line.

**Ex. 55.** Draw a line *XY* about 4" in length. Locate a point *P* about 2" above the middle of *XY*. Draw five lines from *P* to *XY* two of which shall be equal and one of which shall be a perpendicular. Which is shortest? Measure and record any other facts of interest which you observe.

**8. Axioms and Their Applications.** Through observation, investigation, and reflection we learn that there are many fundamental truths which are axiomatic. These are formulated in groups according to their generality.

Several axioms were cited above, and through additional observation, investigation, and reflection we are able to appreciate the list of axioms on page 32.

These fundamental truths may be summarized under a single statement called the axiom of operations.

**Axiom of operations (Ax. Op.).** An equality is preserved when a like operation is performed upon both members of an equation, otherwise an inequality results.

The Ax. Op. includes addition, subtraction, multiplication, division, power, root, and trigonometric, logarithmic, limiting, derivative, and integral operations. By their application we are enabled to solve equations, i.e., determine the values of the quantities in the equations as

AXIOMS		APPLICATIONS	
NO.	ABBREVIATION	STATEMENT	
			GIVEN
			RESULT
(1)	=ity Ax. (Equality)	Things equal to the same or equal things are equal	$a = b$ and $c = b$ $ax = 3 \times 4$ and $5y = 6 \times 2$ $a = c$ $ax = 5y$
(2)	Add. Ax. (Addition)	When equals are added to equals or to the same thing equal sums result	$a + 3 = b + 3$ $x = 10$
(3)	Sub. Ax. (Subtraction)	When equals are subtracted from equals or the same thing equal remainders result.	$x - 3 = 7$ ( $x - 3 + 3 = x$ ) $a = b$ ( $b - b = 0$ ) $x + 3 = 8$ $a - b = 0$ $x = 5$
(4)	Mul. Ax. (Multiplication)	When equals are multiplied by equals or the same thing equal products result	$ac = bd$ $\frac{x}{5} = y$ ( $\frac{5x}{5} = x$ ) $x = 5y$
(5)	Div. Ax. (Division)	When equals are divided by equals or by the same thing equal quotients result	$\frac{a}{c} = \frac{b}{d}$ $a = b$ and $c = d$ $\frac{y}{5} = \frac{y}{5}$ $5x = y$
(6)	{ Power } Ax. { Root }	Like powers or roots of equals produce equal results	$a^2 = b^2$ $\sqrt{a} = \sqrt{b}$ $81 = 16x^4$ $3 = 2x$

shown below, by separating a specified letter from all the others.

GIVEN	RESULT	AXIOM WHICH WAS APPLIED	OPERATOR
$P = 3p$	$p = \frac{P}{3}$	Division	Divisor 3
$P = 3mp$	$p = \frac{P}{3m}$	"	" 3m
$3mp = P$	$m = \frac{P}{3p}$	"	" 3p
$P = EI$	$E = \frac{P}{I}$	"	" I
"	$I = \frac{P}{E}$	"	" E
$r = T - R$	$T = r + R$	Addition	Adjunct R
$I = \frac{V}{r}$	$V = Ir$	Multiplication	Multiplier r

**Ex. 56.** Prepare a list of the equations given below; show the solution for each letter, stating the axiom which was applied and the operator used.

- $I = \frac{E}{R}$ , solve for  $E$ , and from this result solve for  $R$ .
- $V = rI$ , solve for  $I$ , and also for  $r$ .
- $R + r = T$ , solve for  $r$ , also for  $R$ .
- $I = EG$ , solve for  $E$ , also for  $G$ .
- $B = \frac{rb}{a}$ , solve for  $rb$ , and from the result solve for  $a$ ,  $r$  and  $b$ .
- $G = \frac{rb}{a}$ , proceed as in (e) and solve for  $a$ ,  $r$ , and  $b$ .
- $x = \frac{rb}{a}$ , proceed as in (e) and solve for  $a$ ,  $r$ , and  $b$ .

(h)  $x = \frac{bR}{a}$ , proceed as in (e) and solve for  $a$ ,  $b$  and  $R$ .

(i)  $R = \frac{rl}{A}$ , solve for  $A$ ,  $r$  and  $L$ .

(j)  $c = \frac{1}{r}$ , solve for  $r$ .

(k)  $E = \frac{Re}{r}$ , solve for  $r$ ,  $R$ , and  $e$ .

(l)  $C = \frac{cD}{d}$ , solve for  $c$ ,  $d$ ,  $D$ ,  $\frac{C}{c}$ ,  $\frac{c}{C}$ .

A line drawn through a symbol negatives that symbol, thus  $a \neq b$  means  $a$  is unequal to  $b$ . If we wish to give the **sense** of the inequality, the symbol  $>$  or  $<$  must be written between the quantities with the larger quantity in the opening of the wedge symbol.  $a > b$  means  $a$  is greater than  $b$  and  $\pi < \frac{2}{7}$  means  $\pi$  is less than  $\frac{2}{7}$ .

An **inequality** is obtained from an equality when any of the above axioms is violated. Suppose we have  $25a = 25a$  and subtract  $3a$  from one side and  $6a$  from the other. Then these two subtractions have like minuends but unlike subtrahends, and the greater remainder results from the subtraction of the lesser subtrahend.

$$\text{Equality } 25a = 25a,$$

$$\text{lesser subtrahend } 3a < 6a \text{ greater subtrahend,}$$

$$\text{greater remainder } 22a > 19a \text{ lesser remainder.}$$

**Ex. 57.** Perform the operations of addition, multiplication division, power and root with unequal operators on the equation  $25a = 25a$ . State the relation of the results. Can the Axiom of Operations be extended to inequalities providing the operators are equal?

In addition to the axioms of operation we have those observed from the construction work as follows:

**Magnitude Axiom.** One quantity is either less than, equal to or greater than another quantity. The whole is equal to the sum of its parts. A quantity may be substituted for its equal in any equation or inequality.

**Direction Axiom.** Two lines are either parallel, perpendicular, or oblique. Any line has the same direction as a line with which it coincides.

**Point line Axiom.** A point is or is not on a line. Between two points only one straight line can be drawn. Two straight lines intersect in a point. Through a point only one line can be drawn parallel to a given line.

**Construction Axiom.** Any construction line may be added to a geometric figure provided its relation does not contradict or violate the relation of the parts of the given figure.

**Motion Axiom.** An entire figure or part thereof may be moved about in a plane or in space and restored or superposed in any other position provided the essential relations of the parts are unaltered.

**Plane Axiom.** A plane is determined by three points not in the same straight line; by a line and a point outside of it; by two intersecting lines; by two parallel lines.

**Reductio ad absurdum** (Red. ad ab.) Axiom. A statement of relationship is validated when all contradictory or opposite constructions have been falsified, i.e., reduced to an absurdity.

**9. Summary of Theorems on Lines and Angles.** A **theorem** is unlike an axiom because it is a truth admitting of or requiring demonstration. A **demonstration** is a formal array of statements each in logical sequence and each validated by the sufficient necessary authorities. The list of proper **mathematic** authorities to substantiate **mathematic** statements includes:

**First**, the given stated condition called the hypothesis.

Second, axioms of equations, inequalities, constructions, relation, motion, and position.

Third, accurately formulated definitions.

Fourth, a theorem previously demonstrated.

**Geometry** is one of the most beautiful structures of thought in which the above cited authorities secure the safety of the mind's edifice. Although the theorems may be discovered in any order, it is essential that they should be demonstrated in a systematic manner, owing to their interdependence. By definition a straight angle, as its name implies, is an angle whose sides are in the same straight line with the vertex. In Fig. 6,  $AEB$  is a straight angle, because its sides  $EA$  and  $EB$  are in the same straight line with the vertex  $E$ . By measurement we discover that  $AEB = 180^\circ$ . Do all straight angles  $= 180^\circ$ ? The theorem establishes the truth when demonstrated that every straight angle equals  $180^\circ$ .

The following theorems are formulated from the observations made above. They are so simple, in fact, that they are often accepted without demonstration. They are given in the order in which they would have to be demonstrated. If the student has not recognized them, he should review the preceding work until he is entirely familiar with their significance, by experiment, construction, and measurement.

1. A straight angle equals  $180^\circ$ .
2. Adjacent angles whose exterior sides are in a straight line are supplementary.
3. Supplementary adjacent angles have their exterior sides in a straight line.
4. Vertical angles are equal.
5. At a point in a line only one perpendicular can be drawn through the line.
6. From a point outside a line only one perpendicular can be drawn to the line.
7. Any point on a perpendicular erected at the middle

of a line is equally distant from the extremities of the line.

8. Any point not on a perpendicular erected at the middle of a line is unequally distant from the extremities of the line.
9. A perpendicular is the shortest line which can be drawn from a point to a line.
10. Lines perpendicular to the same line are parallel.
11. If one of a number of parallel lines is perpendicular to a line, the others are perpendicular to the same line.
12. Lines parallel to the same line are parallel to each other.

**When**

**If**

*two parallel lines are cut by a transversal, then*

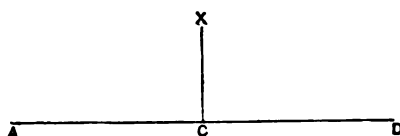
- |  |  |
|--|--|
| 13. Alternate-interior angles are equal  | 14. Alternate - interior angles are equal, or                  |
| 15. Exterior-interior angles are equal   | 16. Exterior-interior angles are equal, or                     |
| 17. Unilateral-interior angles are supplementary   | 18. Unilateral-interior angles are supplementary, or           |
| 19. Alternate-exterior angles are equal  | 20. Alternate-exterior angles are equal,                       |
|  | <i>then the two lines cut by the transversal are parallel.</i> |
| 21. Angles are equal when their pairs of sides are parallel and extend in the same or opposite direction. They are supplementary when one pair of parallel sides extend in the opposite direction from the vertex. |  |
| 22. Angles are equal when their pairs of sides are perpendicular and both are acute or both obtuse. They are supplementary when one of these angles is acute and the other obtuse.                                 |  |

23. Every point in the bisector of an angle is equally distant from the sides of the angle.

10. **Demonstration.** The demonstration of theorem 1 which follows illustrates the rigorous formal proof and is typical of all logical methods of mental discipline.

### I THEOREM 1

(Theorem) A STRAIGHT ANGLE EQUALS  $180^\circ$



(hypothesis) Given St  $\angle ACD$

(conclusion) Prove  $\angle ACD = 180^\circ$

(authorities)

- |  |  |                 |
|--|--|-----------------|
| <div style="display: flex; align-items: center;"> <div style="border-left: 1px dashed black; height: 100px; margin-right: 5px;"></div> <div style="display: flex; flex-direction: column; justify-content: space-around; padding-left: 5px;"> <div>(proof)</div> <div></div> <div></div> <div></div> <div></div> <div></div> <div></div> <div></div> </div> </div> | (1) $ACD$ is a St $\angle$ _____                           | Hyp             |
|  | (2) $\therefore AD$ is a St line _____                     | Def St $\angle$ |
|  | (3) Draw $XC \perp AD$ _____                               | Cons Ax         |
|  | (4) Then $\angle ACX = 90^\circ$ _____                     | Def $\perp$     |
|  | (5) and $\angle XCD = 90^\circ$ _____                      |                 |
|  | (6) $\therefore \angle ACX + \angle XCD = 180^\circ$ _____ | Add Ax          |
|  | (7) but $\angle ACD = \angle ACX + \angle XCD$ _____       | Mag Ax          |
|  | (8) $\therefore \angle ACD = 180^\circ$ _____              | =ty Ax          |

Therefore a straight angle equals  $180^\circ$ .

*Explanation.* The Arabic numeral 1 in the heading stands for the first theorem in Book I. A book was the Greek equivalent of our modern designation of a chapter or part, and in geometry has been tenaciously adhered to since the time of Euclid.

The theorem states the facts as they have been observed, developed, and formulated in the constructions and measurements.

The demonstration proves the fact generally, i.e., for all straight angles

The **figure** is a linear representation of the thing mentioned in the hypothesis, and in this case the picture of the solid lined straight angle  $ACD$  is a mental aid in keeping the condition before us.

A theorem is always analyzed into two parts: hypothesis and conclusion. These are stated immediately after the figure.

The **hypothesis** begins with the words given, and therefore relates the known things both by definition name and by figure name.

The **conclusion** is the fact which is to be established.

The numbered statements begin with a recitation of the fact stated in the hypothesis (Hyp), and are validated by the abbreviated authorities which are placed at the right end of their respective lines.

(1) states that  $\angle ACD$  is a straight  $\angle$ . Upon investigation of the definition of a straight angle (St  $\angle$ ) we are led to make statement (2), which is authorized by quoting the definition of a straight angle (Def St  $\angle$ ).

We would be at a standstill now if it were not for the fact that we have learned that a right angle has  $90^\circ$ . But twice  $90^\circ$  equals  $180^\circ$  and this seems to suggest building two right angles out of  $\angle ACD$ .

In (3) we draw the dash (construction) line  $XC$  perpendicular ( $\perp$ ) to  $AD$  at  $C$ . We are permitted to do this by the Construction Axiom (Cons Ax) which states that the solid lines may be supplemented by any **construction lines (dash)** which do not represent conditions contrary to the hypothesis. This construction creates  $\angle s ACX$  and  $XCD$ .

(4) and (5) state the degree measure of  $\angle s ACX$  and  $XCD$  from the definition of a perpendicular (Def  $\perp$ ).

In (6) the sum of  $\angle s ACX$  and  $XCD$  equals the sum of their equals. Therefore  $\angle ACX + \angle XCD = 90^\circ + 90^\circ = 180^\circ$ . We have quoted the Addition Axiom which says if equals to equals their sums are equal.

(7) brings in an independent fact suggested by the examination of the figure in which it is evident that the whole angle  $ACD$  is composed of the two angles  $ACX$  and  $XCD$ . This statement is therefore backed by the Magnitude Axiom (Mag Ax) which states that the whole equals the sum of its parts.

The Equality Axiom states that things equal to the same thing are equal to each other ( $= ty$  Ax). Taking the statements (6) and (7) together we form the new equality (8).

This ends the proof because we have arrived at the conclusion.

Having conclusively established the theorem it is consistent and even preferable to write it at the end beginning with the word therefore.

**11. Triangles, Quadrilaterals and Polygons.** In general, if we have two or more figures of the same kind, and can make them **coincide**, i.e., place one over the other into exact correspondence in position, they are said to be **equal figures**. In each figure the corresponding lines will be equal and the corresponding angles will be equal. This principle enables us to measure a line or lay off a distance with the dividers. The points of the dividers are adjusted to the extremities of the line and without alteration they are applied to a scale. The measure of the distance between divider points is also the measure of the length of the line. This is illustrated in Fig. 8.

It would be exceedingly laborious were we compelled to resort to this matching process whenever two triangles were to be proven equal. There are **six parts** to every triangle, viz: the three angles and the three sides. If three of these parts, including at least one side, are respectively equal, then the triangles are equal.

**Ex. 58.** Make a list of all the lettered triangles in Fig. 8.

**Ex. 59.** Draw an acute angle 1, with vertex  $O$ , and unequal sides  $AO$  and  $OB$ . Measure 1,  $AO$ ,  $OB$ . On a sheet of tracing paper draw a line  $ED = OB$  and parallel to it. Through  $E$  draw

$EF$  parallel to and  $=$  to  $AO$ , forming angle  $1'$ . Why is  $1 = 1'$ ? Why are the sides equal in length? Join  $A$  with  $B$  forming triangle  $AOB$  and join  $D$  with  $F$  forming triangle  $FED$ . Now place  $AOB$  upon  $DEF$  so that the equal parts coincide or match. Do all six parts of the two triangles coincide? Therefore the two triangles are in what relation?

Construct a triangle  $GHJ$ , measure  $GH$ , and the angles  $GHJ$  and  $HGJ$ . On tracing paper construct a triangle  $KLM$ , making  $KL = GH$ , and angles  $KLM = GHJ$ , and  $LKM = HGJ$ , respectively. Two angles may be laid off equal by making their sides respectively parallel or perpendicular, according to theorems 21 and 22, or using the protractor. We may construct two angles equal by laying off equal arcs of equal radius, using the vertices as centers. See Fig. 9  $I$  and  $I'$ . Will these triangles coincide? Therefore the two triangles are in what relation?

Construct a triangle  $PQR$ , and measure its three sides  $PQ$ ,  $QR$ , and  $RP$ . On tracing paper draw  $ST = PQ$ , from  $S$  as a center strike an arc whose radius equals  $PR$ ; from  $T$  as a center strike an arc whose radius equals  $QR$ . At the intersection of the two arcs place the letter  $V$ . Join  $V$  with  $S$  and  $T$ , forming triangle  $STV$ . Will triangle  $STV$  coincide with  $PQR$ ? Therefore the two triangles are in what relation?

Two triangles are **equal** when they have the following parts respectively equal as illustrated in Fig. 9:

One side and two angles  $I I'$ ,

Two sides and one angle,  $II II'$ ,

Three sides,  $III III'$ .

In a right-angled triangle this reduces to,

One side and an acute angle,  $IV IV'$ ,

A leg and the hypotenuse,  $V V'$ ,

Two legs,  $VI VI'$ .

A **right triangle** is one having a right angle. The side opposite the right angle is called the **hypotenuse** and the other two sides are called legs or arms or simply sides.

**Ex. 60.** Draw two parallel lines  $UW$  and  $XY$ . On  $UW$  locate a point  $Z$ , and connect  $Z$  with  $X$  and  $Y$ . Show that  $UZX = ZXY$ , also  $WZY = ZYX$ .

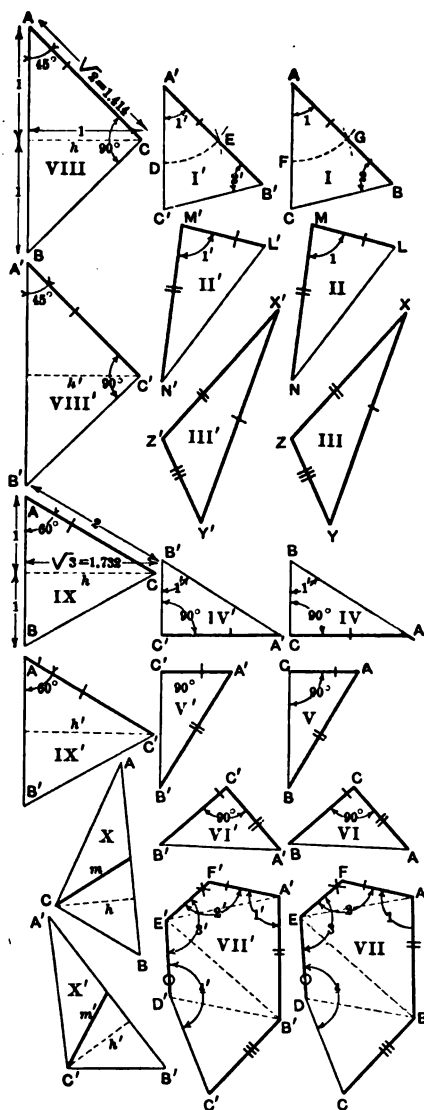


FIG. 9.—Equal Triangles.

See theorem 13. But  $\angle s\ UZX + XZY + WZY = 180^\circ$ . Why? Replace the  $\angle s\ UZX$  and  $WZY$  by their equal angles in the triangles. Therefore:

The sum of the angles of a triangle equals  $180^\circ$ .

**Ex. 61.** In the preceding figure extend  $XY$  to a point  $V$ . An exterior angle  $ZYV$  of a triangle is formed between a side and an adjacent side produced or extended. Show that  $\angle s\ ZYV = ZXY + ZXY$ . Therefore:

The exterior angle of a triangle is equal to the sum of the two non-adjacent interior angles.

A four-sided figure is called a **quadrilateral**.

A quadrilateral whose opposite sides are parallel is called a **parallelogram**. Construct a parallelogram and measure its opposite sides. What is their relation? The **diagonals** are constructed by joining the opposite vertices. The intersection of the diagonals divides them in what relation? One diagonal will divide the parallelogram into two equal triangles. Why? A **trapezoid** is a quadrilateral with one pair of parallel sides.

**Ex. 62.** Specify all the lettered quadrilaterals in Fig. 8 both by title and figure name.

**Ex. 63.** In Fig. 8, the dash lines are drawn parallel through the equidistant points 1, 1, 1 . . . 1 on  $MN$ . Every transversal of these parallels will be subdivided (multisected) into uniformly spaced divisions. If two transversals are parallel their sections called **segments** are mutually equal. Check the point spacing on all transversals by means of bow dividers as shown in the figure.

**Ex. 64.** In Fig. 8 how many parallelograms are formed by the dash lines?

**Ex. 65.** In Fig. 8 how many trapezoids are formed by the dash lines?

**Ex. 66.** In Fig. 8 how many triangles are formed by the dash lines?

Triangles are **similar**, i.e., of the same shape under any one of the following conditions:

When their **homologous** (like placed) angles are equal,  
When their **homologous sides** are parallel,

When the homologous sides are perpendicular,  
When their homologous sides are in the same ratio, i.e.,  
proportional.

**Ex. 67.** In Fig. 8 there are how many groups of similar triangles and how many similar triangles in each group?

A many sided figure is called a **polygon** ( $n$ -gon). A limited number of polygons have special names according to the following list:

NUMBER OF SIDES OR ANGLES	NAME	FIGURE-GON
3	triangle, trilateral, or trigon	3-gon
4	quadrilateral, quadrigon	4-gon
5	pentagon	5-gon
6	hexagon	6-gon
7	heptagon	7-gon
8	octagon	8-gon
9	nonagon	9-gon
10	decagon	10-gon
11	undecagon	11-gon
12	duodecagon	12-gon
15	pentadecagon	15-gon
20	icosagon	20-gon

**Regular** polygons have equal sides and equal angles. Regular polygons are **inscribed** (constructed internally) in a circle by laying off equal chords, striking equal arcs, forming equal central angles.

An irregular polygon may result from an inequality of sides, or an inequality of angles, or by the substitution of arcs for any of its sides.

**Ex. 68.** Name the pairs of regular and irregular polygons in Fig. 10.

Polygons are similar when their corresponding sides are proportional and their corresponding angles are equal. Regular polygons of the same name are similar. A set of similar figures can be decomposed.

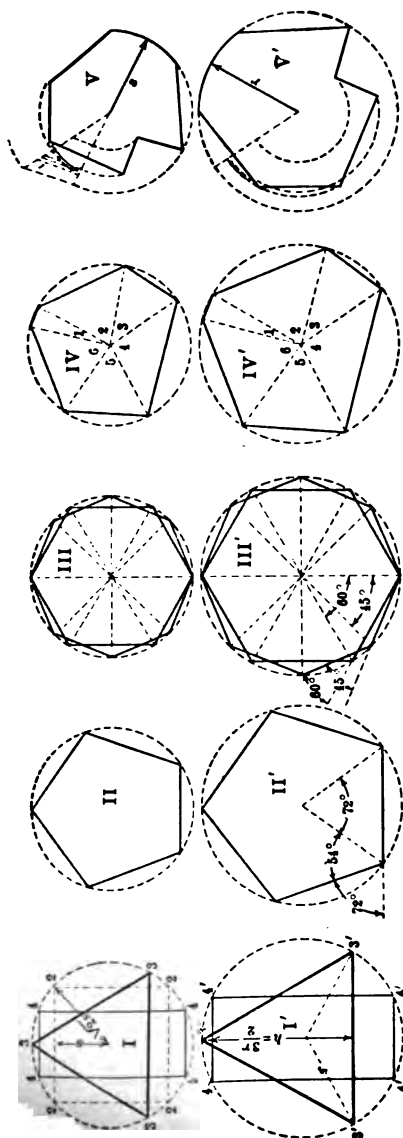


FIG. 10.—Regular and Irregular Polygons.

similar figures by joining corresponding points in the first set.

**Ex. 69.** Name the similar polygons in Fig. 10? Are polygons similar when their sides are proportional and also either parallel or perpendicular?

**Ex. 70.** Why are the triangles in Fig. 10 similar?

**Ex. 71.** Which of the polygons in Fig. 10 show an exterior angle, what is its relation to the central angle subtended by the side which was prolonged?

**Ex. 72.** What is the value of the central angle for constructing the chord of each regular polygon in Fig. 10?

**Ex. 73.** What is the value of the angle formed by adjacent sides in each polygon of Fig. 10?

**Ex. 74.** Suppose we designate the number of sides of the polygon by  $n$ , then show that the sum of the angles of the polygons in each case will be  $(n-2)$  times  $180^\circ$ . Verify by constructing enlarged figures and measuring. Also do the same for irregular figures.

**Ex. 75.** Construct the exterior angles to the figures in Ex. 74 and show that the sum of the exterior angles of a polygon equals two straight angles.

**Ex. 76.** Construct a parallelogram. Measure two non-parallel sides and the angle formed, i.e., included by these two sides. With these measurements construct an equal parallelogram. Therefore:

Two parallelograms are equal when they have two sides and the included angle respectively equal.

When the sides of a parallelogram are rectified, i.e., drawn perpendicular, then the figure is a **rectangle**.

When the sides of a rectangle are made equal, then the figure is called a **square**.

## 12. Areas of Figures.

**Ex. 77.** On a sheet of squared paper ink in one of the smallest squares. If we consider its sides to be one unit of length then the surface bounded by the four unit lines is called a **unit of area**. Construct four other squares whose respective sides are two units, three units, five units, ten units. How many unit squares, i.e., units of area are there in each of the larger squares? Designate these figures with Roman notation calling the unit

other squares II, III, IV, and V respectively. The ratio of the magnitudes obtained by measuring the area of a square to the magnitude obtained by measuring the sides of a square is the square of the side. If the side of a square measures (contains)  $a$  unit areas are there in each row and how many columns or files? Irrespective of how we count we shall find there are  $a$  times  $a$  of them, or  $a^2$ . Originally due to this fact that the exponent of a number is read as the square of that letter before:

Area measured in square units is the square of the side measured in linear units.

The unit, i.e., unit of length, may be a foot, meter, or any other arbitrary length which is convenient for convenience. A centimeter is approximately 0.3937 ins. and is one one-hundredth of a meter. A meter is approximately equal to 39.37 centimeters.

The side of a square V is one unit in length, so the area of square V is one unit. Why? The area of a square is a subunit and is numerically equal to the square of the side of the unit area V.

Express the area of squares II, III, IV, in terms of the area of square I.

Consider square II as the unit of area and express the area of square IV in terms of II.

Define the area of a figure as the ratio of that figure to a unit figure. A ratio is written as a fraction; the numerator expresses the magnitude of the first mentioned figure or quantity, and the denominator expresses the magnitude of the second mentioned figure or quantity. In this case the unit of area we have:

$$\frac{\text{Area of IV}}{\text{Area of II}} = \frac{\text{magnitude of IV}}{\text{magnitude of II}} = \frac{25}{4} = 6.25 \text{ units.}$$

The area of a unit is one.

**Ex. 80.** On squared paper construct a rectangle 3.5 ins. long and 1.5 ins. wide. Select any convenient unit for measuring the sides, and the corresponding unit for measuring the area. Repeat using a different linear unit with a corresponding square unit. By means of the ruled lines on the paper measure the sides and area in all possible ways. Write the law for expressing the area of the rectangle in terms of its length and width (breadth). Designate the length by  $l$ , the width by  $w$  and the area by  $A$ . When the law is abbreviated by using letters and symbols it becomes a **formula**. Show why  $A = lw$  is the formula for obtaining the area of a rectangle. Determine the area of the above rectangle by substituting the values of  $l$  and  $w$  in inch units?  $A = 3.5 \times 1.5 = 5.25$  sq.in.

**Ex. 81.** In the Ex. 80, if either  $l$  or  $w$  is in error by 1 per cent, what is the error in  $A$ ? **Per cent** is a contraction of the Latin per centum, which means by the hundred. One per cent means one one-hundredth or one part in one hundred, and is usually written 1%. Therefore 1% of 3.5 means 3.5 times  $1/100$  or  $3.5/100$ , or 0.035, and is the amount by which  $l$  is too large or too small.

**Ex. 82.** In the Ex. 80, if both  $l$  and  $w$  are 1% in error, what is the per cent error in  $A$ ?

*Observation.* The product of several factors has as many dependable figures as the least number of dependable figures in any of its factors.

**Ex. 83.** Near the left end of a horizontal line of the squared paper locate a point  $A$  and 3 ins. to the right of  $A$  locate  $B$ . One inch vertically above  $A$  and  $B$  respectively, locate  $D$  and  $C$ . What kind of figure is  $ABCD$ ? How many unit squares does it contain? What is its area in square inches? Through  $A$  draw a line to the right, making an angle of  $60^\circ$  with  $AB$ , and mark  $E$  at its intersection with  $DC$ . Through  $B$  draw a parallel to  $AE$  intersecting  $DC$  produced in the point  $F$ . What kind of a figure is  $ABFE$ ? The quadrilateral  $ABCE$  is a trapezoid. Which pair of opposite sides in  $ABCE$  is parallel? What magnitude relations exist between triangles  $ADE$  and  $BCF$ ? Why? By means of the trapezoid  $ABCE$ , and the addition of the one or the other of the triangles  $ADE$  and  $BCF$ , show that the parallelogram  $ABFE$  equals the rectangle  $ABCD$ . Check by counting the unit squares and their fractional parts included in  $ABFE$ .

The dimensions of  $ABCD$  are its length and width, which are the measure of its two perpendicular sides  $AB$  and  $AD$ . In the case of the parallelogram  $ABFE$ , we can measure the side  $AB$  for its length, but its width, also called **altitude**, is measured on a line perpendicular to  $AB$  and  $DC$ . The **base** is the side on which the figure rests. Therefore:

The area of a parallelogram is equal to the product of its length times its width or the product of its base times its altitude.

**Ex. 84.** Supplement the construction of Ex. 83 by connecting  $A$  with  $F$ , dividing  $ABFE$  into equal triangles  $ABF$  and  $AEF$ . Why? Therefore:

The area of a triangle equals one-half the product of its base times its altitude.

The base of a triangle is any one of its sides, and correspondingly its altitude is its perpendicular distance from this side to the opposite vertex.

Since  $ABFD$  is not a parallelogram, the diagonal  $AF$  does not bisect it. The area of  $ABFD$  is the sum of the areas of triangles  $ABF$  and  $ADF$ . Designate the bases  $AB$  and  $DF$  by  $a$  and  $b$  respectively, and the altitude by  $h$ . What two lines in the figure can be used to measure the altitude of both triangles? Why? The area of triangles  $ABF$  and  $ADF$  are  $\frac{1}{2}ah$  and  $\frac{1}{2}bh$ , respectively. Therefore the area of  $ABFD = \frac{1}{2}ah + \frac{1}{2}bh = \frac{1}{2}h(a+b)$ ; from this formula we interpret:

The area of a trapezoid equals one-half the product of its altitude times the sum of the bases or parallel sides.

The altitude of the trapezoid is the perpendicular distance between its parallel sides. The median of a trapezoid

joins the mid-points of its non-parallel sides and is parallel to its bases. See Fig. 11.

**Ex. 85.** Supplement the figure of Ex. 84 by connecting  $B$  with  $D$  and also with  $E$ , and connect  $A$  with  $C$ . Join  $G$  and  $H$  the midpoints of  $DA$  and  $FB$  respectively. Rule and label the following "TABLE I. LENGTHS." Under the columns headed lines, write the figure names of all lines, after measuring each

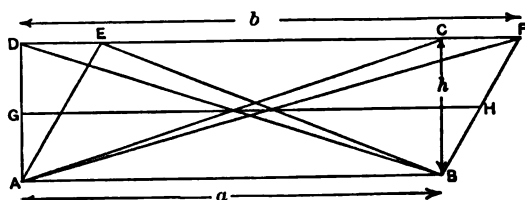


FIG. 11.

line write its numeric value in the next column, alongside of the name to which it belongs. Rule the table with a straightedge.

TABLE I. LENGTHS

Line.	Length, Inches.	Line.	Length, Inches.
$AB$	3.00	$AD$	1.00
$DF$		$AE$	
$DE$		$AC$	
$DC$		$AF$	
$EC$		$BD$	
$EF$		$BE$	
$CF$		$BC$	
$GH$		$BF$	

Rule and label the following "TABLE II. AREAS OF QUADRILATERALS." Under the heading bases, write the numeric values, being careful to note that the trapezoid has two bases which are recorded with a comma between them. Under the columns headed altitude and area, write the respective values in the same horizontal line with the figure to which they belong. Fill in other blank spaces with the name of the figure.

TABLE II. AREAS OF QUADRILATERALS

Quadrilateral.	Figure.	Bases.	Altitude.	Area.
trapezoid	<i>DGHF</i>			
"	<i>GABH</i>			
	<i>ABCD</i>			
	<i>ABFE</i>			
	<i>ABED</i>			
	<i>ABCE</i>			
	<i>ADFB</i>			
	<i>ABFC</i>			

Rule and label the following "TABLE III. AREAS OF TRIANGLES," and make the entries in the respective columns. The headings of the last four columns are a duplication of those of the first four columns. This is done to save space in the vertical direction when there are many entries. The symbol  $\Delta$  read "delta," means triangle; base, altitude, and area are abbreviated by  $b$ ,  $h$ , and  $A$ , respectively.

TABLE III. AREAS OF TRIANGLES

$\Delta$	$b$	$h$	$A$	$\Delta$	$b$	$h$	$A$
<i>ADE</i>				<i>ACB</i>			
<i>ADC</i>				<i>AFB</i>			
<i>ADF</i>				<i>BDE</i>			
<i>AEC</i>				<i>BDC</i>			
<i>AEF</i>				<i>BDF</i>			
<i>ACF</i>				<i>BEF</i>			
<i>AEB</i>				<i>BEC</i>			
				<i>BCF</i>			

**13. Variation, Ratio, and Proportion.** In Table III we observe that all the triangles have an equal altitude. As a check on the work of multiplication, show that for all triangles of equal altitude the area divided by the base equals the constant, i.e., fixed value of half the altitude. In comparing a number of these triangles, what words should be supplied to complete the following statement: in triangles of equal altitude, that one which has the

greatest base has the . . . area and correspondingly that one which has the . . . base has the least area. This is a law of **direct variation**, and may be symbolized by use of the **variation symbol** ( $\propto$ ) written between the area and the base. Therefore from the law of variation we write:

$$A \propto b, \text{ but } A = \frac{1}{2}hb = \text{constant times } b.$$

*Observation.* The variation symbol may be replaced by an equal sign and a constant.

Again referring to the earlier statement above we write:

$$\frac{\text{area of a triangle}}{\text{base of same triangle}} = \frac{A}{b} = \frac{h}{2} = \frac{\text{altitude}}{2}.$$

Since  $A$  is the symbol for area in general, a necessity arises for distinguishing the  $A$ 's of the different triangles. This is provided for by attaching a small number called a **subscript** at the lower right side of the letter thus:  $A_1$ ,  $A_2$ ,  $A_3$ , would represent the respective areas of triangles  $ADC$ ,  $ADE$ ,  $ADF$ . In corresponding manner the bases of these triangles would be designated by  $b_1$ ,  $b_2$ ,  $b_3$ , and the altitudes by  $h_1$ ,  $h_2$ ,  $h_3$ . Therefore we write like subscripts for all parts of the same figure:

$$\frac{A_1}{b_1} = \frac{h_1}{2}, \quad \frac{A_2}{b_2} = \frac{h_2}{2}, \quad \frac{A_3}{b_3} = \frac{h_3}{2}.$$

but since  $h_1 = h_2 = h_3$ , then by =ty  $A_x$ , the three ratios  $\frac{A_1}{b_1}$ ,  $\frac{A_2}{b_2}$ ,  $\frac{A_3}{b_3}$ , are equal; or  $\frac{A_1}{b_1} = \frac{A_2}{b_2} = \frac{A_3}{b_3}$ . This is a **continued equality**, also a **continued proportion**, and is at the same time a condensed form of three distinct equations, viz.:

$$\frac{A_1}{b_1} = \frac{A_2}{b_2}, \quad \frac{A_1}{b_1} = \frac{A_3}{b_3}, \quad \frac{A_2}{b_2} = \frac{A_3}{b_3}$$

The above equations are proportions because a **proportion** is an equality between equal ratios. Every **ratio** is

a fraction, hence the equality symbol between the fractions. Formerly a proportion was written  $A_1-b_1=A_2-b_2$ ; the colon : and double colon :: were used later, giving the more familiar but now obsolete form  $A_1:b_1::A_2:b_2$ . The points were later joined by straight lines allowing  $A_1/b_1=A_2/b_2$ , which in turn has been replaced by the modern equation form written  $\frac{A_1}{b_1}=\frac{A_2}{b_2}$ .

A proportion contains four parts, viz., its two numerators and its two denominators. For convenience in referring to these parts, the two numerators are called **antecedents**, and the two denominators are called **consequents**, or the former are **causes**, and the latter are **effects**. The first numerator **A** and second denominator **b**, i.e.,  $A_1=\frac{\quad}{b_2}$ , are called **extremes**, whereas the first denominator **b**<sub>1</sub> and second numerator **A**<sub>2</sub>, i.e.,  $\frac{\quad}{b_1}=A_2$ , are called the **means**.

**14. Theorems on Proportions.** A **literal** proportion is one in which the four parts or quantities are represented by letters. When the numeric values of these parts are substituted, it becomes a numeric proportion. When letters and numerals are both present it becomes a mixed proportion. The following theorems apply alike to any **simple proportion**, i.e., a proportion of four parts. Consider the proportions representing the headings of the first five columns in Number 1 of Table IV and immediately under them their altered form due to the application of the theorem quoted in the column to their right. If the means of a proportion are equal either mean is called a **mean proportional**.

$$\frac{\text{1st antecedent}}{\text{1st consequent}} = \frac{\text{2d antecedent}}{\text{2d consequent}}; \quad \frac{\text{mean}}{\text{extreme}} = \frac{\text{extreme}}{\text{mean}};$$

$$\frac{\text{1st cause}}{\text{1st effect}} = \frac{\text{2d cause}}{\text{2d effect}}.$$

TABLE IV

SIMPLE PROPORTIONS					THEIR THEOREMS
1	$\frac{a}{b} = \frac{c}{d}$	$\frac{A_1}{b_1} = \frac{A_2}{b_2}$	$\frac{2}{6} = \frac{3}{9}$	$\frac{E}{R} = \frac{I}{1}$	$\frac{C}{D} = \frac{22}{7}$
2	$ad = bc$	$A_1 \times b_2 = A_2 \times b_1$	$2 \times 9 = 3 \times 6$	$E = RI$	$7C = 22D$
3	$\frac{a}{c} = \frac{b}{d}$ $\frac{d}{b} = \frac{c}{a}$	$\frac{A_1}{A_2} = \frac{b_1}{b_2}$ $\frac{b_2}{b_1} = \frac{A_2}{A_1}$	$\frac{2}{3} = \frac{6}{9}$ $\frac{9}{6} = \frac{3}{2}$	$\frac{E}{I} = \frac{R}{1}$ $\frac{1}{R} = \frac{I}{E}$	$\frac{C}{22} = \frac{D}{7}$ $\frac{7}{D} = \frac{22}{C}$
4	$\frac{b}{a} = \frac{d}{c}$	$\frac{b_1}{A_1} = \frac{b_2}{A_2}$	$\frac{6}{2} = \frac{9}{3}$	$\frac{R}{E} = \frac{1}{I}$	$\frac{D}{C} = \frac{7}{22}$

When four quantities are in simple proportion then:

the product of the means equals the product of extremes.

they are in proportion by alternation (interchange of means or extremes).

they are in proportion by inversion, i.e., reciprocation, of the ratios.

5	$\frac{a+b}{b} = \frac{c+d}{d} \quad \text{from 1}$ $\frac{a-b}{b} = \frac{c-d}{d} \quad \text{from 4}$ $\frac{a+b}{a-b} = \frac{c+d}{c-d} \quad \text{from 3}$ $\frac{a+c}{c} = \frac{b+d}{d} \quad \text{from 1}$ $\frac{a-c}{c} = \frac{b-d}{d} \quad \text{from 4}$ $\frac{a+c}{a-c} = \frac{b+d}{b-d} \quad \text{from 3}$ $\frac{2+6}{6} = \frac{3+9}{9}, \text{ i.e., } \frac{8}{6} = \frac{12}{9}$ $\frac{2-6}{6} = \frac{3-9}{9}, \text{ i.e., } \frac{-4}{6} = \frac{-6}{9}$ $\frac{2+6}{2-6} = \frac{3+9}{3-9}, \text{ i.e., } \frac{8}{-4} = \frac{12}{-6}$	<p>they are in proportion by accretion (composition), i.e., by the addition of the consequents to their respective antecedents, or by decretion (division), i.e., by the subtraction of the consequents from their respective antecedents, or they are in proportion collectively, i.e., by accretion and decretion.</p>
6	$\text{Given } \frac{a}{b} = \frac{b}{c}, \text{ then } b = \sqrt{ac}; \frac{2}{4} = \frac{4}{8}, \text{ then } 4 = \sqrt{2 \times 8}.$	<p>A mean proportional between two quantities equals the square root of their product.</p>
7	<p>Given the forms in 2, then 3 and 4 follow.</p>	<p>From two equal products of two factors each, we may write either set of factors as the means and the other set as the extremes of a proportion.</p>

The proof of these theorems is substantiated by the axiom of operations.

**15. Projection.** The projection of a figure or object is its **image** or **shadow**. Projection involves four essential considerations, viz., an imaginary or real **source** of illumination, such as a lamp or the sun; a bundle or system of straight lines, representing **rays** emanating, i.e., sent out from the light source; an **object** or figure in the path of the rays; and lastly a **screen** or suitable surface, plane, or line upon which to cast a shadow, image or projection.

Some of the rays are intercepted, i.e., stopped by the physical obstruction of the object, others pass around the object, i.e., are contiguous to its boundary, and still others pass beyond contact with the object.

A shadow indicates the interception of the rays which fall upon the object. In obtaining the outline of the shadow, it is sufficient to consider those rays which touch the boundary of the figure or object. Every point or corner of the object projects into a point or corner of the shadow. The

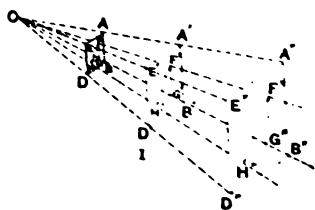


FIG. 12.— Central Projection.

boundary lines or edges of the object project into the boundary lines or edges of the shadow.

When the source of illumination is a point  $C$ , the rays spread into a cone and we have **central** projection as illustrated in Fig. 12.

The square  $ABDE$  has its image or projection at  $A'B'D'E'$  and  $A''B''D''E''$ . The points with like **accented** letters are on a single ray showing clearly how a point of the object will project along a ray into a point of the projection. In Fig. 12 the object and its images are parallel, and in such cases they will be similar figures. If a screen is inserted obliquely to the object the shadow or projection will not be a square.

**Ex. 86.** From a sheet of cardboard cut a square which will be called the **die**. The remaining part of the cardboard will be called a **matrix**.

Place a lighted candle in a darkened room. Hold the die and then the matrix in different positions before the candle light and measure the shadows on a wall. What kind of figures result?

**Ex. 87.** Cut other dies in the cardboard, leaving triangular, circular and irregular matrixes and repeat the detail of Ex. 86.

**Ex. 88.** Repeat Ex. 86, keeping the screen fixed but change the position of the candle.

**Ex. 89.** Place a lighted candle in a cylindrical glass vessel and observe the projections of the rim of the mouth on the walls and ceiling as the vessel is inclined in various positions.

When a matrix is placed before the light, the rays diverge, i.e., the illumination spreads through the opening. The area of illumination upon a parallel screen placed beyond the matrix will vary as the square of the distance of the former from the light. Let  $A$  and  $d$  represent the area and distance respectively, of the projection, then  $A \propto d^2$ . In other words if the area is 9 sq.ft. at a distance of 3 ft., then the area will be 16 sq.ft. at a distance of 4 ft. If two areas  $A_1$  and  $A_2$  are at the respective distances  $d_1$  and  $d_2$ , then

$$\frac{A_1}{A_2} = \frac{d_1^2}{d_2^2} \quad \text{or} \quad \frac{A_2}{A_1} = \frac{d_2^2}{d_1^2} \quad (\text{direct proportion}).$$

The intensity of illumination, i.e., the amount of light falling upon each square foot of screen, will decrease as the areas increase. Let  $I_1$  and  $I_2$  be the intensity of illumination on areas  $A_1$  and  $A_2$ , then

$$\frac{I_1}{I_2} = \frac{A_2}{A_1} \quad (\text{inverse or indirect proportion}),$$

but since  $\frac{A_2}{A_1} = \frac{d_2^2}{d_1^2}$  we can substitute and obtain

$$\frac{I_1}{I_2} = \frac{d_2^2}{d_1^2}, \quad \text{which means } I \propto \frac{1}{d^2},$$

The intensity of illumination varies indirectly, i.e., inversely or reciprocally as the square of the distance.

*Observation. In a direct proportion, the ratio of two values of one quantity, equals the direct or like ratio of two corresponding values of the other quantity. In an inverse or indirect proportion one of these ratios is inverted or reciprocated.*

In central projection the image or shadow is a magnification of the object, die or matrix. The intensity of any other phenomena, such as sound, heat, electricity, magnetism, which is dissipated through space obeys the same law of the inverse squares.

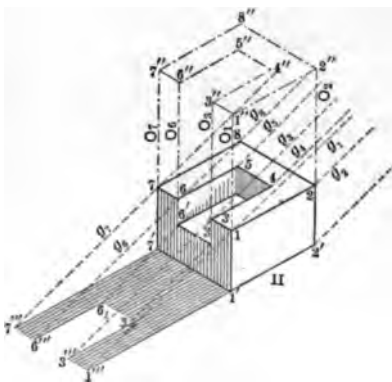


FIG. 13.—Geometric Projection.

When the source of illumination emanates parallel rays (sunlight and parabolic reflectors) we have geometric projection as illustrated in Fig. 13. If the rays meet the screen at right angles (**orthogonal projection**) the boundary of the projection is an exact duplication or reproduction in size and shape, of the boundary of the die. If the rays are not at right angles to the screen, the boundary of the projection is a distortion of the boundary of the die.

The shadow (7'1'1'''3'''3'6'6'''7''') is obtained by parallel rays which are oblique to the rear and upper surfaces of the solid. The shadow (1''2''8''7''6''5''4''3'') is obtained by the orthogonal rays which give a projection of the upper surface.

The shadow (7'1'1'''3'''3'6'6'''7''') is obtained by parallel rays which are oblique to the rear and upper surfaces of the solid. The shadow (1''2''8''7''6''5''4''3'') is obtained by the orthogonal rays which give a projection of the upper surface.

**Mechanical drawings** are geometric projections which enable the workman to gain a picture of a constructive

job as it will appear from the top, front and side when completed.

**Ex. 90.** Make mechanical drawings after obtaining the measurements of a rectangular box; a fly-wheel; and the parts

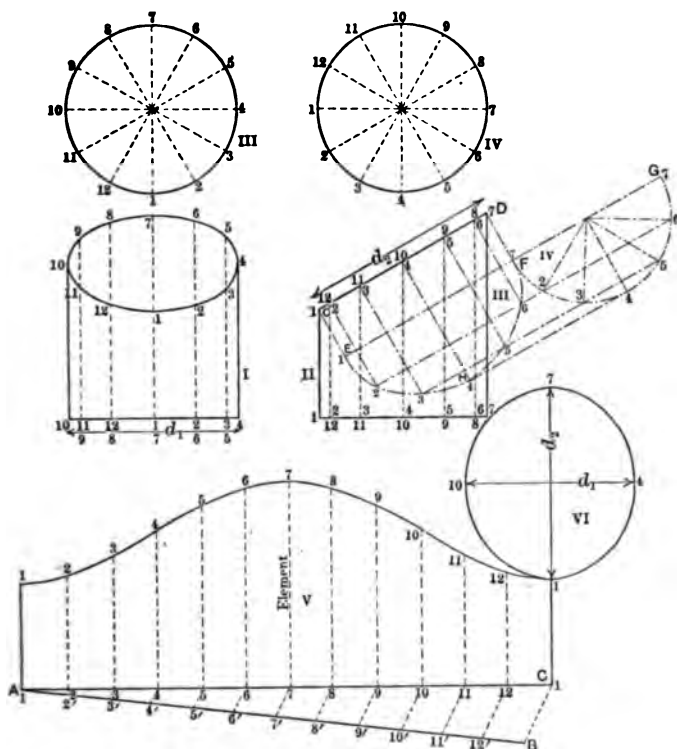


FIG. 14.—Truncated Cylinder and Spread Surface.

of a generator. Show the orthogonal projections of the top, front, and side of the object.

Fig. 14 shows the mechanical drawing of a cylinder which has been truncated, i.e., cut obliquely. Corresponding points and lines are like numbered in each view. Viewed

from above, the **section**, i.e., the cut surface is a circle as shown in III, IV, but viewed orthogonally the section I, is shown in its true shape as an ellipse. The dotted lines represent rays. The bottom or base of the cylinder is a circle. The **lateral** or vertical surface of an entire cylinder would open into a rectangle, i.e., a rectangular strip of paper would cover its lateral surface. The effect of truncating the cylinder is to remove a portion of the rectangular surface as shown in the **development**, i.e., **spread surface** V. The development including the base and elliptic section, VI, represents the amount of material required to be cut from metal and bent into shape in order to cover the solid or build a like hollow vessel.

The vertical lines in the lateral surface are called **elements**. They are parallel and equally spaced, and being twelve in number, they intersect the base at points which divide the circumference into twelve equal arcs. The length of the elements in the development are equal to the length of the corresponding elements in I and II.

Fig. 15 shows a cone with its upper and lower half tapering to the **apex**, *O*. The bases are circles. A line joining the apex with the base is called an element. The development of the lateral surfaces will be two sectors of circles called **nappes**, drawn with a radius equal to an element of the cone, and a length of arc equal to the circumference of the circular base. All sections parallel to the base will be circles as shown at *MNPQ*.

A section inclined slightly non-parallel to the base is an **ellipse**, *CDBE*.

A section parallel to an element is a **parabola**, *CFKLG*.

A section perpendicular to the base will appear in both halves of the cone. Every section which cuts both halves of the cone is called an **hyperbola**, *KBL* and *HAI*.

Each of the four sections of the cone removes part of the conical surface which is illustrated in the development V and VI.



$\frac{h_2}{s_2}$  is a constant, and the height of all objects in the vicinity can be quickly obtained by multiplying the length of an object's shadow by  $\frac{h_2}{s_2}$ . Verify by experiment and make a descriptive report.

**Ex. 93.** Repeat the experiment of Ex. 92 at a later hour of the day and report your observations.

In mathematics work orthogonal projection only is used. Instead of projecting a line called a **projector** upon a surface, it is usually sufficient to project it upon another line called the **shadow line**, as illustrated in Fig. 16. The

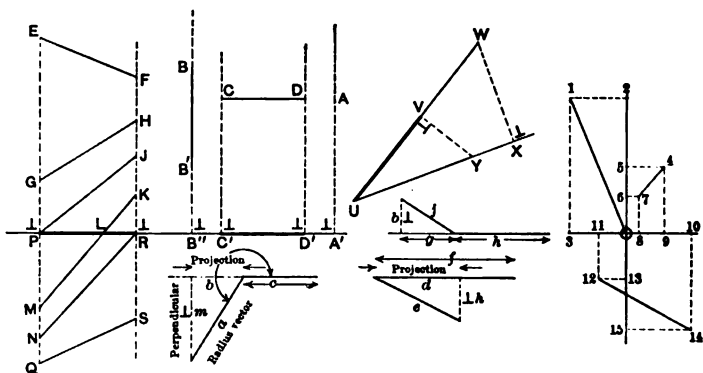


FIG. 16.—Orthogonal Projection.

rays are always orthogonal to the shadow-line, and the **projection** is the **segment**, i.e., the portion of the shadow line lying between the extreme rays.

*Observation.* In projecting a line it is sufficient to draw the rays through its extremities. If an extremity lies in the shadow-line, one ray, called a **perpendicular**, drawn through the other extremity, is sufficient to determine the projection.

The projection is equal in length to the projector when they are parallel,  $C'D'$  and  $CD$ . If we rotate the projector obliquely to the rays, the projection is shortened,  $EF$  and  $PR$ . In the extreme position,  $BB'$  with the projector parallel

the rays, the projection is a point. When the projector rotates it is called a **radius vector (R.V.)**, or moving arm.

**Ex. 94.** Make a list of projectors and their corresponding projections in Fig. 16, giving their figure names.

**Ex. 95.** In Fig. 16 state the vertical and horizontal projections of the lines passing through the point 1, 0; 7, 4; 12, 14.

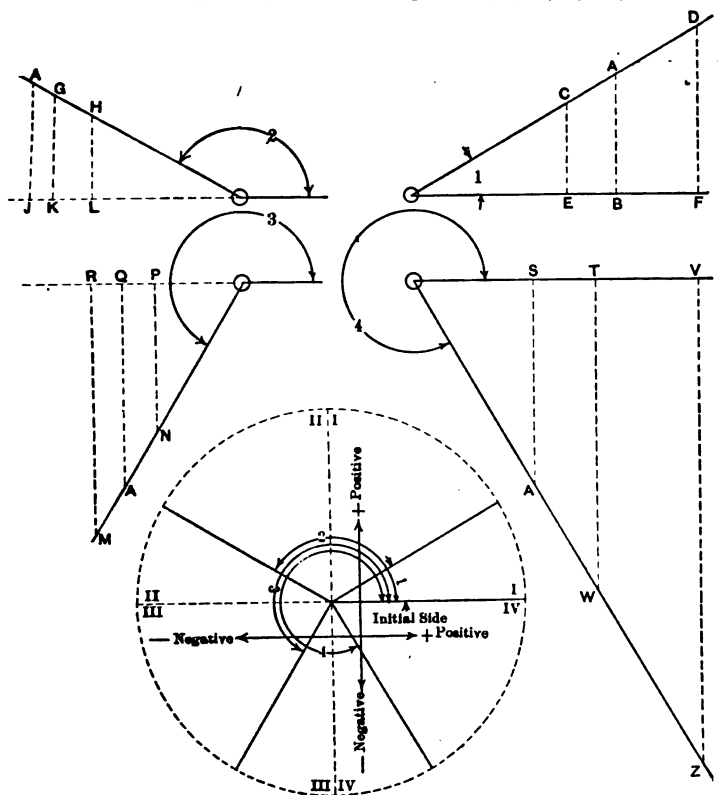


FIG. 17.—Angles of Four Quadrants.

**16. Trigonometric Functions.** Fig. 17 represents two diameters which divide the circle into called **quadrants**, I, II, III, IV. The

numbers of the quadrants begin at the upper right-hand quarter or corner and are ordered in a **counter-clockwise direction**, i.e., opposite to the motion of clock hands. The angles 1, 2, 3, and 4 are called respectively angles of the first, second, third and fourth quadrants, because their terminal sides, i.e., radii vectors, lie in these respective quadrants. Their initial sides are the same horizontal line and their common vertex is the center of the circle. Angles 1, 2, 3, 4 are represented above the circle separately, with their sides prolonged to facilitate measurement. A number of lines or rays are drawn from the terminal sides perpendicular to the initial side or the initial side produced. A radius vector may be considered as any portion of the terminal side measured from the vertex to the beginning of a ray or perpendicular. The intersection of the perpendicular with the initial side is called its **foot**. The distance between the foot of the perpendicular and the vertex is the projection of that particular radius vector. By these rays we have formed four groups of similar figures.

Measure and record the length of a radius vector  $OC$ , its corresponding perpendicular  $EC$ , and projection  $OE$  for angle 1.

From the three measured lines it will be possible to write three distinct ratios and their three reciprocals. In terms of radius vector, perpendicular and projection they are named as follows:

The **sine** of  $\angle 1$  (abbreviated sin 1)

$$= \frac{\text{perpendicular}}{\text{radius vector}} = \frac{\perp}{\text{R.V.}} = \frac{EC}{OC}.$$

The **cosine** of  $\angle 1$  (abbreviated cos 1)

$$= \frac{\text{projection}}{\text{radius vector}} = \frac{\text{Proj.}}{\text{R.V.}} = \frac{OE}{OC}.$$

The **tangent** of  $\angle 1$  (abbreviated tan 1)

$$= \frac{\text{perpendicular}}{\text{projection}} = \frac{\perp}{\text{Proj.}} = \frac{EC}{OE}.$$

The reciprocal of a sine is called a **cosecant** (csc.).

The reciprocal of a cosine is called a **secant** (sec.).

The reciprocal of a tangent is called a **cotangent** (cot.).

$$\frac{1}{\sin \angle 1} = \csc \angle 1; \quad \frac{1}{\cos \angle 1} = \sec \angle 1; \quad \frac{1}{\tan \angle 1} = \cot \angle 1.$$

$$\frac{1}{\csc \angle 1} = \sin \angle 1; \quad \frac{1}{\sec \angle 1} = \cos \angle 1; \quad \frac{1}{\cot \angle 1} = \tan \angle 1.$$

These six ratios are called **trigonometric functions**, because their numeric values depend upon the magnitude of the angle under observation. Any distance, such as  $OA$  or  $OD$ , may be used as the radius vector provided it is measured outward from the vertex. For every such radius vector, there will be a corresponding perpendicular such as  $AB$  or  $DF$ , and in like manner a corresponding projection  $OB$  or  $OF$ .

**Ex. 96.** Determine the sine, cosine, and tangent of angle 1, using two new sets of radii vectors with their corresponding perpendiculars and projections. The ratios obtained for each function should be numerically the same, as the three measured parts form similar triangles in each instance. Why does this make the respective ratios constant for angle 1?

*Repeating the determination of the values of functions for different angles, leads us to the observation, that each function of a given angle is a numeric constant, i.e., a definite fixed number.*

**17. The Use of Trigonometric Tables.** The numeric values of the trigonometric functions of acute angles are given in Table VIII. Reading from the top down, the first column gives the number of degrees of angles from  $0^\circ$  to  $45^\circ$ . The fourth, fifth, sixth, and seventh columns give respectively the sine, tangent, cotangent, and cosine of the angles, and are to be read on the same horizontal line with the corresponding angle.

**Ex. 97.** Obtain the sine of  $20^\circ$  from the table. Locate 20 in the degree column, carry the finger horizontally across the page and in the column headed sin read the decimal .3420. If a card is placed horizontally under 20 it will assist the eye in keeping to the horizontal line.

**Ex. 98.** Obtain the tangent of  $38^\circ$ . In the degree column locate 38 and horizontally across and in the tangent column read .7813.

**Ex. 99.** Obtain the cotangent of  $9^\circ$ . Look up 9 in the degree column, and horizontally across and in the cotangent column read 6.3138.

**Ex. 100.** Obtain the cosine of  $41^\circ$ ,  $41.5^\circ$ , and  $42^\circ$ .

$$\cos 41^\circ = .7547$$

$$\cos 42^\circ = .7431$$

$$.0116 = \text{decrease for } 1^\circ \text{ between } 41^\circ \text{ and } 42^\circ.$$

$$.0116 \times .5 = .0058 = \text{decrease for } .5^\circ \text{ between } 41^\circ \text{ and } 42^\circ.$$

$$\therefore \cos 41.5^\circ = .7431 + .0058 = .7489 = .7547 - .0058.$$

*Observation. To interpolate, i.e., make the correction for a decimal angle, multiply the decimal excess by the difference in the numeric values of the same function of the next lower and next higher angle. Add the correction for sines and tangents, because sines and tangents increase with increasing angles. Subtract the correction for cosines and cotangents, because cosines and cotangents decrease with increasing angles.*

**Ex. 101.** Obtain the sines of  $30^\circ$ ,  $27^\circ$ ,  $32.3^\circ$ ,  $44^\circ$ ,  $8.8^\circ$ ; the cosines of  $36.2^\circ$ ,  $30^\circ$ ,  $18^\circ$ ,  $23.7^\circ$ ,  $1.5^\circ$ ; the tangents of  $30^\circ$ ,  $45^\circ$ ,  $15.8^\circ$ ,  $20^\circ$ ,  $2^\circ$ ; the cotangents  $30^\circ$ ,  $16^\circ$ ,  $7^\circ$ ,  $42.5^\circ$ ,  $3^\circ$ ,  $15^\circ$ .

A further examination of Table VIII shows that functions of acute angles from  $45^\circ$  to  $90^\circ$  are read from the **bottom upward**. Instead of reading the headings of the columns, we read the **footings**, i.e., the designation at the bottom of the columns. This arises from the fact that the numeric value of the function of an angle is also the numeric value of the **cofunction** of the complementary angle. Sines and cosines are cofunctions, and so are tangents and cotangents, also secants and cosecants. Thus the sine  $60^\circ =$

cosine  $30 = .8660$ ;  $\tan 28^\circ = \cot 62^\circ = .5317$ . To read functions of an acute angle greater than  $45^\circ$ , look up the angle in the degree column, carry the finger horizontally to the left until within the column with the function name at the bottom. Thus the  $\sin 77^\circ = .9744$ ;  $\tan 65^\circ = 2.1445$ ;  $\cos 53.5^\circ = .5948$ .

**Ex. 102.** Obtain the sine, cosine, tangent and cotangent of the following angles:  $57.3^\circ$ ,  $87.6^\circ$ ,  $47.6^\circ$ ,  $72.9^\circ$ .

**Ex. 103.** What is the angle whose sin is (.8660)? In order to obtain the angle the table is used inversely, i.e., in a reverse manner, and therefore the angle is called the *inverse* or *antifunction*. It is customary to abbreviate the angle by some Greek letter such as  $\theta$  (theta) and the words "whose sin is" by writing  $(-1)$  above the function name. Using symbols the example may be stated  $\theta = \sin^{-1}(.8660)$ . In the seventh column we find .8660, with the word sine at the foot of the column. This means we read the angle from the bottom, and therefore in the last column we find  $60^\circ$  on the same horizontal line with .8660. Therefore  $\theta = 60^\circ$ .

**Ex. 104.**  $\phi = \tan^{-1}(2.1445)$ , which means  $\phi$  is the angle whose tangent is (2.1445). What is the degree measure of  $\phi$ ?

**Ex. 105.**  $\alpha = \cos^{-1}(.5446)$ , which means  $\alpha$  is the angle whose cosine is (.5446). What is the radian measure of  $\alpha$ ? Instead of reading the angle in degrees in the first column, read its radian measure on the same horizontal line of the second column. One radian of angle equals  $57.3^\circ$  approximately. Since there are  $2\pi$  radians to every  $360^\circ$  of angle, then, one degree equals .0175 radian approximately.

**18. Functions of Angles of Any Quadrant.** To obtain the numeric value of the function of an angle greater than  $90^\circ$ , we use Table VIII.

The functions of angles of any quadrant are ratios, and are defined in exactly the same way as functions of acute angles. They may be obtained by drawing and measuring radii vectors, and their corresponding perpendiculars and projections as shown in Fig. 17, and expressing these as ratios. These ratios vary through the same range

of numeric values given in Table VIII. Therefore, in order to use Table VIII to determine the functions of an obtuse angle, look up, i.e., obtain the functions of its supplement. In order to obtain the functions of a III quadrant angle **subtract  $180^\circ$  from the angle** and look up the functions of the remainder. In order to obtain the functions of a IV quadrant angle **subtract the angle from  $360^\circ$**  and look up the functions of the remainder.

Thus the sine of  $137^\circ$  equals the sine of its supplement.

$$\sin 137^\circ = \sin(180^\circ - 137^\circ) = \sin 43^\circ = .6820.$$

$$\text{The } \tan 212^\circ = \tan(212^\circ - 180^\circ) = \tan 32^\circ = .6249.$$

$$\text{The } \cos 314^\circ = \cos(360^\circ - 314^\circ) = \cos 46^\circ = .6947.$$

In any case the angle is combined by subtraction with  $180^\circ$  or a multiple of  $180^\circ$ .

There is this slight difference, however, that we must observe a convention of signs in measuring perpendiculars and projections, i.e., vertical and horizontal lines as shown in Fig. 17. Accordingly:

A perpendicular is **positive** if it extends above the initial side or the initial side produced to the left of the vertex, i.e., it must be above the projection, *EC* and *HL*. It is **negative** if it extends below the projection, *PN* and *SA*.

A projection is **positive** if it extends to the right of the vertex, *OF* and *OV*. It is **negative** if it extends to the left of the vertex, *OJ* and *OR*.

Since the radius vector is a moving arm, it is a line without definite direction. It is regarded without sign, which is equivalent to saying it is positive in sign.

In consequence of the convention of signs, all functions of acute angles are positive. In the other three quadrants, some of these ratios will be negative, i.e., when either their numerators or their denominators are negative. In any quadrant these ratios will be positive if numerator and denominator have like signs, i.e., both positive or both negative. There will be cycles of change in the signs of the functions as shown in Table V.

**Ex. 106.** Verify the following Table V for the signs of functions of angles of quadrants I, II, III, and IV, by observing the convention of signs.

TABLE V. SIGNS OF QUADRANTS

Functions.	Quadrants.			
	I	II	III	IV
sin, csc .....	+	+	-	-
tan, cot. ....	+	-	+	-
cos, sec. ....	+	-	-	+

**Ex. 107.** The  $\sin 212^\circ = -\sin 32^\circ = -.5299$ . Why?

The  $\cos 137^\circ = -\cos 43^\circ = -.7314$ . Why?

The  $\tan 314^\circ = -\tan 46^\circ = -1.0355$ . Why?

Write the values of the following:  $\tan 263^\circ$ ;  $\sin 281^\circ$ ;  $\cos 157^\circ$ ;  $\cos 300^\circ$ ;  $\sin 300^\circ$ ;  $\tan 300^\circ$ .

In Table VIII the numeric value of  $\cot 0^\circ$  and  $\tan 90^\circ$  is not written, but is expressed by the symbol  $\infty$ , called *infinity*. This means that the numeric value is so excessively large as to be inexpressible with figures. Secants and cosecants are not given, as they are more easily replaced by their respective reciprocals cosines and sines.

**Ex. 108.** Look up the values of the sine and the cosine of  $30^\circ$ . Divide the former by the latter, and observe their quotient in the tangent column on the same line. Perform this division for angles of  $15^\circ$ ,  $20^\circ$ ,  $25^\circ$ ,  $35^\circ$ ,  $40^\circ$ ,  $45^\circ$ . The last or fourth figure of Table V is not dependable, so there may be a slight variance in the fourth figure of the quotient.

*Observation.* The tangent of an angle is the ratio of its sine to its cosine.

**Ex. 109.** Construct a right triangle. Label the right angle  $C$  (a conventional method) and the acute angles  $A$  and  $B$  respectively. At the middle of each side place a lower case letter corresponding to the capital letter at the opposite vertex.

Formulate, i.e., write, the names of the functions of angles  $A$  and  $B$ , connecting them with an equal sign to the following ratios:  $\frac{a}{c}$ ;  $\frac{a}{b}$ ;  $\frac{b}{c}$ ;  $\frac{b}{a}$ .

**19. The Right Triangle.** Each of these four statements contains three quantities, viz., two sides and a function of an angle. If any two of these three quantities are known, the third may be obtained by substituting the numeric values of the former in the proper equation. The simplified result will be the value of the latter. If each of the remaining parts of the triangle has its numeric value determined in this way, the triangle is said to be solved by trigonometry.

**Ex. 110.** A right triangle has a side  $a = 4.33''$ , and its hypotenuse  $c = 5''$ . Determine angles  $A$  and  $B$ , and the side  $b$ .  $\sin A = \frac{a}{c} = \frac{4.33}{5} = .8660$ . Referring to Table VIII under the sin column for .8660 we locate it horizontally across from angle  $60^\circ$ . Therefore  $A = 60^\circ$ , but  $C = 90^\circ$ , and since the sum of the angles of a triangle equals  $180^\circ$ , then  $A$  and  $B$  are complementary. Therefore  $B = 90^\circ - A = 90^\circ - 60^\circ = 30^\circ$ .  $b$  may be obtained by substituting in the tangent formula  $\tan A = \frac{a}{b}$ , or in the cosine formula  $\cos A = \frac{b}{c}$ . Using the latter we have  $c \cos A = b$  (Mul. ax.). Therefore substituting for  $c$  and  $\cos A$ , we have  $b = 5 \times \cos 60^\circ = 5 \times .5 = 2.5''$ .

**Ex. 111.** A right triangle has angle  $A = 30^\circ$  and hypotenuse  $c = 20''$ .

Solution:

$$\begin{array}{ll} \sin A = \frac{a}{c}, \text{ therefore } a = c \sin A = 20 \times .5 = 10''; & \begin{array}{l} A = 30^\circ \\ c = 20'' \\ \hline a = 10'' \end{array} \\ \cos A = \frac{b}{c}, \text{ therefore } b = c \cos A = 20 \times .866 = 17.32''; & \begin{array}{l} b = 17.32'' \\ B = 60^\circ \\ \hline \hline \end{array} \end{array}$$

$$B = 90^\circ - A = 90^\circ - 30^\circ = 60^\circ.$$

**Ex. 112.** A right triangle has an angle  $A = 29^\circ$  and side  $a = 380$ .  
 $\tan A = \frac{a}{b}$ .  $\therefore b \tan A = a$  (Mul. ax.). The multiplier of a function precedes the function. By applying division axiom,  $b = \frac{a}{\tan A}$   
 $= \frac{380}{.554} = 685$ . The value of  $c$  may be obtained by using the formula  
 $\sin A = \frac{a}{c}$ , from which we obtain  $c \sin A = a$ , and therefore  $c = \frac{a}{\sin A}$   
 $= \frac{380}{.4848} = 785$ ;  $B = 90^\circ - 29^\circ = 61^\circ$ .

*Observation.* The number of dependable figures in a quotient is the same as the least number of dependable figures in either the numerator or denominator of the corresponding fraction.

**Ex. 113.** Construct the following right triangles and solve them graphically, i.e., by measuring the sides and angles. In order to obtain graphic results which are comparable with calculated ones, it is necessary that the figure should be drawn with hair lines, by using a hard pencil and then making all measurements with accuracy before inking the figure. Determine the solution by calculation, using trigonometric formulas:

- |                |          |              |          |
|----------------|----------|--------------|----------|
| (1) $a = 4$    | $b = 3$  | (4) $c = 26$ | $b = 24$ |
| (2) $a = 15$   | $c = 25$ | (5) $c = 39$ | $b = 15$ |
| (3) $c = 12.5$ | $b = 10$ | (6) $a = 12$ | $b = 5$  |

If the work is performed carefully, it will be observed that triangles (1), (2), (3) are a group of similar triangles, in which the sides have the ratio 5:4:3. Triangles (4), (5), (6) form another group of similar triangles in which the sides have the ratio 13:12:5. There are very few right triangles of the type where the sides and hypotenuse are integral and commensurable, i.e., have a common unit of measure. The 3-4-5 triangle is well known by builders, who use it in constructing right-angled framework. It is

used also for staking out rough surveys in which the **courses**, i.e., the boundaries, are at right angles.

In each one of the above examples, square the three sides of each triangle, and show that in a right triangle the square of the hypotenuse equals the sum of the squares of the two legs.

This theorem may be recognized from the construction shown in Fig. 18. It is due to its constructive proof that

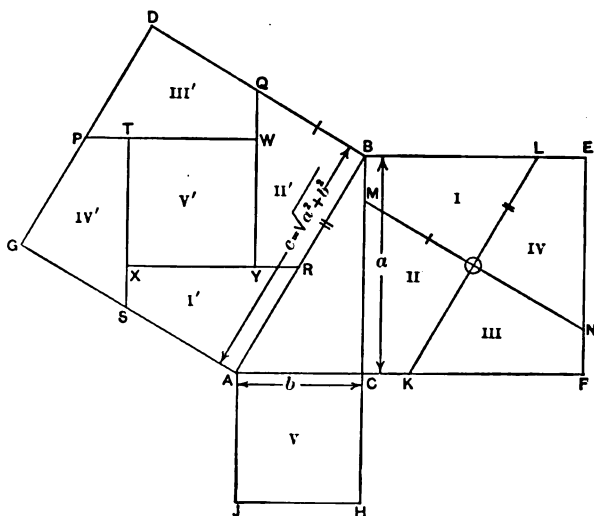


FIG. 18.—Squares on a Right Triangle.

it is known after a Greek philosopher as the **Pythagorean Theorem**. Construct a right triangle on squared paper. Using the hypotenuse and each leg as respective sides, carefully construct squares on these sides as shown in Fig. 18. Count the small squares to verify the theorem. Note the division of the largest square into trapezoids and an included square. Identify the trapezoids with the divisions of the square constructed on the longer of the two legs. The trapezoids include a square equal to the

square constructed on the shorter leg, The formulation of this theorem in terms of hypotenuse  $c$  and legs  $a$  and  $b$  is expressed as:

$$c^2 = a^2 + b^2 \quad \text{or} \quad c = \sqrt{a^2 + b^2} \quad (\text{Root Ax.});$$

let  $c=5$ ,  $b=4$ ,  $a=3$ , then;

$$25 = 9 + 16 \quad \text{or} \quad 5 = \sqrt{9 + 16} = \sqrt{25}.$$

$$a^2 = c^2 - b^2; \quad a = \sqrt{c^2 - b^2}; \quad b^2 = c^2 - a^2; \quad b = \sqrt{c^2 - a^2}.$$

**Ex. 114.** Determine the third side, given the following data for right triangles:

$$(1) \quad a=6 \quad c=10$$

$$(4) \quad a=3.2 \quad c=5.4$$

$$(2) \quad b=7 \quad c=12$$

$$(5) \quad b=3.5 \quad c=7$$

$$(3) \quad a=5 \quad b=6$$

$$(6) \quad a=150 \quad b=175$$

**20. Square Root.** To obtain the square root of a number, such as 600.25, proceed as follows:

1. To the left and right of, and beginning at the decimal point, place the figures in **groups of two**. 6'00'.25'.

2. Under the extreme left-hand group (6), write and subtract the nearest **squared number** (4), and place (2) the **root** of (4), as the **first figure** of the **quotient**.

$$\begin{array}{r|l} 6'00'.25' & 2 \text{ quotient} \\ \underline{4} & 40 \text{ trial divisor.} \\ 2 \ 00 & 4 \text{ second term root} \\ & \underline{44} \text{ complete divisor.} \end{array}$$

3. Bring down the **next group** (00) alongside the remainder (2) to form the **new dividend** (200).

4. The **trial divisor** (40) is twenty times the root obtained so far.

5. Divide the trial divisor (40) into the new dividend (200), and obtain the next, i.e., **second figure** (4) of the **quotient**, and also **add the same** (4) to **complete the divisor**.

6. From the new dividend, subtract (176) the **product** of the **second figure** (4) of the **quotient** **times the complete divisor** (44)

7. Bring down the next group alongside the last remainder.

Continue the process by repeating in order 4, 5, 6, 7, until the groups are exhausted. The root has as many integral and decimal figures, respectively, as the given number has integral and decimal groups. A zero may be written after the last decimal figure to complete a decimal group, if an odd number of decimal figures are given.

2	4	5	Root
6'00'	25'		$2 \times 20 = 40$ , twenty times root obtained so far
4			40 trial divisor
2	00		+ 4 second term root
1	76		44 complete divisor ( $44 \times 4 = 176$ )
24	25		$24 \times 20 = 480$ , twenty times root obtained so far
24	25		480 = trial divisor
			+ 5 = third term root
			485 = complete divisor ( $485 \times 5 = 2425$ ).

$$\therefore \sqrt{600.25} = 24.5.$$

## 21. Operations upon Fractions.

**Ex. 115.** An electric circuit contains a battery which has an electromotive force of 10.36 volts. The resistance of the entire circuit is 93.2 ohms. What current in amperes is flowing through the circuit?

By **Ohm's Law**, the intensity (**I**) of current in amperes which flows through a circuit, equals the ratio of the voltage (**E**) applied, i.e., put in the circuit to the resistance (**R**) of the circuit.

This law is **formulated**, i.e., expressed by a formula as follows:

$$(1) \text{ The current (amperes)} = \frac{\text{electromotive force (volts)}}{\text{resistance (ohms)}}.$$

Using the notation, i.e., the symbols which abbreviate current, electromotive force, and resistance, to replace these words in equation (1) we have (2):

$$(2) \quad I = \frac{E}{R}.$$

This is a working formula, and therefore is an equation, i.e., an expression of equality between quantities.

The data, i.e., the numeric values of quantities given in the example, are substituted in (2), and since  $E=10.36$ , and  $R=93.2$ , we obtain (3):

$$(3) \quad I = \frac{10.36}{93.2}.$$

Performing the indicated division  $\frac{10.36}{93.2}=0.11+$ , the value of the current is expressed in (4).

$$(4) \quad I = \underline{\underline{0.11 \text{ amperes}}}.$$

The answer is indicated not by writing Ans. or Res., but by underscoring and writing the meaning of the answer, and the unit of measure, after the numeric value of the solution.

The equality sign separates the equation into two parts, called respectively the left-hand, or first **member**, and the right-hand or second **member**.

**Ex. 116.** A dynamo supplies a current at 115 volts, to a circuit containing a resistance of 220 ohms. What is the value of the current in amperes flowing through the circuit? Use equation (2).

**Ex. 117.** A dynamo supplies a current of 115 volts, to a circuit containing a resistance of 110 ohms. What is the value of the current in amperes flowing through the circuit?

**Observation.** How does the value of a fraction change, when the value in its numerator is kept constant, but the value in the denominator is decreased?

*How does the value of a fraction change, when the value in the numerator is kept constant, but the value in the denominator is increased?*

**Ex. 118.** A battery of storage cells supplies a current of 50 volts E.M.F. to a circuit having a resistance of 220 volts. What is the value of the current flowing through the circuit? Compare with the result of Ex. 116.

*Observation. How does the value of a fraction change, when the numeric value of its denominator is kept constant, but the value of the numerator is decreased?*

*Observation. How does the value of a fraction change, when the numeric value of its denominator is kept constant, but the value of the numerator is increased?*

**Ex. 119.** A resistance of 2420 ohms is connected to the overhead wire and also the track of a trolley line. Between the trolley wire and track is a voltage difference (P.D.) of 550. What current will flow through the resistance? Compare with the result of Ex. 118.

**Ex. 120.** A battery of 10 volts, E.M. F. is attached to complete a circuit of 44 ohms resistance. What current passes, i.e., flows through the circuit? Compare with the result of Exs. 118 and 119.

*Observation. What is the effect upon the numeric value of a fraction when its numerator and denominator are either both multiplied or both divided by the same factor? This statement is known as the **axiom of fractions**, and is abbreviated by *ax. frac.**

It is the axiom of fractions which is applied in simplifying, reducing, and uniting fractions, in the operations of addition, subtraction, multiplication, and division.

**Ex. 121.** Draw a unit square I (one inch on a side). Join the middle of the upper side to the middle of the lower side. The unit square is divided into two equal parts, i.e., rectangles. One of these parts equals a unit divided by 2, i.e.,  $\frac{1}{2}$ . The fraction  $\frac{1}{2}$  is made up of the numerator 1, which stands for the whole unit, i.e., the magnitude of the unit square. Under it is a dividing line indicating a division of the unit, and beneath the latter is the

2 which indicates the number of parts. If the unit square were divided into three, four, five or six equal rectangles; each of the parts of the unit would be represented by the respective fractions  $\frac{1}{3}$ ,  $\frac{1}{4}$ ,  $\frac{1}{5}$ , or  $\frac{1}{6}$ . If the unit square is cut apart along the lines of division, and the parts reassembled to produce any other figure, the area of the latter would be one unit. Why? Therefore a unit of area of any shape, such as a unit circle, i.e., a circle of unit radius when divided into two, three, four, five, six parts will have these parts represented by the fractional units, i.e., the fractions  $\frac{1}{2}$ ,  $\frac{1}{3}$ ,  $\frac{1}{4}$ ,  $\frac{1}{5}$ ,  $\frac{1}{6}$ .

**Ex. 122.** Construct a square I of 1 unit area, another square II of 2 units area, another square III of 3 units area. Divide each square into 6 parts. How many parts of I equal one part of II?

$$\therefore 2 \text{ times } \frac{1}{6} = \frac{1}{6} \text{ of } 2 = \frac{2}{6}.$$

To get equal divisions the two-unit area II would be divided into 6 parts and the one-unit area I would be divided into 3 parts. In other words  $\frac{2}{3} = \frac{1}{3}$ .

How many parts of I equal one part of III?

$$\therefore 3 \text{ times } \frac{1}{6} = \frac{1}{6} \text{ of } 3 = \frac{3}{6} = \frac{1}{2}.$$

**Ex. 123.** Construct a square. Call its area  $a$  units. Divide the square into two, three, four, five six, equal rectangles. These parts are designated by  $\frac{a}{2}$ ,  $\frac{a}{3}$ ,  $\frac{a}{4}$ ,  $\frac{a}{5}$ ,  $\frac{a}{6}$ , respectively.

Interpret each of the following fractions in two ways:  
 $\frac{1}{2}$ ,  $\frac{3}{4}$ ,  $\frac{5}{8}$ ,  $\frac{7}{16}$ ,  $\frac{9}{32}$ ,  $\frac{11}{64}$ .

Write the fractions which correspond to the divisions of a unit area into  $a$ ,  $b$ ,  $c$ ,  $d$ ,  $e$  parts respectively.

**Ex. 124.** Construct a square of  $K$  units, and divide it into  $m$  equal rectangles. What fraction represents the area of one of its rectangles?

**Ex. 125.** What is the interpretation of the following fractions:

$$\frac{a}{b}, \frac{c}{3}, \frac{d}{e}, \frac{5}{f}, \frac{3a}{b}, \frac{2c}{3}, \frac{d}{2e}, \frac{5}{3f}, \frac{3a}{2b}.$$

**Ex. 126.** Arrange the fractions  $\frac{1}{2}, \frac{3}{4}, \frac{5}{8}, \frac{7}{16}, \frac{9}{32}, \frac{11}{64}$  according to the order of their magnitudes? If we interpret these fractions as multiple values of different subdivisions of a unit area, then the smallest division has been obtained by dividing the unit area into 64 rectangles. Two of these 64 rectangles would unite into a larger rectangle, which could be obtained by dividing the unit area into 32 rectangles. In other words,  $\frac{2}{64} = \frac{1}{32}$ . In like manner our reasoning would show us that  $\frac{2}{32} = \frac{1}{16}$  and  $\frac{4}{64} = \frac{1}{16}$ ; further,  $\frac{2}{16} = \frac{1}{8} = \frac{4}{32} = \frac{8}{64}$ ; further  $\frac{2}{8} = \frac{1}{4} = \frac{4}{16} = \frac{8}{32} = \frac{16}{64}$ ; and further  $\frac{2}{4} = \frac{1}{2} = \frac{4}{8} = \frac{8}{16} = \frac{16}{32} = \frac{32}{64}$ .

All these facts verify the **axiom of fractions**, viz.:

*If the numerator and denominator of a fraction are either both multiplied or both divided by a like factor, the fraction remains unaltered in value, although it is changed in form.*

By applying the axiom of fractions, we have:

$$\frac{1}{2} = \frac{1 \times 32}{2 \times 32} = \frac{32}{64};$$

$$\frac{3}{4} = \frac{3 \times 16}{4 \times 16} = \frac{48}{64};$$

$$\frac{5}{8} = \frac{5 \times 8}{8 \times 8} = \frac{40}{64};$$

$$\frac{7}{16} = \frac{7 \times 4}{16 \times 4} = \frac{28}{64};$$

$$\frac{9}{32} = \frac{9 \times 2}{32 \times 2} = \frac{18}{64};$$

$$\frac{11}{64} = \frac{11 \times 1}{64 \times 1} = \frac{11}{64}.$$

In the order of ascending, i.e., increasing magnitudes, these fractions should be written as follows:

$$\frac{11}{64}, \frac{9}{32}, \frac{7}{16}, \frac{1}{2}, \frac{5}{8}, \frac{3}{4}.$$

Literal and mixed (literal and numeric combined) fractions are treated alike according to the laws for adding, subtracting, multiplying, and dividing numeric fractions.

**Ex. 127.** Unite  $\frac{5}{8} + \frac{7}{4} + \frac{a}{16} + \frac{2b}{16} - \frac{3}{4}$ . The first step is to apply

the axiom of fractions, in order to have a like denominator for each fraction. Therefore the example becomes

$$\frac{10}{16} + \frac{28}{16} + \frac{a}{16} + \frac{2b}{16} - \frac{12}{16}.$$

In other words we have  $(10+28+a+2b-12)$  of the same parts, i.e., common denominator, or  $\frac{10+28+a+2b-12}{16}$ , by combining numerators and writing their sum over the common denominator. This simplifies and reduces to  $\frac{26-a+2b}{16}$ . Since the

literal and numeric terms cannot be united, their sum is indicated by the plus sign connecting them.

**Ex. 128.** Unite  $\frac{5}{8a} + \frac{7}{4b} + \frac{1}{16a} + \frac{2}{16b} - \frac{3}{4ab} + 2$ . In order to obtain a common denominator for the given fractions, we observe the different factors in each denominator. These factors must be contained in the least common denominator (L.C.D.). Factoring the denominators, we have:

$$8a = 2 \times 2 \times 2 \times a,$$

$$4b = 2 \times 2 \times b,$$

$$16a = 2 \times 2 \times 2 \times 2 \times a,$$

$$16b = 2 \times 2 \times 2 \times 2 \times b,$$

$$4ab = 2 \times 2 \times a \times b,$$

$$16ab = 2 \times 2 \times 2 \times 2 \times a \times b = \text{L.C.D.}$$

The last term in the given sum may be written  $\frac{2}{1}$  instead of 2. Each fraction of the example is multiplied by the deficient, i.e., lacking factors in its denominator, hence:

$$\begin{aligned} \frac{5 \times 2b}{8a \times 2b} + \frac{7 \times 4a}{4b \times 4a} + \frac{1 \times b}{16a \times b} + \frac{2 \times a}{16b \times a} - \frac{3 \times 4}{4ab \times 4} + \frac{2 \times 16ab}{1 \times 16ab} \\ = \frac{10b}{16ab} + \frac{28a}{16ab} + \frac{b}{16ab} + \frac{2a}{16ab} - \frac{12}{16ab} + \frac{32ab}{16ab} \end{aligned}$$

Since each fraction now appears with a like denominator, we may collect the numerators, and write their sum with the same divisor or common denominator, which gives  $\frac{11b + 30a + 32ab - 12}{16ab}$ .

When fractions appear in an equation, they may be removed by multiplying the equation by the L.C.D.

**Ex. 129.**

- (1)  $\frac{5}{3} + \frac{6x}{4} = x - \frac{7}{6}$ , ————— the L.C.D. = 12.
- (2)  $\frac{5 \times 12}{3} + \frac{6x \times 12}{4} = 12 \times x - \frac{7 \times 12}{6}$ , — by multiplying (1) by L.C.D.
- (3)  $20 + 18x = 12x - 14$ , ————— simplifying (2).
- (4)  $20 + 6x = -14$ , ————— subtracting  $12x$  from (3).
- (5)  $6x = -34$ , ————— subtracting 20 from (4).
- (6)  $x = -\frac{34}{6} = -\frac{17}{3}$ , ————— dividing (5) by 6.

The operation in (4) is often called **transposition**, because a term appears to be carried across the equality sign with a change in its sign.

In multiplying fractions, write the answer as a fraction, with a numerator equal to the product of the individual numerators, and with a denominator equal to the product of the individual denominators.

**Ex. 130.**  $\frac{5}{8} \times \frac{3}{4} \times \frac{6}{7} \times \frac{14}{25} = \frac{5 \times 3 \times 6 \times 14}{8 \times 4 \times 7 \times 25}$ , but by the axiom of fractions, this simplifies by striking out like factors in numerator and denominator, and becomes  $\frac{1 \times 3 \times 3 \times 1}{2 \times 4 \times 1 \times 5} = \frac{9}{40}$ . This result would have been obtained by striking out numerator factors with equal denominator factors in the different fractions:

$$\frac{\overset{5}{\cancel{5}}}{\underset{4}{\cancel{4}}} \times \frac{\overset{3}{\cancel{3}}}{\underset{2}{\cancel{2}}} \times \frac{\overset{6}{\cancel{6}}}{\underset{7}{\cancel{7}}} \times \frac{\overset{14}{\cancel{14}}}{\underset{25}{\cancel{25}}} = \frac{9}{40}.$$

In multiplication, the multiplication symbol is often replaced by the equivalents "of" and "times." Thus:

$$\frac{1}{2} \times \frac{5}{8} = \frac{1}{2} \text{ times } \frac{5}{8} = \frac{1}{2} \text{ of } \frac{5}{8} = \frac{5}{16}.$$

*Observation.* To multiply a quantity by a number is equivalent to dividing the quantity by its reciprocal.

Therefore

$$\frac{\frac{5}{8}}{\frac{2}{1}} = \frac{5}{8} \times \frac{1}{2},$$

this may be written

$$\frac{\frac{5}{8}}{\frac{6}{3}} = \frac{5}{8} \times \frac{3}{6} = \frac{5}{16}.$$

*Observation.* To simplify a compound fraction, i.e., the ratio of one fraction to another fraction, multiply the numerator fraction by the denominator fraction inverted.

**Ex. 131.** Simplify the following:

$$(a) \frac{1}{2} + 2k + \frac{3}{p} + \frac{5k}{3} - \frac{6}{2p} - \frac{3k}{2},$$

$$(d) \frac{\frac{5}{8}}{\frac{6}{8}}$$

$$(b) \frac{7}{8} \text{ of } \frac{1}{2} \text{ times } \frac{3}{16} \times \frac{48}{42},$$

$$(c) \frac{\frac{3}{4}}{\frac{5}{6}}$$

$$(e) \frac{\frac{3}{2}}{\frac{1}{2}}$$

$$(f) \frac{\frac{2}{3}}{\frac{2}{9}}$$

**Ex. 132.** State why the following forms are not equivalent.

$$(a) \frac{3+x}{5+x} \neq \frac{3}{5},$$

$$(e) \frac{6-x}{6+x} \neq 1,$$

$$(b) \frac{3+2x}{5+4x} \neq \frac{3+x}{5+2x},$$

$$(f) \frac{\sqrt{A+B}}{A+B} \neq 1,$$

$$(c) \frac{5+5x}{5+3} \neq \frac{1+x}{1+3},$$

$$(g) ab^2 + ab + a^2b \neq 3a^2b^2,$$

$$(h) 7ab + 6cb \neq 13acb,$$

$$(d) \frac{20+x}{20} \neq x,$$

$$(i) 16a^2 + \frac{ab^2}{4} \times \frac{a^2}{b^4} \neq 4a^2b.$$

**22. Laws of Numbers.** A polynomial is an expression of several algebraic terms, such as  $a-b-c-d$ . When the polynomial consists of two terms only, such as  $a+b$  or  $-b$ , it is called a **binomial**, and when it consists of three terms, such as  $a-b-c$ , it is called a **trinomial**.

**Ex. 133.** Multiply  $(a+b)$  by  $(a+b)$ , i.e., square the binomial  $(a+b)$ .  $a$  and  $b$  are the symbols for any two distinct quantities or numbers.

$$\begin{array}{rcl}
 a+b & & \\
 a+b & & \\
 \hline
 a^2 & \text{the product of } a \text{ times } a, & \\
 +ab & \text{the product of } a \text{ times } b, & \\
 +ab & \text{the product of } b \text{ times } a, & \\
 +b^2 & \text{the product of } b \text{ times } b, & \\
 \hline
 \text{The product} = a^2 + 2ab + b^2 = \text{the sum of the partial} & & \\
 & \text{products.} & 
 \end{array}$$

In multiplication, every term of the minuend is multiplied by every term of the multiplier. The **product** of terms contains the factors of the terms, with the exponents of like letters added.  $(a+b)$  is treated like a single letter when applying the principle of adding exponents.

$$\therefore (a+b) \text{ times } (a+b) = (a+b)^2 = \underline{a^2 + 2ab + b^2} = \underline{a^2 + b^2 + 2ab}.$$

*Observation.* The square of the sum of two quantities is a trinomial and equals the sum of the squares of the two quantities, plus twice the product of the quantities.

**Ex. 134.** Write the values of the following without performing the multiplications:

- |                 |                  |
|-----------------|------------------|
| (a) $(x+y)^2$ , | (d) $(2a+5)^2$ , |
| (b) $(2+y)^2$ , | (e) $(5+20)^2$ , |
| (c) $(a+5)^2$ , | (f) $(20+1)^2$ . |

**Ex. 135.** Multiply  $(a-b)$  by  $(a-b)$ , i.e., square the binomial  $(a-b)$ .  $a$  and  $b$  may represent any two quantities or numbers.

$$\begin{array}{rcl}
 a-b & & \\
 a-b & & \\
 \hline
 a^2 & \text{the product of } a \text{ times } a, & \\
 -ab & \text{the product of } a \text{ times } (-b), & \\
 -ab & \text{the product of } (-b) \text{ times } a & \\
 +b^2 & \text{the product of } b \text{ times } b. & \\
 \hline
 \end{array}$$

The product  $= a^2 - 2ab + b^2 =$  the sum of the partial products.

$$\therefore (a-b) \text{ times } (a-b) = (a-b)^2 = \underline{a^2 - 2ab + b^2} = \underline{a^2 + b^2 - 2ab}.$$

*Observation.* The square of the difference between two quantities is a trinomial, and equals the sum of the squares of the two quantities, minus twice the product of the two quantities.

**Ex. 136.** Write the values of the following without multiplying:

- (a)  $(x-y)^2$ , (d)  $(2a-5)^2$ ,  
 (b)  $(2-y)^2$ , (e)  $(5-20)^2$ ,  
 (c)  $(a-5)^2$ , (f)  $(20-1)^2$ .

In (e)  $(5-20)^2 = 25 - 200 + 400 = 225$ , but  $(5-20)^2 = (-15)^2$ .

*Observation.* The product of two negative terms or the square of a negative term is positive. The product of a positive and a negative term is negative.

**Ex. 137.** Multiply  $(a+b)$  by  $(c+d)$ .

$$\begin{array}{r}
 a+b \\
 c+d \\
 \hline
 ac \qquad \text{the product of } c \text{ times } a, \\
 +cb \qquad \text{the product of } c \text{ times } b, \\
 +ad \qquad \text{the product of } d \text{ times } a, \\
 +bd \qquad \text{the product of } d \text{ times } b, \\
 \hline
 \end{array}$$

The product =  $\underline{ac+cb+ad+bd}$  = the sum of the partial products.

**Ex. 138.** Multiply  $(a+b)$  by  $(a-b)$ .

$$\begin{array}{r}
 a-b \\
 a+b \\
 \hline
 a^2 \qquad \text{the product of } a \text{ times } a, \\
 -ab \qquad \text{the product of } a \text{ times } (-b), \\
 +ab \qquad \text{the product of } b \text{ times } a, \\
 -b^2 \qquad \text{the product of } b \text{ times } (-b), \\
 \hline
 \end{array}$$

The product =  $\underline{a^2-b^2}$  = the sum of the partial products.

*Observation.* The product of the sum of two quantities times the difference of the same quantities, equals the square of the first quantity, minus the square of the second quantity.

**Ex. 139.** Write the values of the following without multiplying:

- |                    |                        |
|--------------------|------------------------|
| (a) $(a+k)(a-k)$ , | (e) $(5d+3r)(5d-3r)$ , |
| (b) $(a+k)(b-k)$ , | (f) $(5d+3r)(7k-3c)$ , |
| (c) $(5+d)(5-d)$ , | (g) $(16+10)(16-10)$ , |
| (d) $(7-2)(7+2)$ , | (h) $(10+3)(17-4)$ .   |

**Ex. 140.** Multiply  $(a^2-ab+b)$  times  $(a+b) = \underline{\underline{a^3+b^3}}$ .

$$\begin{array}{r}
 a^2-ab+b^2 \\
 a+b \\
 \hline
 a^3-a^2b+ab^2 \quad \text{the product of } a \text{ times the mul-} \\
 \quad \quad \quad \quad \quad \quad \quad \quad \text{tiplicand,} \\
 +a^2b-ab^2+b^3 \quad \text{the product of } b \text{ times the mul-} \\
 \quad \quad \quad \quad \quad \quad \quad \quad \text{tiplicand,} \\
 \hline
 \end{array}$$

The product =  $\underline{\underline{a^3+b^3}}$  = the sum of the partial products.

**Ex. 141.** Multiply  $(a^2+ab+b^2)$  times  $(a-b) = \underline{\underline{a^3-b^3}}$ .

$$\begin{array}{r}
 a^2+ab+b^2 \\
 a-b \\
 \hline
 a^3+a^2b+ab^2 \quad \text{the product of } a \text{ times the mul-} \\
 \quad \quad \quad \quad \quad \quad \quad \quad \text{tiplicand,} \\
 -a^2b-ab^2-b^3 \quad \text{the product of } (-b) \text{ times the} \\
 \quad \quad \quad \quad \quad \quad \quad \quad \text{multiplicand,} \\
 \hline
 \end{array}$$

The product =  $\underline{\underline{a^3-b^3}}$  = the sum of the partial products.

**Ex. 142.** Divide  $(a^2+2ab+b^2)$  by  $(a+b)$ .

In division, set down dividend and divisor in descending, i.e., decreasing powers of one of the letters. The first term of the quotient is obtained by dividing the first term of the divisor into the first term of the dividend. The quotient will contain the same letters as are contained in both divisor and dividend, with exponents of the former subtracted from the exponents of like letters of the latter. The quotient times the divisor gives the first subtrahend.

Subtract the subtrahend from the dividend, and set down the remainder. The remainder with the remaining terms of the dividend constitutes a new dividend, and the operation outlined above is repeated, giving the second term of the quotient. The product of the new term of the quotient times the divisor is the new subtrahend.

$$\begin{array}{r|l}
 \text{dividend} & a+b, \text{ divisor} \\
 a^2+2ab+b^2 & a \quad \text{first term of quotient,} \\
 a \text{ times } (a+b), \underline{a^2+ab} & +b \text{ second term of quotient,} \\
 \text{first remainder, } ab & \\
 b \text{ times } (a+b), \underline{ab+b^2} & \\
 \text{no remainder.} & 
 \end{array}$$

$$\therefore \frac{a^2+2ab+b^2}{a+b} = \underline{\underline{a+b.}}$$

**Ex. 143.** Divide  $a^3+b^3$  by  $(a^2-ab+b^2)$

$$\begin{array}{r|l}
 \text{dividend} & a^2-ab+b^2 \text{ divisor,} \\
 a^3 & +b^3 \quad a \quad \text{first term of quotient,} \\
 \text{first subtrahend, } \underline{a^3-a^2b+ab^2} & +b \text{ second term of quotient,} \\
 \text{first remainder, } a^2b-ab^2 & \\
 \text{second subtrahend, } \underline{a^2b-ab^2+b^3} & \\
 \text{no remainder.} & 
 \end{array}$$

$$\therefore \frac{a^3+b^3}{a^2-ab+b^2} = \underline{\underline{a+b.}}$$

**Ex. 144.** Perform and verify the following divisions:

$$(a) \frac{a^2-2ab+b^2}{a-b} = a-b,$$

$$(e) \frac{a^2+5a+6}{a+2} = a+3,$$

$$(b) \frac{a^2-b^2}{a-b} = a+b,$$

$$(f) \frac{a^2-a-6}{a+2} = a-3,$$

$$(c) \frac{a^2-b^2}{a+b} = a-b,$$

$$(g) \frac{a^2+a-6}{a-2} = a+3,$$

$$(d) \frac{a^2+6a+9}{a+3} = a+3,$$

$$(h) \frac{a^2-b^2}{a-b} = a+ab+b^2.$$

**23. Factoring.** Any product may be resolved into its constituent factors, by performing a division which leaves no remainder. Both quotient and divisor are the factors of the dividend.

Often the work is simplified without division, by recognizing the factors by inspection. The simplest case is one in which each term possesses a factor common to every other term. The product of the common factors (H.C.F.), is written outside a parenthesis and the remaining factors of each term are written within the parenthesis.

GIVEN EXPRESSION	H.C.F. REMAINING FACTORS
$3ax^2b + 9a^2x^3b^2 + 15ax^4b^3$	$= \underline{\underline{3ax^2b(1 + 3axb + 5x^2b^2)}}.$

*Observation.* The highest common factor of an expression is the product of the common numeric and literal factors. The least exponent of any letter in the H.C.F. is the least exponent of that letter found in any term of the expression.

**Ex. 145.** Factor the H.C.F. from the following expressions:

- (a)  $5ax^3 + 25x^3 + 75axb$ ,
- (b)  $25cgs^2 + 12.5c^2g^2s^4 + 62.5c^2gs$ ,
- (c)  $216ab^2 + 864ab + 864a$ .

**Ex. 146.** Factor  $6x^2 - 14xb - 9xb + 21b^2$ .

There is no H.C.F. for this expression, although there is a common factor of  $x$  for all the terms except the last, and a common factor  $b$  for all the terms except the first. There is also a numeric factor of 3 for all the terms except the second. In such cases it is advisable to factor the expression in two groups of two terms each as indicated by the parentheses.

$$6x^2 - 14xb - 9xb + 21b^2 = (6x^2 - 9xb) + (-14xb + 21b^2).$$

$3x$  is the common factor in the first parenthesis, and therefore

$$(6x^2 - 9xb) = 3x(2x - 3b).$$

$-7b$  is the common factor in the second parenthesis, and therefore

$$(-14xb + 21b^2) = -7b(2x - 3b).$$

( $-7b$ ) was factored instead of ( $+7b$ ) so that both parenthetical factors might be equal.

$$\therefore 6x^2 - 14xb - 9xb + 21b^2 = 3x(2x - 3b) - 7b(2x - 3b).$$

The last expression is therefore a difference of two products each having a common factor ( $2x - 3b$ ) and therefore we obtain

$$3x(2x - 3b) - 7b(2x - 3b) = (2x - 3b)(3x - 7b),$$

$$\therefore 6x^2 - 14xb - 9xb + 21b^2 = \underline{\underline{(2x - 3b)(3x - 7b)}}.$$

Verify this result by multiplying the factors.

**Ex. 147.** Factor the following expressions by grouping in two different ways:

(a)  $ac + ad + cb + bd,$

(b)  $5g + kg + ks + 5s,$

(c)  $9sa + 21st + 6ap + 14tp.$

The factors of any product may be detected by inspection or by division. A familiarity with the types of factors that produce binomial and trinomial products suffices for most cases.

If the expression is the difference between two squares, such as  $x^2 - y^2$ , then the factors are  $(x + y)$  and  $(x - y)$ . There are no factors for the sum of two squares. Why?

If the expression is the difference between two cubes, such as  $x^3 - y^3$ , then the factors are  $(x - y)$  and  $(x^2 + xy + y^2)$ .

If the expression is the sum of two cubes, such as  $x^3 + y^3$ , then the factors are  $(x + y)$  and  $(x^2 - xy + y^2)$ .

If trinomial expressions are to be factored, we observe first if they are the squares of binomials, from either of the following forms and their corresponding factors:

$$x^2 + y^2 + 2xy = (x + y)(x + y) = (x + y)^2,$$

$$x^2 + y^2 - 2xy = (x - y)(x - y) = (x - y)^2.$$

The sign of the product terms  $(+2xy)$  or  $(-2xy)$  is also the connecting sign inside the binomial parentheses.

**Ex. 148.** Write the factors of the following expression by inspection:

$$(a) 49 - 16b^2,$$

$$(d) a^2b^2 + 2ab + 1,$$

$$(b) 8 - 27b^3,$$

$$(e) x^2 + 4x + 4,$$

$$(c) 64y^2 + 8x^3,$$

$$(f) (a+b)^2 - (a-b)^2.$$

If the trinomial is not a perfect square, test it for a common factor. Then resort to the following procedure in order to put it into a four-term polynomial, after which it can be factored by grouping.

$$\text{Factor } 6x^2 - 23xb + 21b^2.$$

There is no common factor for this expression.

Multiply the coefficients of the square terms  $6 \times 21 = 126$ . Resolve **126** into two factors whose sum is **23**, i.e., equals the coefficient of the remaining term **23xb**.

$$126 = 53 \times 2 = 42 \times 3 = 21 \times 6 = 18 \times 7 = 14 \times 9, \text{ but } 14 + 9 = 23$$

therefore  $(-23xb)$  may be replaced by  $(-9xb - 14xb)$ . The example becomes  $6x^2 - 14xb - 9xb + 21b^2$ , and therefore the factors are  $(2x - 3b)(3x - 7b)$ . See Ex. 146.

**Ex. 149.** Factor the following trinomials.

$$(a) 3x^2 - 17x + 10,$$

$$(b) 10x^2 - 17x + 3,$$

$$(c) 6x^2y^2 - 23xyb + 20b^2,$$

$$(d) 6a^2 + 7a - 20.$$

In the last example 6 times  $(-20) = -120$ , therefore one of the factors is positive and the other negative. The excess of one factor over the other equals 7, i.e.,  $7a = 15a - 8a$ .

## 24. The Oblique Triangle.

**Ex. 150.** Draw an oblique triangle, i.e., a triangle whose sides are oblique, so that all of its angles are acute. Draw the three perpendiculars by joining each vertex to its opposite side. If a perpendicular falls outside the triangle produce the side to meet it.

Draw an oblique triangle one of whose angles is obtuse. Draw the three perpendiculars by joining the vertices to the opposite sides.

Each of these constructions gives the projections of two sides of a triangle upon the third side.

**Ex. 151.** Draw two unequal adjacent right triangles  $ABD$  and  $BDC$ , with a common equal altitude  $BD$ . The right triangles form an oblique triangle  $ABC$ . In triangle  $ABC$ , the  $\sin A = \frac{BD}{AB}$ ,

and therefore  $BD = AB \sin A$ . In triangle  $BDC$ , the  $\sin C = \frac{BD}{BC}$ , and therefore  $BD = BC \sin C$ . Since  $BD$  is an identity, i.e., a common side of both triangles, we say  $BD$  of triangle  $ABD = BD$  of triangle  $BDC$ . Then by equality axiom,  $AB \sin A = BC \sin C$ . In triangle  $ABC$ ,  $AB$  is opposite angle  $C$ , and may be abbreviated by  $a$ , and  $BC$  is opposite angle  $A$ , and may be abbreviated by  $c$ . Therefore

$$c \sin A = a \sin C.$$

$$\therefore \frac{a}{\sin A} = \frac{c}{\sin C}$$

by equal products theorem in proportion.

Construct perpendiculars from  $A$ , and also from  $C$ , to the opposite sides of triangle  $ABC$ , and establish the complete fact

$$\frac{a}{\sin A} = \frac{c}{\sin C} = \frac{b}{\sin B}$$

*Observation.* In any triangle the ratio of any side to the sine of its opposite angle is a constant. This is known as the **Law of Sines**.

If any three parts of a triangle are given, including two sides and an angle opposite one of them, or two angles and a side opposite one of them, then this formula may be used to solve the triangle.

**Ex. 152.** Given  $a=6$ ,  $b=3$ , and  $A=30^\circ$ .

To determine  $B$ , we use  $\frac{a}{\sin A} = \frac{b}{\sin B}$ , from which we obtain  $\sin B = \frac{b \sin A}{a}$ , and substituting the above values we obtain  $\sin B = \frac{3 \times \sin 30^\circ}{6} = \frac{3 \times .5}{6} = .25$ .

In Table VIII locate .2500 under the heading sines, and determine the angle  $B = \underline{\underline{14^\circ 30'}}$ .

$$C = 180^\circ - (30^\circ + 14^\circ 30') = \underline{\underline{145^\circ 30'}}.$$

Solve for  $c$ .

**Ex. 153.** Construct an oblique triangle  $ABC$ , draw a perpendicular  $h$  from the vertex  $B$  to the opposite side  $b$ . The side  $b$  will be divided into two segments  $x$  and  $y$ , which are the respective projections of  $c$  and  $a$ , upon  $b$ . Then

$$c^2 = h^2 + x^2,$$

from the law of the right triangle,

$$\text{and} \quad x = b - y,$$

because the sum of the parts equals the whole,

$$\therefore x^2 = (b - y)^2 = b^2 + y^2 - 2by,$$

by squaring both members of the preceding equation. This value of  $x^2$  is now substituted in the first equation.

$$\therefore c^2 = h^2 + y^2 + b^2 - 2by;$$

in the latter we replace  $h^2 + y^2$  by  $a^2$  because  $h^2 + y^2 = a^2$ , also replace  $y$  in  $2by$ , by its equal,  $a \cos C$ , because  $y = a \cos C$  from the definition of a cosine. These substitutions will give the final form

$$c^2 = a^2 + b^2 - 2ab \cos C.$$

*Observation.* The square of any side of a triangle equals the sum of the squares of the other two sides, minus twice the product of the two sides times the cosine of the angle included by them.

This is known as the **Law of Cosines**.

$$a^2 = b^2 + c^2 - 2bc \cos A \quad \text{and} \quad b^2 = a^2 + c^2 - 2ac \cos B$$

are two other forms for the law of cosines.

**Ex. 154.** Solve the triangle  $ABC$ , given  $a=6$ ,  $b=10$ ,  $C=30^\circ$ . The working formula is  $c^2 = a^2 + b^2 - 2ab \cos 30^\circ$ . Substituting the values of  $a$ ,  $b$ , and  $C$  from the data we obtain:

$$c^2 = 36 + 100 - 2 \times 6 \times 10 \times .866 = 136 - 104 = 32,$$

$$c = \underline{\underline{5.66}}.$$

The angles  $A$  and  $B$  may then be obtained by the law of sines.

$$\frac{a}{\sin A} = \frac{c}{\sin C}, \text{ from which we obtain } \sin A = \frac{a \sin C}{c},$$

$$\frac{b}{\sin B} = \frac{c}{\sin C}, \text{ from which we obtain } \sin B = \frac{b \sin C}{c}.$$

Solve for  $A$  and  $B$ .

$$\begin{array}{l} \text{Ex. 155.} \quad a=8 \\ \quad \quad \quad b=6 \\ \quad \quad \quad c=5.5 \end{array}$$

---


$$A = ?$$

$$B = ?$$

$$C = ?$$


---

The working formulas are:

$$a^2 = b^2 + c^2 - 2bc \cos A, \text{ from which we obtain } \cos A = \frac{b^2 + c^2 - a^2}{2bc}.$$

$$b^2 = a^2 + c^2 - 2ac \cos B, \text{ from which we obtain } \cos B = \frac{a^2 + c^2 - b^2}{2ac}.$$

$$C = 180^\circ - (A + B).$$

Substitute the numeric values from the data and check by graphic solution.

**Ex. 156.** The triangle in Ex. 155 may be solved by constructing the perpendicular  $h$ , from  $A$  to the opposite side  $a$ . The side  $a$  will be divided into two segments  $x$  and  $y$ , which are the projections of  $b$  and  $c$  respectively. Then  $x+y=a$ . By the law of right triangles  $b^2-x^2=h^2$ , and  $c^2-y^2=h^2$ .

$$\therefore b^2-x^2=c^2-y^2, \quad \text{or} \quad b^2-c^2=x^2-y^2.$$

*Observation.* The difference of the squares of two sides of a triangle is equal to the difference of the squares of their respective projections on the third side. This is known as the **Law of Projective Segments**.

From the above we may also write  $(b+c)(b-c) = (x+y)(x-y)$ , by factoring, or  $(x-y) = \frac{(b+c)(b-c)}{(x+y)} = \frac{b^2-c^2}{a}$ .

The last formula enables us to compute the difference between the segments. By combining the sum, with the difference of the segments, either by addition or subtraction, we obtain either twice  $x$  or twice  $y$  respectively.

Solving for  $(x-y)$  by substituting the data from Ex. 155, we obtain

$$x-y = \frac{b^2-c^2}{a} = \frac{36-30.25}{8} = .718, \quad \text{but} \quad x+y=a=8,$$

$$x+y=8$$

$$x+y=8$$

by addition  $x-y = .718$       by subtraction  $x-y = .718$

$$2x = 8.718$$

$$2y = 7.282$$

$$x = 4.359$$

$$y = 3.641$$

$$\cos B = \frac{y}{c} = \frac{3.641}{5.5} = .662. \quad \therefore B = \underline{\underline{48.6^\circ}}.$$

$$\cos C = \frac{x}{b} = \frac{4.359}{5.5} = .792. \quad \therefore C = \underline{\underline{37.7^\circ}}.$$

$$A = 180^\circ - (A+B) = 180^\circ - 86.3^\circ = \underline{\underline{93.7^\circ}}.$$

**Ex. 157.** Solve the following triangles given:

- (a)  $a = 5$ ,  $b = 6$ ,  $A = 45^\circ$ ,  
 (b)  $a = 3$ ,  $b = 3.2$ ,  $C = 120^\circ$ ,  
 (c)  $a = 12$ ,  $b = 11$ ,  $C = 40^\circ$ ,  
 (d)  $b = 11$ ,  $c = 5.3$ ,  $A = 55.8^\circ$ ,  
 (e)  $a = 13$ ,  $b = 12$ ,  $c = 5$ ,  
 (f)  $b = .10$ ,  $c = .25$ ,  $C = 105^\circ$ .

**25. Logarithms.** We have seen how one number may result from uniting other numbers by addition, subtraction, multiplication, or division. A number may be obtained as a result of a root or a power of another number. Thus 64 may result from adding 50 and 14 or by subtracting 36 from 100 or by multiplying 16 times 4 or dividing 320 by 5. Again  $64 = \sqrt{4096}$ , and further  $64 = 8^2 = 4^3 = 2^6$ . In like manner 729 may be considered the square root of 53'14'41, or  $729 = 3^6 = 9^3 = 27^2$ .

By the law of multiplication, we add exponents of like factors, and write this sum over the like factor in the product. Therefore  $3^{2.1} \times 3^{5.8} \times 3^1 = 3^{8.4}$ . If one or more of the exponents should be negative, the sum of the exponents will bear the sign of the excess. Thus  $10^{3.2} \times 10^{-2.6} \times 10^{2.4} = 10^3 = 1000$ , since  $3.2 - 2.6 + 2.4 = 3$ , and  $10^3 = 1000$ . 1000 is the product of three numbers  $a$ ,  $b$ , and  $c$ , corresponding respectively to  $10^{3.2}$ ,  $10^{-2.6}$ , and  $10^{2.4}$ , in other words,  $abc = 1000$ , where  $a = 10^{3.2}$ ,  $b = 10^{-2.6}$ , and  $c = 10^{2.4}$ .

*Observation.* An entire multiplication may be made by considering the factors and the product as powers of 10.

Our familiarity with powers and roots of 10 is limited as follows:

The first power of 10  $= 10^1 = 10$ .

The second power of 10  $= 10^2 = 100$ .

The third power of 10  $= 10^3 = 1000$ .

The fourth power of 10  $= 10^4 = 10000$ .

The fifth power of 10  $= 10^5 = 100000$ .

The square root of  $10 = \sqrt{10} = 3.162$  approx.  $= 10^{\frac{1}{2}} = 10^{.5}$ ,  
the one-half power of 10.

The cube root of  $10 = \sqrt[3]{10} = 2.154$  approx.  $= 10^{\frac{1}{3}} = 10^{.333}$ ,  
the one-third power of 10.

The fourth root of  $10 = \sqrt[4]{10} = 1.778$  approx.  $= 10^{\frac{1}{4}} = 10^{.25}$ ,  
the one-fourth power of 10.

$$\sqrt{10} \times \sqrt{10} = 10^{.5} \times 10^{.5} = 10^1 \times 10^1 = 10$$

$$\sqrt[3]{10} \times \sqrt[3]{10} \times \sqrt[3]{10} = 10^{\frac{1}{3}} \times 10^{\frac{1}{3}} \times 10^{\frac{1}{3}} = 10.$$

$$\therefore 10 \times 10^{.5} = 10^{1.5} = 10 \times 3.162 = 31.62.$$

The logarithm of  $31.62 = 1.5$ .

$$10^2 \times \sqrt[3]{10} = 10^{2.333} = 100 \times 2.154 = 215.4.$$

The logarithm of  $215.4 = 2.333$ .

$$10^3 \times \sqrt[4]{10} = 10^{3.25} = 1000 \times 1.778 = 1778.$$

The logarithm of  $1778 = 3.25$ .

From the above it will be seen that all numbers may be considered as powers of 10. Such a foundation number as 10, to which all numbers are referred as powers, is called a **base**, and the **exponent** of the base takes the special name of **logarithm**.

*Observation.* To obtain the logarithm of a number means, therefore, to determine the corresponding exponent of the base 10.

The power of 10 and the given number are equivalent, i.e.,  $10^{3.25} = 1778$ , and therefore the logarithm of 1778 is 3.25.

Considering the integral powers of 10 in the above table, we observe that the logarithms increase by a constant difference of 1, for every higher multiple of 10. There are as many zeros following 1 in the given number as the corresponding exponent of 10. In other words, a six-figure number has a logarithm of 5; a five-figure number has a logarithm of 4; a four-figure number has a logarithm of 3; a three-figure number has a logarithm of 2; a two-

figure number has a logarithm of 1. In each case the logarithm is one less than the number of integral figures. If we divide a number by 10, we decrease its integral figure by 1, and therefore decrease its logarithm by 1.

$$\therefore 10^2 = 100.$$

$$\frac{a}{a} = a^{1-1} = a^0 \text{ but } \frac{a}{a} = 1.$$

$$10^1 = 10.$$

$$\therefore a^0 = 1.$$

$$10^{1-1} = 10^0 = 1. = \frac{10}{10} = \frac{10^1}{10^1}. \therefore 10^0 = 1.$$

$$10^{0-1} = 10^{-1} = 0.1$$

$$10^{-1-1} = 10^{-2} = 0.01$$

*Observation. The zero power of 10 or any base equals unity. Logarithms of decimals are negative.*

A number of two figures, such as 27, will have a logarithm greater than 1 but less than 2, because 27 is greater than 10, but less than 100. A number of three figures, such as 546, will have a logarithm greater than 2 but less than 3. The logarithm of a digit is a decimal number, i.e., it is greater than 0 but less than 1.

The logarithm of any number, except a multiple of 10, consists of an integral part, called the **characteristic**, and a decimal part, called the **mantissa**. The characteristic is a positive integer for all numbers greater than unity, and is numerically one less in unit value than the number of **integral figures**, i.e., figures written to the left of the decimal point. The characteristic is negative for any number less than one. Its unit value is numerically the same as the decimal position of the first **significant figure**, i.e., first digit to the right of the decimal point. Mantissas are read in a table of logarithms such as Table IX. Any multiple of 10 has a zero mantissa.

The position of the decimal point in a given number affects the unit value of the characteristic only. The

mantissa depends upon the order, i.e., succession or sequence, of figures only. Thus the  $\log 1778 = 3.25$ . Dividing 1778 by 10 gives 177.8, and therefore  $\log 177.8 = 2.25$ ; again dividing 177.8 by 10 gives 17.78 and therefore  $\log 17.78 = 1.25$ .  $\therefore \log .1778 = \bar{1}.25$ . The minus sign is put over the characteristic because the mantissa is always positive.

Table IX consists of twenty vertical columns. Column one is headed N, columns 2 to 11 are headed "the third figure of your number," and columns 12 to 20 inclusive are headed proportional parts. The decimal points are omitted from the table, and should be prefixed to the four figures of the mantissa. Table IX is a four-place table. Other tables may give 3, 5, 6, 7, 10, or 15 figures for the mantissa. The degree of accuracy required in a given computation determines the proper number-place table which should be used. For practical purposes a four-place table suffices.

To use Table IX, locate the first two figures of the given number under the column headed N, carry the finger horizontally across the page, and in the column headed by the third figure read the mantissa, i.e., the four figures. Precede these figures by a decimal point, and the latter by the proper characteristic determined by inspection. Thus to find the log of 325, locate 32 under the N column, and in the seventh column headed 5 read 5119. The characteristic equals 2.  $\therefore \log 325 = 2.5119$ . To logarize a number means to determine its logarithm.

**Ex. 158.** Logarize and verify the following:

$$(a) \log 320 = 2.4771, \quad \therefore \log 3.2 = 1.4771,$$

$$(b) \log 408 = 2.6107, \quad \therefore \log 4080 = 3.6107,$$

$$(c) \log 912 = 2.9600, \quad \therefore \log 0.912 = \bar{1}.9600,$$

$$(d) \log 5.55 = 0.7443, \quad \therefore \log 0.0555 = \bar{2}.7443,$$

$$(e) \log \pi = 0.4971, \quad \therefore \log \frac{\pi}{10} = \bar{1}.4971.$$

The mantissa of a two-figure number, such as 12, is the same as the mantissa of the number 120. Therefore the mantissa of a two-figure number is read in the second or zero column, opposite the two figures in the N column. The mantissa of a one-figure number, such as 7, is the same as the mantissa of 700. Any number of zeros preceding or following a given number does not change its mantissa (man).

$$\text{man } 7 = \text{man } 70 = \text{man } 700 = \text{man } 7000 = \text{man } .0007, \text{ etc.}$$

To read the mantissa of a four-figure number, such as 238.4, we must appreciate the fact that the number 238.4 lies between 238 and 239. In other words, the man 238 is smaller than the man 238.4, and the man 239 is greater than the man 238.4. The required man 238.4 may be obtained by interpolation, i.e., adding a correction to man 238 or subtracting a correction from man 239.

$$\begin{array}{r} \text{man } 239 = 3784 \\ \text{man } 238 = 3766 \\ \hline \text{difference} = 18 \end{array}$$

An increase of 1 in the numbers corresponds to an increase of 18 in the mantissas. Therefore an increase of:

.1 in numbers  $\equiv 18 \times .1 = 2$  approximately to be added to the man 238, and an increase of .2 in numbers  $\equiv 18 \times .2 = 4$  approximately to be added to the man 238, and an increase of .4 in numbers  $\equiv 18 \times .4 = 7$  approximately to be added to the man 238.

$\equiv$  means corresponds to.

$$\therefore \text{man } 238.4 = \text{man } 238 + \text{correction for } .4 = 3766 + 7 = 3773.$$

$$\therefore \text{man } 238.4 = \text{man } 239 - \text{correction for } .6 = 3784 - 11 = 3773.$$

Instead of calculating the correction for the fourth figure of a number, we may read the correction under the columns 12-20 headed **proportional parts**. After locating the mantissa for the first three figures, continue the finger horizontally across the page, and in columns of proportional parts headed by the fourth figure read the correction.

The mantissa of 238 is 3766, and under column 4 of proportional parts is 7.  $\therefore \text{man } 238.4 = 3766 + 7 = 3773$ .

**Ex. 159.** Verify the following:

$$(a) \log 55.55 = 1.7447, \quad (d) \log 0.1764 = \bar{1}.2465,$$

$$(b) \log 9097 = 3.9589, \quad (e) \log 1.985 = 0.2978,$$

$$(c) \log 12330 = 4.0910, \quad (f) \log 700.7 = 2.8455.$$

**Ex. 160.** Multiply 55.55 by 700.7. Since  $55.55 = 10^{1.7447}$  and  $700.7 = 10^{2.8455}$ , then

$$55.55 \times 700.7 = 10^{1.7447} \times 10^{2.8455} = 10^{1.7447 + 2.8455} = 10^{4.5902} \\ = \text{product.}$$

$$\therefore \text{man product} = .5902.$$

Table IX serves a new purpose, viz., to find the number when the mantissa is known. The process is the reverse of finding a logarithm and is called **antilogarization**. The number is the antilogarithm ( $\log^{-1}$  or **antilog**) of a logarithm.

Locate 5902 under columns 2 to 11 inclusive, and on the same horizontal line, read the antilogarithm under the *N* column. When the mantissa is not given in the table, read that antilogarithm in the *N* column which corresponds to the next lowest mantissa, and correct the antilogarithm by an accretion taken from the proportional parts column. 5899 is the next lowest mantissa to 5902, and this corresponds to the antilogarithm 389. The difference between 5902 and 5899 is 3, which is located under column headed 3 of proportional parts. Therefore the fourth figure of the antilogarithm of 5902 is 3, and accordingly the product is 38930. Five integral figures appear to the left of the decimal point because the characteristic was given as 4.

$$\therefore \log^{-1} 4.5902 = 38930 = \text{product } 55.55 \times 700.7.$$

Verify this result by actual multiplication.

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*Observation. The logarithm of a product equals the sum of the logarithms of its factors.*

**Ex. 161.** Perform the following multiplications by logarithms:

- |                           |                                |
|---------------------------|--------------------------------|
| (a) $38.92 \times 6005$ , | (d) $72 \times 72 \times 72$ , |
| (b) $7891 \times .8032$ , | (e) $100 \times 62.54$ ,       |
| (c) $360 \times 360$ ,    | (f) $.0001 \times 83620$ .     |

**Ex. 162.** Divide 55.55 by 700.7. This may be written:

$$\frac{55.55}{700.7} = \frac{10^{1.7447}}{10^{2.8445}} = 10^{1.7447 - 2.8455},$$

by the law of division.

$$1.7447 - 2.8445 = 1.7447 - .8455 - 2 = .8992 - 2 = \bar{2}.8992.$$

The subtraction must be performed so as to leave a positive mantissa.

$$\text{Quotient} = 10^{\bar{2}.8992}.$$

Looking up the antilog of  $\bar{2}.8992$  we obtain 7929.

The negative characteristic indicates that the first significant figure is in the second decimal place.

Therefore

$$\log^{-1} \bar{2}.8992 = \underline{\underline{.07929}} = \frac{55.55}{700.7}$$

Verify this result by actual division.

**Ex. 163.**

- |                            |                             |
|----------------------------|-----------------------------|
| (a) $\frac{38.92}{6005}$ , | (d) $\frac{1}{.6254}$ ,     |
| (b) $\frac{7891}{.8032}$ , | (e) $\frac{1}{6.254}$ ,     |
| (c) $\frac{81}{9}$ ,       | (f) $\frac{.0001}{83620}$ . |

*Observation. The logarithm of a quotient equals the logarithm of the numerator minus the logarithm of the denominator.*

In example 161 (c),  $360 \times 360$  may have been written  $360^2$ . The logarithm (5.1136) of the power  $360^2$  is twice the logarithm (2.5563) of the number 360.

In Ex. 161 (d),  $72 \times 72 \times 72$  may have been written  $72^3$ . The logarithm (5.5719) of  $72^3$  is three times the logarithm (1.8573) of the number 72.

*Observation.* The logarithm of a power of a number is equal to the exponent times the logarithm of the number.

**Ex. 164.** Perform the following:

- |                    |                           |
|--------------------|---------------------------|
| (a) $5360^{1.6}$ , | (d) $360^{\frac{1}{3}}$ , |
| (b) $8.5^{2.5}$ ,  | (e) $560^{-2}$ ,          |
| (c) $92.1^{-5}$ ,  | (f) $.018^{-2}$ .         |

**Ex. 165.** (c), Ex. 164, may be written  $92.1^{\frac{1}{5}} = \sqrt[5]{92.1}$  and since  $\log 92.1^{\frac{1}{5}} = \frac{1}{5} \log 92.1 = \frac{\log 92.1}{5}$ , then  $\log \sqrt[5]{92.1} = \frac{\log 92.1}{5}$ .

Ex. 164 (d) may be written  $360^{\frac{1}{3}} = \sqrt[3]{360}$ , and since  $\log 360^{\frac{1}{3}} = \frac{1}{3} \log 360 = \frac{\log 360}{3}$ , then  $\log \sqrt[3]{360} = \frac{\log 360}{3}$ .

*Observation.* The logarithm of the root of a number equals the logarithm of the number divided by the root index, i.e., root numeral. A root of a number is equivalent to a power of the same number whose exponent is the reciprocal of the root index.

**Ex. 166.** Perform the following:

- |   |  |
|---|--|
| (a) $\frac{\sqrt{360}}{\sqrt[3]{92.1}}$ , | (d) $\sqrt[5]{3}$ ,                    |
| (b) $5\sqrt{389}$ ,                       | (e) $\sqrt[.01]{1.009}$ ,              |
| (c) $\sqrt[2.5]{2.5}$                     | (f) $\frac{1.01^{3.7}}{\sqrt{1.01}}$ . |

In the above outline on logarithms, all numbers are considered as powers of the base 10. This system is most

commonly used, and the ordinary logarithm table is constructed upon this so-called **common base**. It is less frequently designated as the **Briggs system**. Its principal advantage lies in its decimal characteristic.

There are circumstances in limited branches of science where the bases 2, 2.4, 2.5, and other numbers are used instead of the base 10.

Natural phenomena are very often formulated in terms of an unending number 2.71828+, which is abbreviated by the Greek letter  $\epsilon$ , and sometimes by the English  $e$ . It becomes desirable, therefore, to express numbers as powers of  $\epsilon$ , and hence we have tables of logarithms based upon  $\epsilon$  instead of 10. These are called by various names as either **natural** or **Napierian** or **hyperbolic** logarithms.

The Napierian tables are used in the same manner as the Briggs tables. The former are distinguished by the fact that they give both the characteristic as well as the mantissa of a number, and comparatively larger numeric values than the corresponding Briggs logarithm. The Napierian logarithm is abbreviated  $\log_e$  in order to distinguish it from the common log or  $\log_{10}$ .

$\log 1000 = 3$  by the Briggs system,

$\log_e 1000 = 6.9078$  by the Napierian system.

In order to convert the Briggs logarithm of a number into a Napierian logarithm the former must be multiplied by a conversion factor **M** called a **modulus**. In other words,  $\log_e 1000 = M \log 1000$ . Therefore,

$$M = \frac{\log_e 1000}{\log 1000} = \frac{6.9078}{3} = 2.3026 = 2.303 \text{ approx.}$$

$$\log 10 = 1 \quad \log_e 10 = 2.303.$$

$$\log_e 10 = M \log 10, \therefore \log 10 = \frac{1}{M} \log_e 10 = .434 \log_e 10.$$

$$\text{The reciprocal of } M = \frac{1}{M} = \frac{1}{2.303} = .434.$$

*Observation. The smaller base produces the larger logarithm.*

**Ex. 167.** Write the Napierian logarithms of the following by multiplying the modulus into the common logarithm, check the results by consulting a Napierian logarithm table:

- |            |                  |
|------------|------------------|
| (a) 120,   | (d) .125,        |
| (b) 32.5,  | (e) .025,        |
| (c) 1.125, | (f) $\log 325$ . |

**Ex. 168.** In computing the logarithm of a power it is desirable at times to use the logarithmic operation twice in succession. Thus  $B = 6820^{1.63}$  is solved as follows:

$$\log B = 1.63 \log 6820$$

by law of log of a power.

$$\therefore \log (\log B) = \log 1.63 + \log \log 6820$$

by the law of log of a product.

$\log \log B$  is abbreviated  $\log'' B$  or LLB

$$\therefore \log \log 6820 = \log'' 6820 = \text{LL } 6820.$$

$$\therefore \log'' B = \log 1.63 + \log'' 6820.$$

$$\log 1.63 = .2122; \log 6820 = 3.8338; \log'' 6820 = \log 3.8338 = .5836$$

$$\therefore \log'' B = .2122 + .5836 = .7958.$$

In order to obtain  $B$  the operation of antilogarization must be performed twice

$$\therefore \log B = \log^{-1} .7958 = 6.248,$$

$$B = \log^{-1} 6.2480,$$

$$B = \underline{\underline{1770000}}.$$

This principle may be used in multiplying a common logarithm by the modulus 2.303.

**Ex. 169.** Obtain the Napierian logarithm of .625.

$$\log_e .625 = 2.303 \log .625,$$

$$\log .625 = \bar{1}.7959,$$

$$\therefore \log_e .625 = 2.303 \text{ times } \bar{1}.7959.$$

$\bar{1}.7959$  is part negative and part positive, so that the product will be part negative and part positive. Therefore these two parts must be clearly distinguished by one of the following methods.

(I)  $\bar{1}.7959$  may be united, giving the negative excess =  $-.2041$ .  
 $\therefore 2.303 \times (-.2041) = -\underline{\underline{.4700}}$ . The mantissas of common logarithms are always positive and are so read in the common logarithm table. Napierian tables may have the decimals negative or positive. In the latter case  $-.4700$  could not be read in the table. We may overcome this difficulty by writing  $\bar{1}.5300$ , which is equivalent to  $-.4700$  and is obtained by both adding and subtracting 1 to  $-.4700$ , which becomes:

$$-.4700 + 1 - 1 = (-.4700 + 1) - 1 = .5300 - 1 = \bar{1}.5300,$$

$$\therefore 2.303 \times (\bar{1}.7959) = \bar{1}.5300,$$

$$\therefore \log_e .625 = \underline{\underline{\bar{1}.5300}}.$$

$$(II) 2.303 \times (\bar{1}.7959) = 2.303(-1 + .7959) = -2.303 + (2.303 \times .7959)$$

$$2.303 \times .7959 = 1.833,$$

$$\therefore 2.303 \times (\bar{1}.7959) = -2.303 + 1.833 = -.4700 = \bar{1}.5300,$$

$$\therefore \log_e .625 = -.4700 = \bar{1}.5300.$$

*Observation. Logarithms simplify arithmetic processes by reducing them to more fundamental operations. Addition and subtraction cannot be further simplified. The arithmetic operations of multiplication, division, power, and root, are changed by the use of logarithms to the respective operations of addition, subtraction, multiplication, and division.*

**26. The Slide Rule.** Write a series, i.e., a row or succession or progression of numbers from 0 to 10, in gradations, i.e., by increases of .5. This is abbreviated by 0, .5, 1,

1.5, 2, 2.5 . . . 10; the dotted line . . . means **and so on**. Directly under these numbers write their respective squares, and in the intervening spaces on the following row (3), write the differences of the square, and under the latter, on row (4), write the differences of the latter. Thus:

row 1,	1	1.5	2	2.5	3 . . .	10
row 2,	1	2.25	4	6.25	9 . . .	100
row 3,	1.25	1.75	2.25	2.75	. . .	
row 4,	.5	.5	.5	.5 . . .		

Rows 1 and 3 are called **arithmetic progressions**, i.e., series with a constant difference between their consecutive terms.

**Ex. 170.** Determine the sum of the arithmetic progression (.5, 1, 1.5, 2, 2.5 . . . 10). The first term is 1, which we may designate by  $f$ , the last term is 10, which we may designate by  $l$ , the constant difference is .5, which we may designate by  $d$ , the number of terms is 20, which we designate by  $n$ , and finally designate the sum by  $S$ .

term number 2 = term number 1 + .5 difference,  $= f + d$ ,

term number 3 = term number 2 + .5 difference,  $= f + 2d$ ,

term number 4 = term number 3 + .5 difference,  $= f + 3d$ ,

term number 20 = term number 19 + .5 difference,  $= f + 19d = l$ ,

The sum equals the average value between first and last term, multiplied by the number of terms

$$S = 20 \left( \frac{.5 + 10}{2} \right) = n \left( \frac{f + l}{2} \right) = n \left( \frac{f + f + 19d}{2} \right) = \underline{\underline{105.}}$$

**Ex. 171.** (a) What is the sum of the first ten numbers? (b) What is the sum of the nine digits?

**Ex. 172.** Write a series of digits from 1 to 10. Under these write their respective cubes. On a third row write the differences of the cubes. On the fourth row write the differences of the numbers of the third row. What kind of a series does the fourth row represent?

A **geometric progression** is a series of terms which preserves a constant ratio between consecutive terms. Such a series is represented by the powers of 10, as 10, 100, 1000, 10000, . . .

**Ex. 173.** Prepare a strip of cardboard 1 in. wide by 10 ins. long. Half way between the upper and the lower edges draw a center line. On the latter mark ten points of division 1 in. apart, labeling them 0, 1, 2, . . . 10. Through these points of division, draw vertical lines intersecting the upper and lower edge. At the ten corresponding points of division on the upper edge, write the squares to the numbers on the center line. At the corresponding points on the lower edge, write the cubes of numbers. We now have three scales which we shall designate respectively as the scale of squares (*S*), the scale of proportionate parts (*PP*), and the scale of cubes (*C*). Subdivide the *PP* scale into tenths, and supplement the *S* and *C* scales by adding additional graduations to correspond to the new divisions on *PP*. The additional values for *S* and *C* may be calculated with very little trouble, by multiplying the ten original squares and cubes by a proportionate ratio. This rule serves the double purpose of reading squares and square roots, and also cubes and cube roots. Locate a number on the *S* scale and immediately opposite on the *PP* is its square root. Thus 6.25 on *S* is opposite 2.5 on *PP*. Locate a number on the *C* scale and immediately opposite on the *PP* scale is its cube root. Thus 64 on *C* is opposite 4 on *PP*.

**Ex. 174.** Prepare a strip of cardboard 1 in. by 10 ins. On the lower edge lay off a scale (*L*) of proportionate parts with subdivisions in tenths like the *PP* scale in Ex. 173. On the upper edge lay off a scale (*D*) of logarithms as follows: Opposite the following points of the *L* scale

0   3.01   4.77   6.02   6.99   7.78   8.45   9.03   9.54   10.00

place the following respective graduations on the *D* scale:

1   2   3   4   5   6   7   8   9   10

If the *D* scale is supplemented further by additional graduations the rule may be used in place of a logarithm table. Locate any number on the *D* scale and immediately opposite on the *L* scale is its logarithm. The extreme right-hand reading of the *L* scale is considered 1 in reading logarithms.

To multiply 4 by 2 adjust the needle points of a dividers on the *D* scale so that they span the distance between

1 and 2. This span is the logarithm of 2. A span on the *D* scale between 1 and 4 is the logarithm of 4. These two spans can be added by placing one needle point of the dividers at 4, and then with a span equal to the logarithm of 2, the other needle point will fall on 8. In other words, the  $\log 4 + \log 2 = \log 8$ , and therefore  $4 \times 2 = 8$ . The same result will be obtained by taking the span of the dividers between 1 and 4 on the *D* scale, and then if one of the needle points is placed at 2 on the *D* scale, the other needle point will fall on 8. Again we have  $\log 2 + \log 4 = \log 8$ , or  $2 \times 4 = 8$ . To divide 8 by 4, reverse the process with the dividers, i.e., with a span corresponding to  $\log 4$ , set one needle point at 8, and the other needle point will fall on 2, showing  $\log 8 - \log 4 = \log 2$ , or  $8 \div 4 = 2$ . Explain how you would adjust and set the dividers to divide 8 by 2.

The **slide rule** was invented in order to obviate the use of the dividers as described above and yet retain the advantage of a scale over a table. The slide rule takes its name from the fact that it consists of a grooved or channeled stick called the **staff**, **stock**, or **rule**, and a sliding tongued strip of wood called the **slide** or **tongue**.

The contiguous lower edges of slide and rule bear a pair of like logarithmic scales **C** and **D** respectively, which are identically the same as the *D* scale constructed in Ex. 174. By pulling the slide out to the right it is possible to add a logarithm on the *C* scale to a logarithm on the *D* scale. By pulling the slide to the left it is possible to subtract a logarithm on the *C* scale from a logarithm on the *D* scale. The upper contiguous edges bear two like logarithmic scales **A** and **B**, which are in reality two *C* or *D* scales placed end to end in the corresponding space of *C*. The extreme left-hand mark, the middle mark, and the extreme right-hand mark of any scale, are known respectively as the left index, (lin), middle index (min), and right index (rin).

Additional scales shown in Fig. 19 are placed on slide rules of different makes and their construction and use is described later.

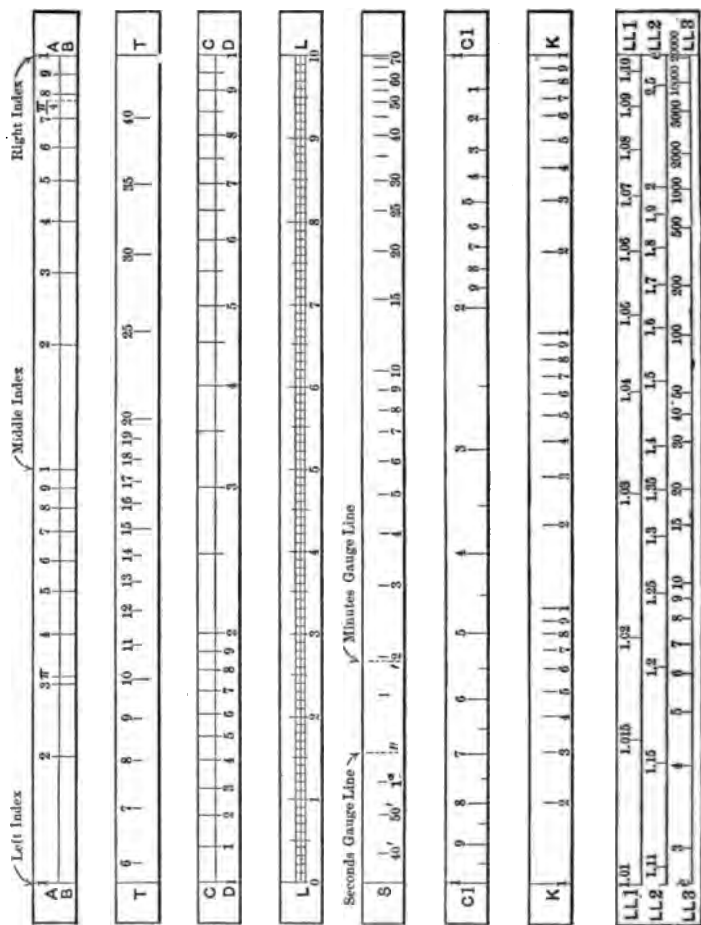


FIG. 19.—Slide Rule Scales

**Ex. 175.** Multiply 2.5 by 3.5. Opposite 2.5 on the *D* scale, set the (lin) left index, of *C*, and under 3.5 on *C*, read 8.75 on *D*. We have added the log of 3.5 to the log of 2.5. One factor was

located on **C**, and the other factor on **D**, while the product is read on **D**. No attention is given to the decimal point in locating a factor, as the scales represent mantissas, and any sequence of figures representing a factor would be located in the same position on the scale irrespective of its decimal value.

	DIAGRAMMATIC SETTING.	PROPORTIONATE SETTING.
<i>C</i> scale	lin lower edge of slide 35	<i>C</i> lin 3.5
<i>D</i> scale	25 contiguous edge of rule 875	<i>D</i> 2.5 = 8.75 or 1 = 3.5 / 2.5 = 8.75

The **diagrammatic setting** shows the position of the respective marks on the *C* and *D* scale for giving the product, and is interpreted as follows: set the left index (lin) of *C*, over the factor 2.5 on the *D* scale, under the factor 3.5 on the *C* scale, read the product 8.75 on the *D* scale. In other words  $2.5 \times 3.5 = 8.75$ .

The equation  $2.5 \times 3.5 = 8.75$  may be written as a **proportion**,  $\frac{1}{2.5} = \frac{3.5}{8.75}$ , which is the form given in the **proportionate setting**. If the **two numerators** of the proportion are read on the *C* scale, then the **two denominators** are read on the *D* scale.

*It will be observed that every number on the C scale bears the same constant ratio to the opposite number on the D scale for any given setting.*

The equation  $2.5 \times 3.5 = 8.75$  may be written as proportions in three other ways, each of which corresponds to a different setting of the rule as follows:  $\frac{C\ 2.5}{D\ lin} = \frac{8.75}{3.5}$ ;  $\frac{C\ lin}{D\ 3.5} = \frac{2.5}{8.75}$ ;  $\frac{C\ 3.5}{D\ lin} = \frac{8.75}{2.5}$ .

These forms follow from the theorem on proportions.

**Ex. 176.** Multiply  $3.5 \times 6.5$ . The diagrammatic setting is  $\frac{C\ 3.5}{D\ 22.75} = \frac{(lin)}{6.5}$ , or any one of its equivalent forms. State the other forms?

Instead of using the *C* and *D* scales the like operation may be performed with the use of the *A* and *B* scales. The latter differ from the former, in so far as they represent two scales like *C* and *D*, placed end to end. The graduations are therefore closer together, and therefore the *A* and *B* scales give less accurate readings than the *C* and *D* scales. The range of values on the

$A$  and  $B$  scales is 1 to 100, whereas the range of the  $C$  and  $D$  scales is 1 to 10. The settings for the  $A$  and  $B$  scales are:

DIAGRAMMATIC SETTING.				PROPORTIONATE SETTING.		
Left half of $A$	3.5	22.75	Right half of $A$	$\frac{A}{B}$	3.5	$\frac{22.75}{6.5}$
Left half of $B$	lin	6.5	Left half of $B$	$B$	lin	6.5

Show the other settings of the  $A$  and  $B$  scales for the same example.

**Ex. 177.** Make a complete outline of four different settings of the  $C$  and  $D$  scales for the following examples. State which index was used and the corresponding extensional direction of the slide.

- |                          |                          |
|--------------------------|--------------------------|
| (a) $3.62 \times 8.16$ , | (e) $4.62 \times 2.08$ , |
| (b) $11.2 \times 19$ ,   | (f) $1728 \times 16$ ,   |
| (c) $9.5 \times 1.1$ ,   | (g) $.163 \times 12$ ,   |
| (d) $1.1 \times 2.1$ ,   | (h) $.147 \times .133$ . |

In two of the four settings the product is read on the  $D$  scale, and they are called **direct settings**. In the other two settings the products are read on the  $C$  scale, and they are called **inverse settings**.

The position of the decimal point of a product may be determined by inspection, otherwise it may be determined by observing the position of the decimal point of each factor, also the extensional direction of the slide, and the direct or inverse setting. The **worth** of a number greater than 1 is equal to the number of digits to the left of its decimal point. The worth of a number less than 1 is negative and is equal to the number of zeros preceding its first significant figure.

NUMBER	WORTH
3596	4
823	3
11.2	2
9.5	1
.16	0
.013	-1
.0072	-2

For direct settings, the worth of a product is the **sum** of the worths of its factors, when the slide extends to the **left**. The worth of a product is **one less** than the **sum** of the worths of its factors, when the slide extends to the **right**.

For inverse settings, the worth of a product is the **sum** of the worths of its factors, when the slide extends to the **right** and **one less** than the **sum** of the worths of its factors, when the slide extends to the **left**.

**Ex. 178.** Since the log of  $2.5^2$  is twice the log of 2.5, we can use scales *A* and *D*, or *B* and *C*, to obtain the square of a number. Scales *A* and *B* have correspondingly twice the range of scales *C* and *D* by construction. Therefore we locate the number on the *D* scale and read its square directly opposite on the *A* scale, or we locate the number on the *C* scale and read its square directly opposite on the *B* scale. Opposite 2.5 on *C* or *D* is 6.25 on *B* or *A* respectively.  $6.25 = 2.5^2$

The slide rule is encircled by a **runner**, which is a sliding piece of glass mounted in a metal frame. The glass is scratched with a straight line and by adjusting the scratch, the graduations on two scales may be brought into alignment, i.e., into a straight line.

**Ex. 179.** Perform the following indicated operations:

- |                     |                                       |
|---------------------|---------------------------------------|
| (a) $1.45^2$ ,      | (g) $\sqrt{6.13}$ ,                   |
| (b) $32.2^2$ ,      | (h) $\sqrt{.3872}$ ,                  |
| (c) $6.13^2$ ,      | (i) $\sqrt{2}\sqrt{2}$ ,              |
| (d) $.3872^2$ ,     | (j) $2 \times 2^2 = 2^3 =$ ,          |
| (e) $\sqrt{1.45}$ , | (k) $2\sqrt{2} = 2^{\frac{3}{2}} =$ , |
| (f) $\sqrt{32.2}$ , | (l) $2^2 2^2 = 2^4 =$ .               |

Division is performed in the reverse manner to multiplication. In the use of the slide rule the principle of multiplication is to add the logarithms of the factors. The **sum of the logarithms** is the logarithm of the product.

The principle of division is to subtract the logarithm of the denominator from the logarithm of the numerator. The difference of the logarithms is the logarithm of the quotient. Thus to divide  $\frac{5.7}{6.8}$  subtract the log 6.8 from

the log 5.7. This may be performed by setting the divisor 6.8 on *C*, over the dividend 5.7 on *D*. The slide extends to the left, so that instead of reading the quotient under the left index, we observe that the right index occupies the same relative position from the end of the scale, therefore, under the right index of *C* read the quotient, .838 on *D*.

The setting is  $\frac{C\ 6.8}{D\ 5.7} = \frac{\text{rin } C}{.838}$ , which follows from proportion

$$\frac{6.8}{5.7} = \frac{1}{.838}.$$

This quotient may be located by three other settings obtained from the equivalent proportions,  $\frac{C\ 5.7}{D\ 6.8} = \frac{.838\ C}{1\ D}$ , or  $\frac{6.8\ C}{1\ D} = \frac{5.7}{.838}$ , or  $\frac{1\ C}{6.8\ D} = \frac{.838\ C}{5.7\ D}$ .

When the quotient is read on the *D* scale we have a direct setting, and when the quotient is read on the *C* scale we have an inverse setting.

**Ex. 180.** Make a complete statement of the settings of the *C* and *D* scales for the following examples, stating which index was used, and also the extensional direction of the slide.

(a)  $3.62 \div 8.16$ ,

(e)  $4.62 \div .208$ ,

(b)  $112 \div 19$ ,

(f)  $1728 \div 16$ ,

(c)  $9.5 \div 1.1$ ,

(g)  $.163 \div 12$ ,

(d)  $1.1 \div 2.1$ ,

(h)  $.147 \div .133$ .

For a direct setting, the worth of the quotient equals the **worth** of the dividend **minus** the **worth** of the divisor when the slide extends to the **left**. This difference is increased by **one** to give the worth of a quotient when the slide extends to the **right**.

The problem  $\frac{5.7}{6.8}$  may be written as a product  $5.7 \times \frac{1}{6.8}$ , and may be solved as a problem in multiplication by adding the log 5.7 to the log of the reciprocal of 6.8. *CI* is a reciprocal scale and when used with *D* the setting is  $\frac{\text{lin } CI}{D 5.7} \equiv \frac{6.8 C}{.838 D}$ .

The cube of a number as well as its cube root may be read directly when the slide rule contains a scale (**K**) of cubes. Scale *K* has three times the range of scales *C* and *D*, i.e., it represents three *C* scales placed end to end. Since the  $\log 3.5^3$  is three times the log of 3.5, we can use scales *K* and *D* to find cubes of numbers. Locate the number 3.5 on *D*, and by means of the scratch on the runner, we find 42.88 aligned on the *K* scale. To find  $\sqrt[3]{60}$  locate 60 on the *K* scale, and opposite on the *D* scale read 3.91.

When the slide rule does not contain a *K* scale, the cube of a number may be read with the *D*, *B*, and *A* scales. Since  $3.5^3 = 3.5 \times 3.5^2$ , then  $\log 3.5^3 = \log 3.5 + \log 3.5^2$ . The *D* scale gives the logs of numbers and the *B* scale gives the logs of squares of numbers. Therefore set the left index of *C* over 3.5 on *D*, and over 3.5 on *B*, read 42.88 on *A*, or  $\frac{C 1}{D 2.5} \equiv \frac{42.88 A}{3.5 B}$ . In other words, by setting the slide we have added the  $\log 3.5^2$  to the  $\log 3.5$ . The reverse operation will give the cube root of a number. Locate 27 on the *A* scale by setting the runner, then pull out the slide until the reading 3 on *D*, under the left index of *C*, is identical with the reading 3 on *B*, in alignment with the runner, or  $\frac{C 1}{D 3} \equiv \frac{27 A}{3 B}$ .

The alignment of scales *CI* and *D* sets all numbers on the one scale to their reciprocals on the other scale. Thus 5 on *D* is opposite .2 on *CI*; 8 on *CI* is opposite

.125 on *D*. The reciprocal of an integral number must be read as a decimal.

The alignment of scale *S* with *A* or *B* enables us to locate an angle on the *S* scale, and read its **sine** directly opposite on the *A* or *B* scale. Thus  $30^\circ$  on *S* is opposite its sine .5 on *A* or *B*.

The alignment of scale *T* with *C* or *D* enables us to locate an angle on the *T* scale, and read its **tangent** directly opposite on the *C* or *D* scale. Thus  $25^\circ$  on *T* is opposite its tangent .4663 on *C* or *D*.

The alignment of scales *D* and *L*, enables us to locate any number on the *D* scale, and read its **logarithm** directly opposite on the *L* scale. Thus 6 on *D* is opposite its logarithm .7782 on *L*.

A log log slide rule contains three additional scales, LL 1, LL 2, LL 3, which are segments of one continuous scale, but which are placed one above the other for convenience. The LL scale represents logarithms of logarithms or second logs, i.e., logs''.

The alignment of the LL scales with *C* or *D*, enables us to locate a number on LL and read its Napierian logarithm on *C* or *D*. Thus 10 on LL 3, is opposite its Napierian logarithm 2.303 on *C* or *D*. 1.25 on LL 2, is opposite its Napierian log .2232 on *C* or *D*. 1.04 on LL 1, is opposite its Napierian logarithm .0392 on *C* or *D*.

The LL scales with the *C* or *D* scales may be used to find any integral or decimal power of a number. Thus to obtain  $4^{1.6}$ , set the left index of *C* under 4 on LL 3, and over 1.6 on *C*, read the power 9.19 on LL 3, or  $\frac{4 \text{ on LL 3}}{1 \text{ on } C}$   $\equiv \frac{9.19 \text{ on LL 3}}{1.6 \text{ on } C}$ . To obtain  $\sqrt[3.24]{1.046}$ , set the runner to 1.046 on LL1, and 3.24 of *C* under the runner, over the left index of *C*, read the root 1.01397 on LL 1, or

$$\frac{1.01397 \text{ LL 1}}{\ln C} \equiv \frac{1.046 \text{ LL 1}}{3.24 C}.$$

The positions of  $\pi$  and  $\frac{\pi}{4}$  are usually indicated on a slide rule because of their constant use. Additional marks, called **gauge points**, are indicated for various other special computations. Two of these appear usually on the *S* scale as  $1'$ ,  $1''$ , and are known respectively as the minute and seconds gauge points. They are used in determining the sines and tangents of angles less than those indicated on the extreme left of the *S* and *T* scales. Sines and tangents of small angles are practically numerically equal.

In reading tangents of angles greater than  $45^\circ$  and less than  $90^\circ$ , locate the tangent of the complementary angle and use its reciprocal.  $\tan 60^\circ = \cot 30^\circ = \frac{1}{\tan 30^\circ} = \frac{1}{.575} =$

1.732. This work may be simplified by locating the complementary angle  $30^\circ$  on *T* and reading 1.732 directly opposite on *CI* scale.

In reading sines of angles between  $3.43'$  and  $34.3'$ , set the minute gauge point  $1'$  of *S* under the number of minutes on *A*, and read the sine on *A*, directly opposite the left index of *S*. This process is applicable to smaller decimals of an angle, but where the angle is given in seconds, its sine may be obtained by using the seconds gauge point  $1''$  on *S*. Set the gauge point  $1''$  of *S* under the number of seconds on *A*, and read the value of the sine on *A*, over the left index of *S*. Thus for  $\sin 25'$  we have  $\frac{.00728 \text{ on } A}{\text{lin on } S}$

$$= \frac{25 \text{ on } A}{1' \text{ on } S}; \text{ for } \sin 35'' \text{ we have } \frac{.0001699 \text{ on } A}{\text{lin on } S} = \frac{35 \text{ on } A}{1 \text{ on } S}.$$

$$\therefore \sin 25' = \underline{\underline{.00728}}; \quad \sin 35'' = \underline{\underline{.0001699}}.$$

In reading tangents of angles less than  $5^\circ 44.9'$  convert the angle measure into minutes or seconds according to the magnitude of the angle. Use the minutes or seconds gauge point, correspondingly adjusted to the *A* scale, in

exactly the same manner as for determining sines of angles less than  $34.3'$  as described in the preceding paragraph.

The worth of the sine or tangent of an angle is determined from the position of the angle in Table VI.  $\sin 90^\circ = 1.0000$ ;  $\tan 45^\circ = 1.0000$ . The sines of all angles and the tangents of angles less than  $45^\circ$  are decimals.

TABLE VI

Angles.	Worth of sin or tan.	sin and tan.
$5^\circ 44.9'$	0	.1
$34.3'$	-1	.01
$3.43'$	-2	.001
$.343' = 20.6''$	-3	.0001
$2.06''$	-4	.00001

**Ex. 181.** With the use of the runner, a **continued product** may be made quickly and accurately. Thus multiply  $8 \times 12 \times 7.5$ .

The product  $8 \times 12 = 96$  is represented in the setting  $\frac{\sin C}{D} \frac{C}{8} = \frac{12 C}{96 D}$ . Therefore  $8 \times 12 \times 7.5 = 96 \times 7.5$ ; the latter is represented by  $\frac{7.5 C}{720 D} = \frac{\sin C}{96 D}$ .

This double operation may be performed with one setting as follows:  $\frac{\sin C}{D} \frac{C}{8} = \text{runner to } 12 \dots \text{runner to } \frac{7.5 C}{720 D}$ , in other words, set left index of  $C$  over 8 on  $D$ , move runner to 12 on  $C$ , slide right index of  $C$  to runner, under 7.5 on  $C$  read the product 720 on  $D$ .  $\therefore 8 \times 12 \times 7.5 = \underline{\underline{720}}$ .

**Ex. 182.** Multiply  $9 \times 10.5 \times 8 \times 6.3$ . Verify the following setting, giving an analysis according to the successive products:

$$\frac{\sin C}{D} \frac{C}{9} = \text{runner to } 10.5 \dots 1 \text{ to runner} \dots \text{runner to } 8 \dots 1 \text{ to runner} = \frac{6.3 C}{4770 D}$$

in other words, set the left index of  $C$  over 9 on  $D$ ; move the runner to 10.5 on  $C$ ; set right index of  $C$  to runner; move runner to 8 on  $C$ ; set right index of  $C$  to runner; under 6.3 on  $C$  read 4770 on  $D$ .  $\therefore 9 \times 10.5 \times 8 \times 6.3 = \underline{\underline{4770}}$ .

**Ex. 183.** Simplify  $\frac{9.1 \times 8.2}{7.3 \times 3.5}$ . This may be analyzed by performing the operation as follows: divide 9.1 by 7.3; multiply the quotient by 8.2; divide the product by 3.5. The setting is

$$\frac{C 7.3}{D \text{ lin}} = \text{runner to 9.1} \dots 1 \text{ to runner} \dots \text{runner to 8.2} \dots 3.5 \text{ to runner} = \frac{\text{rin } C}{2.818 D},$$

in other words, set 7.3 on  $C$  over the left index of  $D$ ; move runner to 9.1 on  $C$ ; set right index of  $C$  to runner; move runner to 8.2; set 3.5 on  $C$  to runner; under the right index of  $C$  read 2.818 on  $D$ .

This operation may be performed by using the  $C$ ,  $CI$ , and  $D$  scales, necessitating but one adjustment of the runner as follows:

$$\frac{8.2 \text{ on } CI}{9.1 \text{ on } D} = \text{runner to lin of } CI \dots 7.3 \text{ on } C \text{ to runner} = \frac{3.5 \text{ on } CI}{2.818 \text{ on } D},$$

in other words, set 8.2 on  $CI$  over 9.1 on  $D$ ; move runner to left index of  $CI$ ; set 7.3 on  $C$  to runner; under 3.5 on  $CI$  read 2.818 on  $D$ .  $\therefore \frac{9.1 \times 8.2}{7.3 \times 3.5} = \underline{\underline{2.818}}$ .

**Ex. 184.** Multiply  $2.4 \times 6.3 \cos 60^\circ = 2.4 \times 6.3 \times \sin 30^\circ$ . The setting requires the  $A$ ,  $B$ , and  $S$  scales, as follows:

$$\frac{A 2.4}{\text{lin } B} = \text{runner to 6.3 on } B \dots \text{rin of } S \text{ to runner} = \frac{7.56 A}{30 S}.$$

**Ex. 185.** Determine  $\phi$  from the equation  $\tan \phi = \frac{5.7}{6.8}$ . Set 6.8 on  $C$  over 5.7 on  $D$ ; move runner to right index of  $C$ ; under runner read  $40^\circ$  on  $T$ .

The following Table VII shows the method of determining the worth of a continued product or the worth of a compound ratio.

TABLE VII

Examples.	Sum of Worths in		Difference of Worths.	Number Times Slide Moves		Worth of Answer.	Answer.
	Numerator.	Denominator.		to Left.	to Right.		
$2.5 \times 3.5$	2				1	1	8.75
$8 \times 12 \times 7.5$	4			(1)	1	3	720
$9 \times 10.5 \times 8 \times 6.3$	5			(2)	1	4	4770
$5.7 \div 6.8$	1	1	0	(1)		0	.838
$.163 \div 12$	0	2	-2		1	-1	.0133
$9.1 \times 8.2$ $7.3 \times 3.5$	2	2	0	(1 div.) (1 mult.)	1 div.	1	2.818

The numbers in ( ) do not affect the result.

**Ex. 186.** Use the slide rule to perform the following examples as a check on the preceding work:

- |                      |                  |
|----------------------|------------------|
| (a) Ex. 16,          | (l) Ex. 115-120, |
| (b) Ex. 31,          | (m) Ex. 126,     |
| (c) Ex. 33,          | (n) Ex. 152,     |
| (d) Ex. 80-82,       | (o) Ex. 154-156, |
| (e) Ex. of areas 85, | (p) Ex. 157,     |
| (f) Ex. 92-93,       | (q) Ex. 158-159, |
| (g) Ex. 97-101,      | (r) Ex. 160-161, |
| (h) Ex. 102,         | (s) Ex. 162-163, |
| (i) Ex. 103-105,     | (t) Ex. 164-165, |
| (j) Ex. 110-113,     | (u) Ex. 166,     |
| (k) Ex. 114,         | (v) Ex. 167-169. |

By means of the slide rule show that:

$$\left. \begin{aligned} (w) \quad & \frac{1}{2} \sin \theta \neq \sin \frac{\theta}{2} \\ & \frac{1}{3} \tan \theta \neq \tan \frac{\theta}{3} \\ & 2 \cos \theta \neq \cos 2\theta \end{aligned} \right\} \text{test these for ten values of } \theta.$$

$$(x) \quad \frac{1}{2} \log N \neq \log \frac{N}{2}. \quad \text{Test this for ten values of } N.$$

$$(y) \quad \log(a+b) \neq \log a + \log b; \text{ test this for ten values of } a \text{ and } b.$$

$$(z) \quad \sin(\theta + \phi) \neq \sin \theta + \sin \phi; \text{ test this for ten values of } \theta \text{ and } \phi.$$

*Observation.* A multiplier of a function must precede the function name and not the quantity. A divisor of a function must be written under the function name and not under the quantity.

A function name extends only to the term or parenthesis immediately following it. The function name cannot be distributed, i.e., written separately before the terms included in a parenthesis.

**Ex. 187.** Prepare a table giving the following forms, their numeric values and their logarithms.



TABLE VIII. TRIGONOMETRIC FUNCTIONS

Angle.		Chord.	X. F. Sine.	$\phi$ F. Tangent.	Cotan.	R. F. Cosine			
Degrees	Radians								
0°	0	0	0	0	$\infty$	1	1.414	1.5708	90°
1	.0175	.017	.0175	.0175	57.2900	.9998	1.402	1.5533	89
2	.0349	.035	.0349	.0349	28.6363	.9994	1.389	1.5359	88
3	.0524	.052	.0523	.0524	19.0811	.9986	1.377	1.5184	87
4	.0698	.070	.0698	.0699	14.3007	.9970	1.364	1.5010	86
5	.0873	.087	.0872	.0875	11.4301	.9962	1.351	1.4835	85
6	.1047	.105	.1045	.1051	9.5144	.9945	1.338	1.4661	84
7	.1222	.122	.1219	.1228	8.1443	.9925	1.325	1.4486	83
8	.1396	.140	.1392	.1405	7.1154	.9903	1.312	1.4312	82
9	.1571	.157	.1564	.1584	6.3138	.9877	1.299	1.4137	81
10	.1745	.174	.1736	.1763	5.6713	.9848	1.286	1.3963	80
11	.1920	.192	.1908	.1944	5.1446	.9816	1.272	1.3788	79
12	.2094	.209	.2079	.2126	4.7046	.9781	1.259	1.3614	78
13	.2269	.226	.2250	.2309	4.3315	.9744	1.245	1.3439	77
14	.2443	.244	.2419	.2493	4.0108	.9703	1.231	1.3265	76
15	.2618	.261	.2588	.2679	3.7321	.9659	1.218	1.3090	75
16	.2793	.278	.2756	.2867	3.4874	.9613	1.204	1.2915	74
17	.2967	.296	.2924	.3057	3.2709	.9563	1.190	1.2741	73
18	.3142	.313	.3090	.3249	3.0777	.9511	1.176	1.2566	72
19	.3316	.330	.3256	.3443	2.9042	.9455	1.161	1.2392	71
20	.3491	.347	.3420	.3640	2.7475	.9397	1.147	1.2217	70
21	.3665	.364	.3584	.3839	2.6051	.9336	1.133	1.2043	69
22	.3840	.382	.3746	.4040	2.4751	.9272	1.118	1.1868	68
23	.4014	.399	.3907	.4245	2.3559	.9205	1.104	1.1694	67
24	.4189	.416	.4067	.4452	2.2460	.9135	1.089	1.1519	66
25	.4363	.433	.4226	.4663	2.1445	.9063	1.075	1.1345	65
26	.4538	.450	.4384	.4877	2.0503	.8988	1.060	1.1170	64
27	.4712	.467	.4540	.5095	1.9626	.8910	1.045	1.0996	63
28	.4887	.484	.4695	.5317	1.8807	.8829	1.030	1.0821	62
29	.5061	.501	.4848	.5543	1.8040	.8746	1.015	1.0647	61
30	.5236	.518	.5000	.5774	1.7321	.8660	1.000	1.0472	60
31	.5411	.534	.5150	.6009	1.6613	.8572	.985	1.0297	59
32	.5585	.551	.5299	.6249	1.6003	.8480	.970	1.0123	58
33	.5760	.568	.5446	.6494	1.5399	.8387	.954	.9948	57
34	.5934	.585	.5592	.6745	1.4826	.8290	.939	.9774	56
35	.6109	.601	.5736	.7002	1.4281	.8192	.923	.9599	55
36	.6283	.618	.5878	.7265	1.3764	.8090	.908	.9425	54
37	.6458	.635	.6018	.7536	1.3270	.7986	.892	.9250	53
38	.6632	.651	.6157	.7813	1.2799	.7880	.877	.9076	52
39	.6807	.668	.6293	.8098	1.2349	.7771	.861	.8901	51
40	.6981	.684	.6428	.8391	1.1918	.7660	.845	.8727	50
41	.7156	.700	.6561	.8693	1.1504	.7547	.829	.8552	49
42	.7330	.717	.6691	.9004	1.1106	.7431	.813	.8378	48
43	.7505	.733	.6820	.9325	1.0724	.7314	.797	.8203	47
44	.7679	.749	.6947	.9657	1.0355	.7193	.781	.8029	46
45°	.7854	.765	.7071	1.0000	1.0000	.7071	.765	.7854	45°
			Cosine R. F.	Co-tan.	Tangent $\phi$ F.	Sine X. F.	Chord.	Radians	Degrees.
									Angle.

TABLE IX. LOGARITHMS

N	The Third Figure of Your Number.										Proportional Parts.									
	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9	
10	0000	0043	0086	0128	0170						4	9	13	17	21	26	30	34	38	
11	0414	0453	0492	0531	0569	0607	0645	0682	0719	0755	4	8	12	16	20	24	28	32	37	
12	0792	0828	0864	0899	0934	0969					4	8	12	15	19	23	27	31	35	
							1004	1038	1072	1106	4	7	11	15	19	22	26	30	33	
											3	7	11	14	18	21	25	28	32	
											3	7	10	14	17	20	24	27	31	
13	1139	1173	1206	1239	1271						3	7	10	13	16	20	23	26	30	
14	1461	1492	1523	1553		1303	1335	1367	1399	1430	3	7	10	12	16	19	22	25	29	
15	1761	1790	1818	1847	1875	1903					3	6	9	12	15	18	21	24	28	
							1584	1614	1644	1673	3	6	9	12	15	17	20	23	26	
											3	6	9	11	14	17	20	23	26	
							1931	1959	1987	2014	3	5	8	11	14	16	19	22	25	
16	2041	2068	2095	2122	2148						3	5	8	11	14	16	19	22	24	
17	2304	2330	2355	2380	2405	2430					3	5	8	10	13	15	18	21	23	
18	2553	2577	2601	2625	2648		2175	2201	2227	2253	3	5	8	10	13	15	18	20	23	
							2455	2480	2504	2529	2	5	7	10	12	15	17	19	22	
							2672	2695	2718	2742	2	5	7	9	11	14	16	19	21	
											2	5	7	9	11	14	16	18	21	
19	2788	2810	2833	2856	2878						2	4	7	9	11	13	16	18	20	
20	3010	3032	3054	3075	3096	3118	3139	3160	3181	3201	2	4	6	8	11	13	15	17	19	
21	3222	3243	3263	3284	3304	3324	3345	3365	3385	3404	2	4	6	8	10	12	14	16	18	
22	3424	3444	3464	3483	3502	3522	3541	3560	3579	3598	2	4	6	8	10	12	14	15	17	
23	3617	3636	3655	3674	3692	3711	3729	3747	3766	3784	2	4	6	7	9	11	13	15	17	
24	3802	3820	3838	3856	3874	3892	3909	3927	3945	3962	2	4	5	7	9	11	12	14	16	
25	3979	3997	4014	4031	4048	4065	4082	4099	4116	4133	2	3	5	7	9	10	12	14	15	
26	4150	4166	4183	4200	4216	4232	4249	4265	4281	4298	2	3	5	7	8	10	11	13	15	
27	4314	4330	4346	4362	4378	4393	4409	4425	4440	4456	2	3	5	6	8	9	11	13	14	
28	4472	4487	4502	4518	4533	4548	4564	4579	4594	4609	2	3	5	6	8	9	11	12	14	
29	4624	4639	4654	4669	4683	4698	4713	4728	4742	4757	1	3	4	6	7	9	10	12	13	
30	4771	4786	4800	4814	4829	4843	4857	4871	4886	4900	1	3	4	6	7	9	10	11	13	
31	4914	4928	4942	4955	4969	4983	4997	5011	5024	5038	1	3	4	6	7	8	10	11	12	
32	5051	5065	5079	5092	5105	5119	5132	5145	5159	5172	1	3	4	5	7	8	9	11	12	
33	5185	5198	5211	5224	5237	5250	5263	5276	5289	5302	1	3	4	5	6	8	9	10	12	
34	5315	5328	5340	5353	5366	5378	5391	5403	5416	5428	1	3	4	5	6	8	9	10	11	
35	5441	5453	5465	5478	5490	5502	5514	5527	5539	5551										
36	5563	5575	5587	5599	5611	5623	5635	5647	5658	5670	1	2	4	5	6	7	8	10	11	
37	5682	5694	5705	5717	5729	5740	5752	5763	5775	5786	1	2	3	5	6	7	8	9	10	
38	5798	5809	5821	5832	5843	5855	5866	5877	5888	5899										
39	5911	5922	5933	5944	5955	5966	5977	5988	5999	6010	1	2	3	4	5	7	8	9	10	
40	6021	6031	6042	6053	6064	6075	6085	6096	6107	6117	1	2	3	4	5	6	8	9	10	
41	6128	6138	6149	6160	6170	6180	6191	6201	6212	6222	1	2	3	4	5	6	7	8	9	
42	6232	6243	6253	6263	6274	6284	6294	6304	6314	6325										
43	6335	6345	6355	6365	6375	6385	6395	6405	6415	6425										
44	6435	6444	6454	6464	6474	6484	6493	6503	6513	6522										
45	6532	6542	6551	6561	6571	6580	6590	6599	6609	6618										
46	6628	6637	6646	6656	6665	6675	6684	6693	6702	6712	1	2	3	4	5	6	7	7	8	
47	6721	6730	6739	6749	6758	6767	6776	6785	6794	6803	1	2	3	4	5	5	6	7	8	
48	6812	6821	6830	6839	6848	6857	6866	6875	6884	6893	1	2	3	4	4	5	6	7	8	
49	6902	6911	6920	6928	6937	6946	6955	6964	6972	6981										
50	6990	6998	7007	7016	7024	7033	7042	7050	7059	7067	1	2	3	3	4	5	6	7	8	

## FUNDAMENTAL OPERATIONS

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TABLE IX. LOGARITHMS

N	The Third Figure of Your Number.										Proportional Parts.									
	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9	
51	7076	7084	7093	7101	7110	7118	7126	7135	7143	7152	1	2	3	3	4	5	6	7	8	
52	7160	7168	7177	7185	7193	7202	7210	7218	7226	7235	1	2	2	3	4	5	6	7	7	
53	7243	7251	7259	7267	7275	7284	7292	7300	7308	7316	1	2	2	3	4	5	6	6	7	
54	7324	7332	7340	7348	7356	7364	7372	7380	7388	7396	1	2	2	3	4	5	5	6	7	
55	7404	7412	7419	7427	7435	7443	7451	7459	7466	7474	1	2	2	3	4	5	5	6	7	
56	7482	7490	7497	7505	7513	7520	7528	7536	7543	7551										
57	7559	7566	7574	7582	7589	7597	7604	7612	7619	7627	1	1	2	3	4	4	5	6	7	
58	7634	7642	7649	7657	7664	7672	7679	7686	7694	7701	1	1	2	3	4	4	5	6	7	
59	7709	7716	7723	7731	7738	7745	7752	7760	7767	7774	1	1	2	3	4	4	5	6	6	
60	7782	7789	7796	7803	7810	7818	7825	7832	7839	7846										
61	7853	7860	7868	7875	7882	7889	7896	7903	7910	7917										
62	7924	7931	7938	7945	7952	7959	7966	7973	7980	7987	1	1	2	3	3	4	5	5	6	
63	7993	8000	8007	8014	8021	8028	8035	8041	8048	8055	1	1	2	3	3	4	5	5	6	
64	8062	8069	8075	8082	8089	8096	8102	8109	8116	8122										
65	8129	8136	8142	8149	8156	8162	8169	8176	8182	8189										
66	8195	8202	8209	8215	8222	8228	8235	8241	8248	8254										
67	8261	8267	8274	8280	8287	8293	8299	8306	8312	8319	1	1	2	3	3	4	4	5	6	
68	8325	8331	8338	8344	8351	8357	8363	8370	8376	8382	1	1	2	2	3	4	4	5	6	
69	8388	8395	8401	8407	8414	8420	8426	8432	8439	8445	1	1	2	2	3	4	4	5	6	
70	8451	8457	8463	8470	8476	8482	8488	8494	8500	8506										
71	8513	8519	8525	8531	8537	8543	8549	8555	8561	8567	1	1	2	2	3	4	4	5	5	
72	8573	8579	8585	8591	8597	8603	8609	8615	8621	8627										
73	8633	8639	8645	8651	8657	8663	8669	8675	8681	8686										
74	8692	8698	8704	8710	8716	8722	8727	8733	8739	8745										
75	8751	8756	8762	8768	8774	8779	8785	8791	8797	8802										
76	8808	8814	8820	8825	8831	8837	8842	8848	8854	8859	1	1	2	2	3	3	4	4	5	
77	8865	8871	8876	8882	8887	8893	8899	8904	8910	8915	1	1	2	2	3	3	4	4	5	
78	8921	8927	8932	8938	8943	8949	8954	8960	8965	8971										
79	8976	8982	8987	8993	8998	9004	9009	9015	9020	9025										
80	9031	9036	9042	9047	9053	9058	9063	9069	9074	9079										
81	9085	9090	9096	9101	9106	9112	9117	9122	9128	9133										
82	9138	9143	9149	9154	9159	9165	9170	9175	9180	9186										
83	9191	9196	9201	9206	9212	9217	9222	9227	9232	9238										
84	9243	9248	9253	9258	9263	9269	9274	9279	9284	9289										
85	9294	9299	9304	9309	9315	9320	9325	9330	9335	9340										
86	9345	9350	9355	9360	9365	9370	9375	9380	9385	9390	0	1	1	2	2	3	3	4	4	
87	9395	9400	9405	9410	9415	9420	9425	9430	9435	9440										
88	9445	9450	9455	9460	9465	9469	9474	9479	9484	9489										
89	9494	9499	9504	9509	9513	9518	9523	9528	9533	9538										
90	9542	9547	9552	9557	9562	9566	9571	9576	9581	9586										
91	9590	9595	9600	9605	9609	9614	9619	9624	9628	9633										
92	9638	9643	9647	9652	9657	9661	9666	9671	9675	9680										
93	9685	9689	9694	9699	9703	9708	9713	9717	9722	9727										
94	9731	9736	9741	9745	9750	9754	9759	9763	9768	9773										
95	9777	9782	9786	9791	9795	9800	9805	9809	9814	9818										
96	9823	9827	9832	9836	9841	9845	9850	9854	9859	9863										
97	9868	9872	9877	9881	9886	9890	9894	9899	9903	9908										
98	9912	9917	9921	9926	9930	9934	9939	9943	9948	9952										
99	9956	9961	9965	9969	9974	9978	9983	9987	9991	9996	0	1	1	2	2	3	3	3	4	

## CHAPTER II

### THE TRANSFORMATION OF FORMULAS

**The Formula.** A formula is a condensed form of the verbal statement of a law of science.

In 1827, G. S. Ohm annunciated his celebrated law: the intensity or strength of an electric current, i.e., the quantity of electricity passing a section of the conductor in a unit of time, is directly proportional to the whole electromotive force in operation, and inversely proportional to the sum of all the resistances in the circuit.

**Ohm's Law** may be abbreviated as follows:

$$\text{Intensity of current (amperes)} = \frac{\text{Electromotive force (volts)}}{\text{Resistance (ohms)}}.$$

In other words, the intensity of the current is equal to the quotient obtained by dividing the electromotive force by the resistance.

**The Notation.** A group of abbreviations with their meanings is called the notation.

NOTATION:

Let  $E$  = the whole electromotive force in operation,

$I$  = the intensity of the current,

$R$  = the sum of the resistances in the circuit.

By the use of the above notation the formula becomes:

$$I = \frac{E}{R}$$

**The Data.** The values which have been assigned to the notation of a formula are called the data. The data

should include the numeric values of all letters but one. That letter may be computed for its numeric value.

**DATA:**

$E = 0.05$  volts = 50 millivolts,

$I = 200$  amperes,

$R = ?$  (the value of  $R$  is to be determined.)

**The Transformation.** The transformation of a formula is the result of performing one or more operations upon a formula in order to obtain the literal solution of any of its letters.

Transformation for  $R$  mathematic authority

(1)  $I = \frac{E}{R}$  ——— quotation of Ohm's Law,

(2)  $RI = E$  ——— multiplying (1) by  $R$  (mul ax),

(3)  $R = \frac{E}{I}$  ——— dividing (2) by  $I$  (div ax).

**The Solution.** The data should include the numeric values of all letters but one. That one letter is said to be an unknown and the determination of its numeric value is called the solution of the formula.

The data may be substituted in (1), giving (4), which leads to (5), or by substituting the data in (3), which gives (5) directly.

$$(4) 200 = \frac{.05}{R},$$

$$(5) R = \frac{.05}{200},$$

simplifying (5) we obtain (6)

$$(6) R = .00025 \text{ ohm} = 250 \text{ microhms.}$$

**The Interpretation.** The notation enables us to make formula work a reversible process. A formula is inter-

preted as a verbal law by reading into the abbreviations their significance as expressed in the notation.

The same current passes through every cross-section of a circuit, so that if we consider any length of the circuit we can use Ohm's Law providing we know the difference of voltage between the ends of a conductor and the corresponding resistance. Therefore, for a segment or any definite length of a circuit or conductor we use the following notation:

$E$  = the difference of potential (P.D.), i.e., the voltage difference or so called voltage drop between two points in an electric circuit or conductor,

$R$  = the corresponding resistance between the two points,

$I$  = the intensity of the current flowing through the circuit or conductor.

Interpretation of formula (2):

(2)  $E = IR$  consists of three distinct elements—

$$[ ] = ( ) \{ \}.$$

[The voltage drop or potential difference between any two points of a conductor] equals the product of {the resistance of the conductor} times (the current flowing through it).

Interpretation of formula (3):

(3)  $R = \frac{E}{I}$  consists of three distinct elements.  $\{ \} = \frac{[ ]}{( )}$ .

{The resistance of a conductor} is equal to the ratio of [the voltage drop across the conductor] to (the current passing through it).

**The Analysis of a Formula.** The outline of the above example is called an analysis and represents the mental and written processes which should be applied to every problem. The following order of procedure suggests the

method of attacking a problem: (a) at the beginning enter the heading which is the title or subject of the formula; (b) under the heading write the formula or formulate the given law or problem; (c) on the right-hand side of the page state the notation, and the (d) data; under the formula enter a series of numbered statements representing the (e) transformations of the formula; (f) the numeric substitution of the data; (g) the simplification of the result; (h) the interpretation of the formula in its various transformations may follow.

*Observation.* The formulation of a law or problem is the focusing or concentration of facts by the use of abbreviations and symbols which also express the relations between things and the forces which act upon them, and thereby enable the mind to appreciate a law in its entirety.

Induced E.M.F. in a moving coil. The E.M.F. which is induced in a coil moving through a magnetic field is proportional to the rate of cutting the magnetic flux, i.e., the number of lines of force cut per second.

An equivalent statement would be: the electromotive force, in volts, generated in a moving coil is equal to the product of the total flux cut times the number of revolutions per second divided by the proportionality factor  $10^8$ .

Voltage generated in a coil:

NOTATION:

- |                                 |  |
|---------------------------------|--|
| (a) $E = \frac{\Phi N}{10^8}$ , | $E$ = E.M.F. generated in the coil,          |
| (b) $\Phi = AH$ ,               | $\Phi$ = total flux in the magnetic circuit  |
| (c) $A = ld$ ,                  | which is cut by the coil,                    |
|                                 | $N$ = the number of revolutions per second   |
|                                 | of the coil,                                 |
|                                 | $A$ = area of the coil in sq.cm.,            |
|                                 | $H$ = the strength of the field in gaussess, |
|                                 | per sq.cm.,                                  |
|                                 | $l$ = the length of the coil in cm.          |
|                                 | $d$ = the breadth of the coil in cm.         |

DATA:

$$E = ?$$

$$H = 10000 \text{ gausses,}$$

$$l = 60 \text{ cm.,}$$

$$N = 180 \text{ r.p.m.} = 3 \text{ r.p.s.,}$$

$$d = 30 \text{ cm.}$$

(a), (b), and (c) are **simultaneous equations**, i.e., equations in which the same letters have the same significance and the same numeric values.

The solution for  $E$  may be performed in one of two ways: substitute the value of  $A$  from (c) in (b) in order to obtain  $\Phi$ , then substitute the latter in (a), which gives the transformed value of  $E$  in terms  $H, N, l, d$ , the numeric values of which are then substituted; another method of proceeding is to substitute the data and calculated values in each of Eqs. (c), (b), and (a) in succession. In both cases we arrive at the same numeric value for  $E$ .

## FIRST METHOD.

$$(b) \Phi = AH,$$

$$(c) A = ld,$$

$$(1) \Phi = ldH,$$

$$(a) E = \frac{\Phi N}{10^8},$$

$$(2) E = \frac{ldHN}{10^8},$$

$$(3) E = \frac{60 \times 30 \times 10000 \times 3}{10^8},$$

$$(4) E = .54 \text{ volt.}$$

## SECOND METHOD.

$$(c) A = ld,$$

$$(1) A = 60 \times 30 = 1800,$$

$$(b) \Phi = AH,$$

$$(2) \Phi = 1800 \times 10000 = 18 \times 10^6,$$

$$(a) E = \frac{\Phi N}{10^8},$$

$$(3) E = \frac{18 \times 10^6 \times 3}{10^8},$$

$$(4) E = .54 \text{ volt.}$$

Interpretation of formula (b):

The total flux in a magnetic circuit equals the product of the area times the strength of the field.

Interpretation of formula (c):

The area of the coil equals the product of its length times its breadth.

By transforming (a) we obtain (5) and (6):

$$(5) \Phi = \frac{10^8 E}{N},$$

$$(6) N = \frac{10^8 E}{\Phi}.$$

Interpretation of (5):

The total flux required to generate a given E.M.F. equals  $10^8$  times the E.M.F. divided by the number of revolutions per second.

Interpretation of (6):

The number of revolutions per second required to generate a given E.M.F. equals  $10^8$  times the E.M.F. divided by the total flux.

**Transformation of Formulas.** The transformation of a formula implies the necessary steps through which it is modified to give the solution for its letters. Every step must be assured by definite mathematic authorities.

A test for the accuracy of transformation work is reversion to the initial formula.

The solution of a formula requires a letter isolated in one member of the equation, so as to have a coefficient and exponent of unity.

Transform the following formulas and express the solution for each letter. Show authorities and then present the work in the form which is in accordance with the standards set for this department.

**Ex. 1.**  $Q = It.$

Solve for  $I, t.$

**Ex. 2.**  $I = \frac{W}{K_2 t}.$

Solve for  $W, K_2, t.$

**Ex. 3.**  $I = \frac{V}{K_1 t}.$

Solve for  $V, K_1, t.$

**Ex. 4.**  $I = \frac{Vh \ 273}{0.1733 \times 76(273 + T)t}$ . Solve for  $V, h, t$ .

Divide (3) by (4).

Solve for  $K$  in terms of  $h$ .

**Ex. 5.**  $C = \frac{5}{9}(F - 32.)$  Solve for  $F$ .

**Ex. 6.**  $R = \frac{KL}{d^2}$ . Solve for  $K, L, d$ .

**Ex. 7.** Ohm = 1000000 microhms. Express microhms in ohms.

**Ex. 8.** Ohm =  $\frac{\text{megohm}}{1000000}$ . Express megohms in ohms.

**Ex. 9.**  $d^2 = CM$ . Solve for  $d$ .

**Ex. 10.** sq. mils =  $CM \ 0.7854$ . Solve for  $CM$ .

There are a few cases where two letters are used to abbreviate a quantity, as illustrated in (9) and (10). In such cases, these compounded letters are to be treated as inseparable.

## CHAPTER III

### INTERPRETATION OF FORMULAS

FOR the formula work of this chapter use the notation given below. Interpret the formulas and complete those statements of laws which are stated in part.

- $t$  = time in seconds which elapses,
- $Q$  = quantity of electricity passing a point in a circuit in coulombs,
- $I$  = rate of flow, or intensity of current in amperes,
- $W$  = weight in grams gained or deposited in a voltmeter,
- $K_2$  = weight in grams deposited by one amp. per sec.
- $V$  = volume in cu.cm. evolved in a gas voltameter,
- $K_1$  = volume in cu.cm. evolved in a gas voltameter by one amp. per sec.,
- $h$  = pressure of the atmosphere in cm.,
- $R$  = resistance of a conductor in ohms,
- $L$  = length of a conductor in feet,
- $d$  = diameter of a conductore in mils,
- $K$  = resistance of a mil-foot of wire,
- $CM$  = circular mils,
- $T$  = temperature of a room in degrees centigrade.

In many cases a quantity is abbreviated by its initial letter and in other cases where a conflict might arise by an arbitrarily chosen letter. Wherever possible the notation of convention is adopted in this text.  $I$  in this instance is the initial letter in the expression "intensity of current," because  $C$  is used as the abbreviation for capacity.

**Ex. 1.**  $t = \frac{Q}{I}$ , from (1)

using the notation

$$\text{time (seconds)} = \frac{\text{quantity (coulombs)}}{\text{current strength (amperes)}}$$

$\therefore$  The law to find the time (in seconds) required for a given quantity of electricity (in coulombs) to pass a point in a circuit, divide the quantity of electricity (in coulombs) by the rate of flow (in . . . ).

**Ex. 2.** Interpret  $Q = tI$  as follows:

Law for the quantity of electricity (in coulombs) to pass a point in a circuit: . . . ?

**Ex. 3.** Transform  $I = \frac{W}{K_1 t}$  for  $t$  and interpret as follows:

Law for the time required to deposit electrically a given weight of metal with a given current strength?

**Ex. 4.** Interpret  $I = \frac{V}{K_1 t}$  as follows:

Law to calculate strength of an unknown current when a volume of gas is evolved in a given time: . . . ?

**Ex. 5.** Transform the formula in Ex. 3 for  $W$  and interpret as follows:

Law for the weight of any metal that will be deposited in a voltameter by a given current in a given time: . . . ?

**Ex. 6.** Transform the formula in Ex. 4 for  $V$  and interpret as follows:

Law for the volume of a gas generated by a known current in a given time: . . . ?

## CHAPTER IV

### THE FORMULATION OF PROBLEMS

**The Ampere-hour.** The coulomb is a very small quantity of electricity. For commercial purposes a larger unit, the ampere-hour, is used.

One ampere-hour is the quantity of electricity that would pass any point in a circuit in one hour when the strength of current is one amp. An equivalent would be 2 amps. flowing for  $\frac{1}{2}$  hour.

**Ex. 1.** What is the equivalent of one amp.-hour (a) for a current flowing for 4 hours; (b)  $\frac{1}{4}$  ampere requires what equivalent time; (c) show that 3600 coulombs is the equivalent of an amp.-hour.

**Ex. 2.**  $\text{Amperes} = \frac{\text{ampere-hours}}{\text{hours}}$ . Interpret this formula as follows:

Law for ampere-hours consumed in an electric circuit: multiply the average strength of current (in ?) by the ?

**Ex. 3.** A current of 6.5 amp. was maintained by a cell for 5 hours. What quantity of electricity has been used? Express in coulombs and also in ampere-hours (amp.-hrs.).

**Ex. 4.** Suppose the cell in Ex. 3 has a capacity of 80 amp.-hrs., how long could the above current be maintained?

Use the formula in Ex. 2.

**Ex. 5.** How long a time will be required to deposit 5 grams of silver on a copper-plated teaspoon with a current of 2 amps.?  $K_1$  for silver = 0.001118 gram.

**Ex. 6.** The negative plate of a copper voltameter has increased its weight 1.9 grams in 35 seconds. What was the average rate of current strength?  $K_2$  for copper = 0.0003286.

**Ex. 7.** In an electroplating bath how many grams of zinc will be deposited by a current of 5 amps. in 60 min?  $K_3$  for zinc 0.0003386.

**Ex. 8.** The following data are recorded in a gas voltameter test:

Level of burette before test 2.7 cu.cm.  
 Level of burette after test, 32.2 cu.cm.  
 Vol. of gas evolved =  $32.2 - 2.7 = ?$   
 Time of closing switch 8 hr. 40 min. 15 sec.  
 Time of opening switch 8 hr. 45 min. 15 sec.  
 Length of run = ? min. = ? sec.?  
 Total gas generated per sec. = ?

One amp. in one sec. (one coulomb) generates 0.1733 cu.cm. gas per sec. Substitute in the formula:

$$\therefore I = \frac{V}{K_1 t}$$

**Ex. 9.** For more accurate determinations Formula (4) from Chapter II is used. In an experiment the volume of gas generated in a gas voltameter was found to be 25 cu.cm. in 65 sec., its temperature (taken as temperature of the room) was 20° centigrade, and the atmospheric pressure was equal to 76 cm. What was the current strength?

**Ex. 10.** A piece of sheet iron 6 ins. square is to be plated on both sides in a copper acid bath. What current strength is required?

Current density for Cu cyanides 5–10 amp. per sq.ft.

Allow 7.5 amps. per sq.ft.

Area of plate both sides = ? sq.ft.

$7 \times ? = ?$  amps.

**Ex. 11.** Give the source of information and show range of variation of values of the current strength used in practice to operate the following:

- Current required to operate a 110-volt 16 c.p. inc. lamp.
- Current required to operate an enclosed arc lamp 110-volt.
- Current required to operate an open-air arc lamp.
- Current required to operate a 2-motor trolley car full load.
- Current required to operate a fan motor.
- Current required to operate average electric bell.
- Current required to operate telephone circuit.
- Current required to operate telegraph circuit.
- Current required to operate 150-volt Weston voltmeter for full-scale deflection.
- Current required to operate electric welding.

Current required to operate search light.

Current required to operate electric locomotive.

Current required to operate wireless telegraph outfit.

Current required to operate railroad signals, marine and hotel annunciators.

12. How many amp.-hours will be recorded by a meter in which 150 amps. have passed for  $\frac{1}{4}$  hr.?

13. A 100-amp.-hour Edison cell is discharged through a magnet at a 3.5 amp. rate. How long will the cell last in this current through the magnet?

14. A meter records 700 amp.-hours. It was in circuit for 10 hours each day. What was the average rate of use?

15. How many coulombs have passed through an arc in  $\frac{1}{4}$  hr. if the current is 9.6 amps.?

16. What current strength is required to deposit 4 grams upon an iron spoon in 30 min.

17. A meter records 44000 coulombs in 8 hrs. What was the average strength of current?

18. How many grams of Cu will be deposited on an electrode used for a ship's hull in 10 hrs. if the average current strength is 30 amps.?

19. What quantity of electricity in coulombs passes in a circuit in which a current of 40 amps. flows for 56 secs.?

20. What quantity of electricity in coulombs passes in a circuit of 13 amps. flowing for 15 min.?

21. In 1 hour 36000 coulombs of electricity pass through a circuit. If the flow is uniform during that time, what is the strength of the current?

22. How long will it take 72000 coulombs of electricity to pass in a circuit in which the strength of current is 4 amps.?

## CHAPTER V

### VARIATION PROBLEMS APPLIED TO THE ELECTRIC CIRCUIT

THE most significant fact regarding Ohm's Law is that the resistance of a conductor is independent of the strength of the current flowing through it. The resistance of a conductor depends upon its shape, size, the materials of which it is made, and also upon its temperature and strain.

The resistance of a conductor varies directly as its length. This fact is symbolized in three different ways:

$$(1) R \propto L, \quad (2) \frac{R_1}{R_2} = \frac{L_1}{L_2} \quad (3) R = KL.$$

#### NOTATION:

$R$  = the resistance of a conductor,

$L$  = the length of a conductor,

$K_1$  = a proportionality factor.

**Ex. 1.** 2000 ft. of copper (Cu) wire 0.1 ins. in diameter (dia.) has 2 ohms resistance. What is the resistance of 5000 ft. of the same kind of wire?

$$(1) \frac{R_1}{R_2} = \frac{L_1}{L_2},$$

$$(2) \frac{R_1}{2} = \frac{5000}{2000},$$

$$(3) R_1 = 5,$$

DATA:

$$R_2 = 2\Omega,$$

$$L_2 = 2000 \text{ ft.},$$

$$R_1 = ?$$

$$L_1 = 5000 \text{ ft.}$$

**Ex. 2.** Use the data in Ex. 1 and determine: the resistances for the following amounts of the same kind of wire:

(a) 1000 ft.; (b) 10000 ft.; (c) 1 mile; (d) 50 miles.

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The resistance of a conductor varies inversely (reciprocally) as its cross-section area. This fact is symbolized by:

$$1) R \propto \frac{1}{A},$$

NOTATION:

$A$  = the area of the section,

$$2) \frac{R_1}{R_2} = \frac{A_2}{A_1},$$

$K_2$  = a proportionality factor,

$d$  = the dimensions of a round wire,

$$3) R = \frac{K_2}{A}.$$

$s$  = the side of a square wire.

In the case of square wires  $\frac{A_1}{A_2} = \frac{s_1^2}{s_2^2}$ . Why?  $\therefore$  (4)  $\frac{R_1}{R_2} = \frac{s_2^2}{s_1^2}$ .  
Why?

In the case of round wires  $\frac{A_1}{A_2} = \frac{d_1^2}{d_2^2}$ . Why?  $\therefore$  (5)  $\frac{R_1}{R_2} = \frac{d_2^2}{d_1^2}$ .  
Why?

In the case of conductors of similar cross-section  $\frac{A_1}{A_2} = \frac{?}{?}$ . Why?

$$\therefore R \propto \frac{1}{d^2} \text{ and } R \propto \frac{1}{s^2}. \text{ Why?}$$

**Ex. 3.** Two wires of circular cross-section have resistances in the ratio 1 : 2. What is the ratio of their diameters?

Substituting in (5) we obtain:

$$\frac{l_1^2}{l_2^2} = \frac{1}{2}. \therefore \left(\frac{d_2}{d_1}\right)^2 = \frac{1}{2} \text{ and } \frac{d_2}{d_1} = \sqrt{\frac{1}{2}} = \frac{1}{\sqrt{2}}. \therefore \frac{d_1}{d_2} = \frac{\sqrt{2}}{1}. \text{ Why?}$$

**Ex. 4.** What is the ratio of the diameters of round conductors whose resistances have the ratios: (a) 1 : 3 (b) 1 : 4, (c) 1 : 5, (d) 1 : 9, (e) 1 : 16?

**Ex. 5.** What is the ratio of resistances of round conductors whose diameters have the ratios: (a) 1 : 2, (b) 1 : 3, (c) 1 : 4, (d) 1 :  $\sqrt{2}$ , (e) 1 :  $\sqrt{3}$ ?

**Ex. 6.** Two conductors of square cross-section have resistances in the ratio of 1 : 2. What is the ratio of their dimensions?

Using formula (4) above we obtain

$$\frac{s_2^2}{s_1^2} = \frac{1}{2}. \therefore \frac{s_2}{s_1} = \frac{1}{\sqrt{2}} \text{ and } \frac{s_1}{s_2} = \frac{\sqrt{2}}{1}. \text{ Why?}$$

unit. If the side of the unit square is 1 inch then the unit of area is 1 sq. in. If the side of the unit square is 1 mil then the unit area is 1 sq. mil.

In wire measure the unit of area is a circle having a diameter of one mil and therefore the unit of area is one circular mil (CM). The ratio of any section or plane fig. to the unit CM gives the area in CM, whereas the ratio of any section or plane figure to the unit sq. mil gives the area in sq. mils.

**Ex. 12.** How many circular mils in a wire (a) 2 mils dia., (b) 5 mils dia., (c) 100 mils dia.?

Given the dia. of a wire, how are the circular mils computed?

**Ex. 13.** What is the cir. mil area of a wire  $\frac{1}{4}$  inch in diameter?

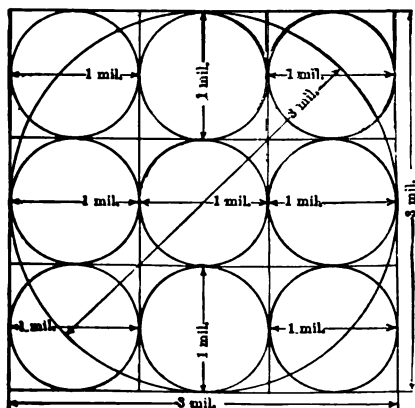


FIG. 20.

**Ex. 14.** Construct a square three mils on a side to some suitable scale. Divide the square into nine smaller squares each one mil on a side. Inscribe a circle in the large square as well as in each of the smaller squares. Show (a) that the sum of the areas of the smaller circles equals the area of the larger one; (b) the No. cir. mils in the larger circle equals 9 times the No. cir. mils in the smaller; (c) the No. of cir. mils in the cir. cross-sections equals the No. of sq. mils in the square cross-sections.

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**Ex. 15.** Show that the cir. mil area of round wires is nearly one-quarter (0.2146) larger than the area expressed in sq. mils.

**Ex. 16.** A wire of 6530 CM has what diameter? Verify by comparison with a wire table.

**Ex. 17.** A copper ribbon for a field coil measures  $\frac{3}{4} \times \frac{1}{4}$  ins. What is its sq. mil area?

**Ex. 18.** What is the equivalent cir. mil area for the cross-section in (17)?

**Ex. 19.** What is the sq. mil area of a wire  $\frac{1}{4}$  ins. dia.?

**Ex. 20.** A Cu wire has a cross-section area of 8234 CM and has a length of 1050 ft. What is its resistance?

**Ex. 21.** A spool is to be wound with No. 20 B. & S. German silver wire. What length is required to give 500Ω?

**Ex. 22.** 1000 ft. of iron wire has a resistance of 30 Ω. What is its CM area and its approximate size?

**Ex. 23.** 10 ft. of Cu wire has a res. = .013Ω. What is the gauge number and the resistance of 1 mile of the wire?

**Ex. 24.** A German silver wire of 11 ins. has 0.022 Ω resistance. What length is required to give 2.4Ω?

**Ex. 25.** 1 mile of wire has a resistance of 14.75Ω. What is its resistance per foot?

**Ex. 26.** The resistance of 18 ins. of wire = 0.027Ω. What is the resistance of 1020 ft.?

**Ex. 27.** The res. of a conductor is 0.32 ohm and sectional area = 0.025 sq.in. What will be the res. of a like conductor whose sectional area = 0.125 sq.in., other conditions being the same?

**Ex. 28.** The sectional area of a conductor is 0.01 sq.in. and its res. = 1 ohm. It is replaced by a wire 0.001 sq.in., other conditions being alike. What will be the res. of the latter?

**Ex. 29.** A round copper wire 0.12 in. diameter has a resistance 0.64 ohm, whereas a round copper wire 0.24 in. diameter has a resistance ? ohms, when other conditions are alike?

**Ex. 30.** The diameter of a round wire is 0.1 in. and its resistance = 2 ohms, whereas the diameter of a round wire ? in., and its resistance 50 ohms, when other conditions are alike?

**Ex. 31.** 1000 ft. of copper wire, diameter = 0.05 in., has a resistance = 4 ohms. 2500 ft. of copper ribbon 0.006 in. thick, 0.02 in. wide, has what resistance?

**Ex. 32.** The resistance of a mil-foot of copper wire is 10.8 ohms. Using the same quality of copper determine the resistance for the following wires and the nearest gauge number in the wire table:

**Ex. 7.** What is the ratio of dimensions of square conductors whose resistances have the ratios: (a) 1 : 3, (b) 1 : 4, (c) 1 : 5, (d) 1 : 9, (e) 1 : 16?

**Ex. 8.** What is the ratio of the resistances of square conductors whose dimensions have the ratios: (a) 1 : 2, (b) 1 : 3, (c) 1 : 4, (d) 1 :  $\sqrt{2}$ , (e) 1 :  $\sqrt{3}$ ?

**Ex. 9.** What is the ratio of the dimensions of two conductors one of circular and one of square cross-section, which have equal areas?

**Ex. 10.** Two conductors are equal in length and in cross-section. One is a round wire and the other a square wire. What is the ratio of their radiating surfaces?

*Observation. The resistance of any round conductor varies jointly, i.e., collectively or together as the length and inversely as the square of its diameter.*

$$R \propto \frac{L}{d^2},$$

from which we obtain:

$$(6) \quad R = \frac{KL}{d^2}.$$

#### NOTATION:

$R$  = the resistance of the conductor in  $\Omega$ ,  
 $L$  = the length of the conductor in ft.,  
 $d$  = the diameter of the conductor in mils,  
 $K$  = a proportionality factor.

The value of  $K$  may be determined from (6) by substituting  $L=1$  and  $d=1$  and then  $K=R$ , i.e.,  $K$  equals the resistance of 1 ft. of wire which is 1 mil in diameter and hence  $K$  is called the circular mil-foot constant or mil-foot resistance.

$$1 \text{ mil} = .001'' = \frac{1''}{1000}.$$

TABLE OF MIL-FOOT RESISTANCES AT 0°C

Silver.....	89.4	Iron.....	63.33
Copper.....	9.35	Ger. silver (30% nickel)...	290
Aluminum.....	17.21	German silver.....	128.29
Manganin.....	258.	Platinoid.....	188.93
Zinc.....	34.69	Mercury.....	586.24
Platinum.....	145.	Nickelin (40% nickel)....	290

**Ex. 11.** Determine the res. of 1000 ft. of Cu wire  $\frac{1}{16}$  inch dia. using  $K = 10.39$ .

$$\text{Resistance} = \frac{\text{res.mil.ft.} \times \text{length}}{(\text{diameter})^2} = \frac{KL}{(\text{mils})^2} = \frac{KL}{CM}.$$

$$(7) \quad R = \frac{KL}{d^2},$$

$$R = \frac{10.39 \times 1000}{100^2},$$

$$R = 1.039.$$

If we examine a wire table we note that the standard sizes of wires are numbered and the nearest size to a 100-mil wire is No. 10 B. & S., which has a diameter of 101.9 mils.

Some of the properties of the wire table are easily remembered by mnemonics of resistance. A mnemonic is any memory device which suggests the association of ideas or their relations.

**Mnemonics of Resistance.** The following approximations serve the electrical man:

1 ohm = 1000 ft. of Cu wire  $\frac{1}{16}$  inch dia. (No. 10 B. & S.)

1 ohm = 250 ft. of Cu wire  $\frac{1}{8}$  inch dia. (No. ?).

Examine a wire table and fill in the blanks.

1 ohm = 2 lbs. of Cu wire  $\frac{1}{16}$  inch dia. (No. ....?).

1000 ft. of Cu wire nearly  $\frac{1}{32}$  inch dia. (0000 B. & S.) = 0.05 ohms approximately.

1000 ft. of Cu wire  $\frac{3}{16}$  inch (No. 40 B. & S.) = 1063 ohms.

The area of any cross-section or plane figure is the ratio of the figure or section to any unit of area. The common unit of area is a square whose sides is one linear

unit. If the side of the unit square is 1 inch then the unit of area is 1 sq. in. If the side of the unit square is 1 mil then the unit area is 1 sq. mil.

In wire measure the unit of area is a circle having a diameter of one mil and therefore the unit of area is one circular mil (*CM*). The ratio of any section or plane fig. to the unit *CM* gives the area in *CM*, whereas the ratio of any section or plane figure to the unit sq. mil gives the area in sq. mils.

**Ex. 12.** How many circular mils in a wire (a) 2 mils dia., (b) 5 mils dia., (c) 100 mils dia.?

Given the dia. of a wire, how are the circular mils computed?

**Ex. 13.** What is the cir. mil area of a wire  $\frac{1}{4}$  inch in diameter?

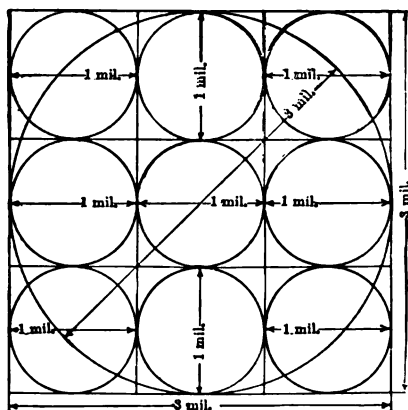


FIG. 20.

**Ex. 14.** Construct a square three mils on a side to some suitable scale. Divide the square into nine smaller squares each one mil on a side. Inscribe a circle in the large square as well as in each of the smaller squares. Show (a) that the sum of the areas of the smaller circles equals the area of the larger one; (b) the No. cir. mils in the larger circle equals 9 times the No. cir. mils in the smaller; (c) the No. of cir. mils in the cir. cross-sections equals the No. of sq. mils in the square cross-sections.

## PROBLEMS APPLIED TO THE ELECTRIC CIRCUIT 141

**Ex. 15.** Show that the cir. mil area of round wires is nearly one-quarter (0.2146) larger than the area expressed in sq. mils.

**Ex. 16.** A wire of 6530 *CM* has what diameter? Verify by comparison with a wire table.

**Ex. 17.** A copper ribbon for a field coil measures  $\frac{3}{8} \times \frac{1}{4}$  ins. What is its sq. mil area?

**Ex. 18.** What is the equivalent cir. mil area for the cross-section in (17)?

**Ex. 19.** What is the sq. mil area of a wire  $\frac{1}{4}$  ins. dia.?

**Ex. 20.** A Cu wire has a cross-section area of 8234 *CM* and has a length of 1050 ft. What is its resistance?

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**Ex. 24.** A German silver wire of 11 ins. has 0.022  $\Omega$  resistance. What length is required to give 2.4 $\Omega$ ?

**Ex. 25.** 1 mile of wire has a resistance of 14.75 $\Omega$ . What is its resistance per foot?

**Ex. 26.** The resistance of 18 ins. of wire = 0.027 $\Omega$ . What is the resistance of 1020 ft.?

**Ex. 27.** The res. of a conductor is 0.32 ohm and sectional area = 0.025 sq.in. What will be the res. of a like conductor whose sectional area = 0.125 sq.in., other conditions being the same?

**Ex. 28.** The sectional area of a conductor is 0.01 sq.in. and its res. = 1 ohm. It is replaced by a wire 0.001 sq.in., other conditions being alike. What will be the res. of the latter?

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**Ex. 30.** The diameter of a round wire is 0.1 in. and its resistance = 2 ohms, whereas the diameter of a round wire ? in., and its resistance 50 ohms, when other conditions are alike?

**Ex. 31.** 1000 ft. of copper wire, diameter = 0.05 in., has a resistance = 4 ohms. 2500 ft. of copper ribbon 0.006 in. thick, 0.02 in. wide, has what resistance?

**Ex. 32.** The resistance of a mil-foot of copper wire is 10.8 ohms. Using the same quality of copper determine the resistance of the following wires and the nearest gauge number in the wire table:

1200 ft. of wire 0.102 in. dia.

1 mile of wire  $\frac{1}{8}$  in. dia.

1500 ft. of square wire 0.1 on a side.

100 yds. of wire 0.12 wide by 0.09 thick.

#### VARIATION LENGTH, $CM$ AND WEIGHT

$$\text{lbs. per mile bare copper wire} = \frac{CM}{62.5'}$$

$$\text{lbs. per foot bare copper wire} = \frac{CM}{62.5 \times 5280'}$$

$$\text{lbs. per foot bare iron wire} = \frac{CM}{72.13 \times 5280'}$$

**Ex. 33.** How many lbs. of No. 10 Cu wire are required if strung 5 miles and return?

**Ex. 34.** Give the equivalent resistance in microhms of 0.00425 ohm.

1 microhm = .000001 $\Omega$ , 1 milliohm = .001 $\Omega$ , 1 megohm = 1000000 $\Omega$ .

Give the equivalent resistance in ohms of 375 microhms.

Give the equivalent resistance in mehgoms of 4560000 ohms.

Give the equivalent resistance in ohms of 62.5 megohms.

Give the equivalent resistance in milliohms of 5 megohms.

Use the abbreviated notation 1000000 =  $10^6$ .

**Ex. 35.** The insulation res. of a wire measures 16.75 megohms. What is its equivalent res. in ohms?

**Ex. 36.** What is the  $CM$  area of a wire  $\frac{3}{8}$  in. in diameter?

**Ex. 37.** The  $CM$  area of a wire is 5625. What is its dia.?

**Ex. 38.** An armature is wound with copper bars  $\frac{1}{8} \times \frac{3}{4}$  of an inch. What is their equivalent area in cir. mils.?

**Ex. 39.** The res. of a series coil of a dynamo is 0.0065 ohm. What is the equivalent in microhms?

**Ex. 40.** What is the sq. mil area of a No. 12 B. & S. copper wire?

**Ex. 41.** What is the res. of 5 lbs. of No. 18 B. & S. wire allowing 5 per cent for insulation?

**Ex. 42.** The coils of a rheostat are constructed of No. 8 iron wire and have a res. of 10 ohms. What length of wire was required?

**Ex. 43.** A rectangular conductor has a sq. mil area of 20616.75. What is its equivalent  $CM$  area?

**Ex. 44.** What size of B. & S. wire has an area equivalent to the wire in the preceding problem?

**Ex. 45.** Calculate the res. of 2000 ft. of No. 6 B. & S. copper wire.

**Ex. 46.** Construct from your own calculations by the use of formulas a wire gauge table for No. 12 B. & S. copper wire.

Give cir. mil area, sq. mil area, lbs. per mile, lbs. per ft., lbs. per ohm, ft. per lb., ft. per ohm, ohms per lb., and ohms per ft. A No. 12 B. & S. wire has a diameter = .08081 in.

**Ex. 47.** What is the area in  $CM$  of a wire having a diameter of 0.46 in.?

**Ex. 48.** Find the area of a copper rod having a diameter of  $\frac{3}{16}$  in.

**Ex. 49.** What is the diameter of a wire having a sectional area of 1021.5  $CM$ ?

**Ex. 50.** Brown & Sharpe or American Gauge. Show that a No. 7 wire has a sectional area equal to two No. 10 wires.

Show that four No. 13 wires have a sectional area equal to eight No. 16 wires.

**Ex. 51.** Show that by subtracting 3 from any gauge number we obtain the number of a wire having very nearly twice the sectional area. Compare Nos. 1, 2, . . . 10 with Nos. 10, 11 . . . 20.

**Ex. 52.** Show that the ratio between the resistance of any wire in the B. & S. gauge and that of the next higher number is that of  $1 : 1.26$ , i.e.,  $1 : \sqrt[4]{2}$ .

**Ex. 53.** How should the above statement read when the ratio is inverted?

**Ex. 54.** What is the res. of 1000 ft. No. 16 wire having given the res. of 1000 ft. No. 10 equals 1 ohm?

**Ex. 55.** The res. of a No. 12 B. & S. gauge copper wire is 8.37 ohms per mile. What is the res. of a No. 11 and also a No. 13 wire?

**Ex. 56.** The res. of a No. 00 copper conductor is 0.411 ohm per mile. What is the res. of a similar conductor of No. 3 gauge?

**Ex. 57.** It is a very convenient fact to remember that the diameter of a No. 10 wire in the B. & S. gauge is very close to  $\frac{1}{10}$  in. (0.10189) and that the res. of a No. 10 annealed copper wire per 1000 ft. is practically 1 ohm (0.9972). Determine the per cent of error for the two approximate values.

**Ex. 58.** Determine the sizes of wire for supplying 200 lamps at 110 volts with an allowable loss of 10 per cent in the mains, when two- and three- wire system are used. Calculate the difference in copper allowing in the 3-wire system a neutral wire of half the cross-section of the outer wires.

Make the necessary drawings to illustrate the problems, and calculate the percentage on the power delivered to the receiving circuit. Fig. 21.

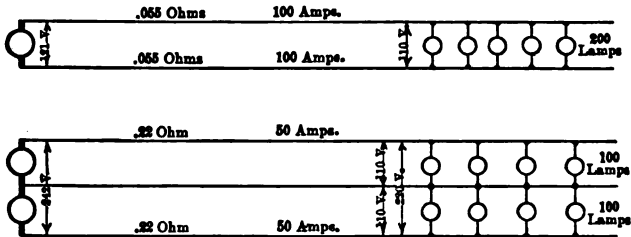


FIG. 21.

**Ex. 59.** The data are the same as in Ex. 58, except the loss in the mains is 5 per cent.

**Ex. 60.** An armature of an alternator has a diameter of 4.5 ft. and runs at 300 revolutions per minute. What is its peripheral speed?

## CHAPTER VI

### THE ALGEBRA OF A SIMPLE ELECTRIC CIRCUIT

The notation used in this chapter is given below.

- $I$  = current amperes; also  $i_1, i_2$ , etc., instantaneous values current,  
 $E$  = E.M.F. or P.D. or drop in volts; also  $e_1, e_2$ , etc., instantaneous values E.M.F.,  
 $R$  = resistance in  $\Omega$ ; also  $X, M, N, R_1, R_2, m, n, r_1, r_2, r_3, r_4$ ,  
 $E_R, E_X$  = drop in volts corresponding to res.  $R$  and  $X$  respectively,  
 $X$  = unknown resistance,  
 $R_T$  = res. at temp.  $T$ ,  
 $R_0$  = res. at temp. zero centigrade,  
 $K$  = res. of a mil-foot wire,  
 $L$  = length of wire,  
 $A$  = area of cross-section,  
 $K_1, K_2$  = constants of instruments,  
 $\alpha$  = angle of deflection,  
 $W$  = watts,  
 $s$  = No. cells in series,  
 $p$  = No. cells in parallel,  
H.P. = horse-power,  
K.W. = kilowatt,  
 $T$  = temperature in centigrade degrees.

The subscripts are often replaced by accents as shown in Ex. 8. In Ex. 11 *sta* and *mov* are subscripts which indicate the respective currents in the stationary and

movable coils. In formula work multiplication symbols are avoided so as not to conflict with  $X$  in the notation.

Transform and interpret the following equations for each letter and solve for the numeric values when data are given in the following:

**Ex. 1.** Measurement of resistance:

$$RI = E.$$

Solve for  $R$  and  $I$ .

**Ex. 2.** Drop of potential in a conductor:

$$I = \frac{E_R}{R} = \frac{E_X}{X}.$$

Solve for  $E_R$ ,  $R$ ,  $E_X$ ,  $X$ , giving all possible values.

**Ex. 3.** Influence of length, cross-section and material of a conductor on its resistance:

$$R = K \frac{L}{A}.$$

Solve for  $K$ ,  $L$ ,  $A$ .

**Ex. 4.** Influence of temperature on the resistance of conductors:

$$(1) \quad R_T = R_0(1 + 0.0042 T).$$

If the resistance  $R_0$  at  $0^\circ$  centigrade be taken as 100 per cent, what is the per cent increase of temperature per each degree centigrade?

For commercial Cu the formula is written

$$(2) \quad R_T = R_0(1 + 0.004 T).$$

Determine the  $R$  at  $25^\circ$  C. and  $15^\circ$  C. in terms of  $R_0$  and show the per cent error in using (2) instead of (1).

**Ex. 5.** In the preceding work eliminate  $R_0$  by dividing the equation for  $R_T$  by the equation for  $R_{15}$ .

Determine  $T$  given  $R_T = 5.468$   $R_{15} = 4.573$ .

**Ex. 6.** Resistances in parallel:

$$I_1 = \frac{E}{R_1}, \quad I_2 = \frac{E}{R_2}.$$

Solve for  $\frac{I_1}{I_2}$ .

## THE ALGEBRA OF A SIMPLE ELECTRIC CIRCUIT 147

The elements of a branch are marked with like subscripts.

**Ex. 7.** Wheatstone bridge. When no current flows through the galvanometer, i.e., a balance, then

$$(1) \quad Ri_1 = Mi_2 = E_1.$$

$$(2) \quad Xi_1 = Ni_2 = E_2.$$

Divide (1) by (2) and solve for  $X$ ,  
 $X$  is the arm of unknown resistance.

DATA:

$$X = ?, \quad R = 10 \text{ ohms,}$$

$$M = 27 \text{ ohms, } N = 73 \text{ ohms.}$$

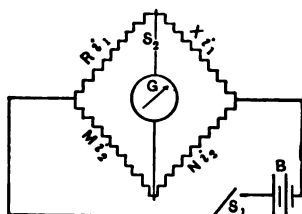


FIG. 22.

**Ex. 8.** Measurement of very small resistances by the Kelvin double bridge: Given

$$(1) \quad \frac{m}{n} = \frac{m'}{n'}$$

and

$$(2) \quad \frac{R}{X} = \frac{m}{n},$$

then

$$(3) \quad \frac{R}{X} = \frac{IR}{IX} = \frac{m}{n} = \frac{mi}{ni} = \frac{m'}{n'} = \frac{m'i'}{n'i'},$$

by the  $Az$  of fractions and  $= tyAz$ .

But we have also given

$$(4) \quad i'm' = IR + im,$$

and

$$(5) \quad i'n' = IX + in;$$

show that

$$\frac{R}{X} = \frac{i'm' - im}{i'n' - in}.$$

**Ex. 9.** Soft iron instrument:

$$\text{Pull} = IK_1,$$

$$\text{Pull} = K_1 I^2.$$

Solve for  $K_1$ ,  $K_2$ ,  $I$ , eliminating pull.

**Ex. 10.** Ammeter shunt:

$$I = \frac{E}{R},$$

DATA:

$$E = 0.050 \text{ volt},$$

$$R = 0.001 \text{ ohm},$$

$$I = ?.$$

**Ex. 11.** Electro-dynamometer instrument:

$$(a) \text{ torque} = K_1 i_{sta} i_{mov},$$

$$(b) \quad i_{sta} = i_{mov},$$

$$(c) \text{ torque} = K_1 \text{ times } ?,$$

$$(d) \text{ torque} = K_2 \alpha,$$

Prove

$$(e) \quad i = \sqrt{\frac{K_2}{K_1} \alpha},$$

$$(f) \quad i = K_3 \sqrt{\alpha}.$$

What is the relation between  $K_1$ ,  $K_2$ ,  $K_3$ , if Eq. (f) is made to replace (e)?

*Observation. The ratio or product of two constants is another constant. The root or power of a constant is another constant.*

**Ex. 12.** Standard cell:

$$E = 1.0186 - 0.000038(T - 20^\circ) - 0.00000065(T - 20^\circ)^2.$$

Compute  $E$  for the following temperatures:

(c)  $20^\circ$ , (d)  $25^\circ$ , (e)  $30^\circ$ .

Ans. (a)  $15^\circ$ .

## THE ALGEBRA OF A SIMPLE ELECTRIC CIRCUIT 149

**Ex. 13.** Indicating wattmeters:

Watts = amperes  $\times$  volts,

$$W = IE.$$

Solve for  $I$  and  $E$  and state the formula as a law.

**Ex. 14.** Compensation for power consumption in the watt-meter:

$$i^2 R = \frac{E^2}{R}.$$

Solve for  $i$ ,  $R$ ,  $E$ .

**Ex. 15.** Circuit with series resistance:

$$I = \frac{E}{R + r_1 + r_2}.$$

Solve for  $E$ ,  $R$ ,  $r_1$ ,  $r_2$ ,  $r_1 + r_2$ .

**Ex. 16.** Resistances in parallel:

$$\frac{1}{R} = \frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} + \frac{1}{r_4}.$$

Solve for  $R$ ,  $r_1$ ,  $r_2$ ,  $r_3$ ,  $r_4$ .

Interpret this formula, naming the reciprocal of a resistance as conductivity. It will be noticed that in any of the transformations of this example the small  $r$ 's are homogeneous, i.e., interchangeable.

**Ex. 17.** Cells in series:

$$I = \frac{Es}{rs + R}.$$

Notation:

$r$  = resistance of a cell,

$R$  = external resistance.

Solve for  $E$ ,  $s$ ,  $r$ ,  $R$ .

**Ex. 18.** Cells in parallel:

$$I = \frac{E}{\frac{r}{p} + R}.$$

$p$ ,  $R$ .

**Ex. 19.** Cells in mixed combination:

$$I = \frac{Es}{\frac{rs}{p} + R}.$$

Solve for  $E$ ,  $s$ ,  $r$ ,  $p$ ,  $R$ .

**Ex. 20.** Formulas for wattage:

$$(a) \quad I = \sqrt{\frac{W}{R}}.$$

In (a) sub. for ( $I$ ) from (1) and solve for  $W$  and  $E$ .

**Ex. 21.** Mechanical horse-power of electric circuit.

$$(a) \quad \text{H.P.} = \frac{I^2 R}{746}.$$

In (a) sub. for  $I$  from (1) and solve for  $E$  and  $R$ .

In (a) sub. for  $R$  from (1) and solve for  $E$  and  $I$ .

$$E = \frac{W}{I} + IR$$

## CHAPTER VII

### THE APPLICATIONS OF OHM'S LAW

THE current flowing through an electric circuit is subject to change arising from a variation in the value of its resistance or in the value of the impressed E.M.F.

Ohm's law contains the three elements  $I$ ,  $E$ , and  $R$ . Write its three forms. Consider  $R$  a constant, i.e., a fixed resistance; (a) how can  $E$  and  $I$  be made to change; (b) what is the effect upon either  $E$  or  $I$  if one of them increases; (c) if one of them decreases; (d) if  $E$  vanishes what is the effect upon the current; (e) if  $I$  is very excessive; what is the corresponding effect upon the value of  $E$ ; (f) does a direct or indirect variation exist between  $E$  and  $I$ ? (g) write the variation, using the variation symbol; (h) what is  $R$  in such a variation?

In the three forms of Ohm's law substitute the value of  $R$  in terms of length, cross-section, and temperature.

Restate all the possibilities for changing the current in a circuit when the impressed E.M.F. remains constant but when the elements of resistance are made to change.

Determine the numeric values in the following examples:

**Ex. 1.**

$$I = \frac{E}{R}.$$

DATA:

Incandescent lamp.

$E = 110$  volts,

$R = 200$  ohms,

$I = ?$ .

**Ex. 2.**

$$I = \frac{E}{R}.$$

DATA:

Arc lamp.

$E = 50$ ,

$R = ?$ ,

$I = 7$ .

**Ex. 3.**

$$I = \frac{E}{R}$$

**DATA:**

Absolute units.

$$E = 10',$$

$$R = 10',$$

$$I = ?.$$

**Ex. 4.**

$$I = \frac{E}{R+r}$$

**DATA:**

Battery circuit.

$$E = ?,$$

 $R = 60$  external resistance, $r = 3$  internal resistance.

$$I = 2 \text{ amperes.}$$

**Ex. 5.** An incandescent lamp has a hot res. 220 ohms and is connected to an electric light main across which 110 volts potential difference is maintained.

**Ex. 6.** A circuit has a resistance of 50 ohms and a pressure of 110 volts. What is the strength of the current in amperes?

**Ex. 7.** The pressure in a conductor is 4 volts and the resistance 15 ohms. How many amperes is flowing?

**Ex. 8.** What current can be made to flow through a circuit having a resistance of 10 ohms, if an E.M.F. of 110 volts is applied?

A **resistor** is any coil, conductor, or mechanical device whose **resistance** is used specifically for controlling the strength of a current of electricity.

**Ex. 9.** A resistor is connected to a 550-volt circuit. What resistance must it have in order that a current of .5 ampere may flow through it?

By Ohm's law  $R = \frac{E}{I}$  and after substituting the values of  $E$  and  $I$  from the data, we obtain:

**DATA:**

$$R = \frac{550}{.5} = 1100\Omega.$$

$$E = 550 \text{ volts,}$$

$$I = .5 \text{ amp.}$$

**Ex. 10.** What is the resistance of a resistor in order that a current of 50 amperes may flow through it if it is connected to 500-volt mains?

**Drop of Potential.** The E.M.F. between two points of a circuit is variously designated as a potential difference (P.D.), or pressure difference in volts or millivolts, and more commonly as the **drop**, meaning the fall or difference of voltage.

**Ex. 11.** Fig. 23 represents a part of a circuit, *A, B, C, D, E* being definite points thereon. A current of 2.5 amperes flows through the conductor. The drops between the two points designated are given as follows:

$AB = 12.5$  volts,  $BC = 6.25$  volts,  $CD = 18$  volts,  $DE = .025$  volt.

The resistances corresponding to these drops are computed by substituting the data in Ohm's law:  $R = \frac{E}{I}$ . Therefore  $AB = 5\Omega$ ,  $BC = 2.5\Omega$ ,  $CD = 7.2\Omega$ ,  $DE = .01\Omega$ .

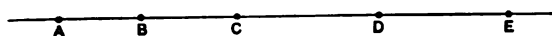


FIG. 23.

**Ex. 12.** It is desired to transmit 10 amperes to supply power 1000 ft. from the source at 110 volts. What size conductor would be required for a 2 per cent loss in the transmission?

$110 \times .02 = 2.2$  volts drop in the line,

$\frac{2.2}{2} = 1.1$  volts drop in each main,

$R = \frac{E}{I} = \frac{1.1}{10} = .11\Omega$ , resistance of each main,

$R = \frac{KL}{d^2}$ .

$\therefore d^2 = \frac{KL}{R} = \frac{10.8 \times 1000}{.11}$ .

$d^2 = 98182$  CM in the cross-section.

From the wire table we find No. 0 wire is required.

**Ex. 13.** The millivolt drop across 3 ft. of rail including a bond is 20 and the corresponding drop for 3 ft. of continuous rail is

15. Taking these as average values what is the estimated drop per mile of track (using 30 ft. rails)?

**Ex. 14.** In a closed circuit the drop caused by the resistance of the conductor is 10 volts. If the current flowing is 4 amps. what is the resistance of that part of the circuit?

**Ex. 15.** The E.M.F. generated in a circuit is 220 volts. The current is 10 amps. The leads have a drop of 10 per cent. What is their resistance, (a) when per cent is figured on generator E.M.F., (b) when per cent is figured on E.M.F. delivered to receiving circuit?

**Ex. 16.** How much pressure will it take to force a current of 18 amps. through a resistor of 5 ohms?

**Ex. 17.** What voltage is required to send a current of 25 amps. through a resistor of 4 ohms?

**Ex. 18.** The total resistance of a closed circuit is 49.3 ohms. If the current is 2.73 amperes, what is the total E.M.F. in volts?

**Ex. 19.** A difference of potential of 110 volts exists between the terminals of a conductor whose resistance is 20 ohms. What is the current flowing through the conductor?

**Ex. 20.** A circuit has an available pressure of 220 volts. What is its resistance if a current of 50 amps. can flow through it?

**Ex. 21.** The two electrodes of a simple voltaic cell are connected together by a copper wire having 1 ohm resistance. If the internal resistance of the cell is 2.4 ohms and E.M.F. 2 volts, what is the strength of current in the circuit?

**Ex. 22.** The total E.M.F. developed in a circuit is 1.2 volts and the strength of the current is 0.3 amp. What is the total resistance of the circuit?

**Ex. 23.** A battery of 10 cells each having an E.M.F. 2 volts and internal resistance  $2.4\Omega$  causes a current of .25 amp. in a circuit. What is the external resistance of the circuit?

**Ex. 24.** Given a battery circuit in which the E.M.F. = 30 volts, current = .6 amp., internal resistance = 40 ohms, what is the external resistance?

**Ex. 25.** The total E.M.F. of a voltaic battery is 22 volts and its total internal resistance is 11 ohms. What is the external resistance of the circuit if 0.5 amp. is flowing through it?

**Ex. 26.** A telegraph circuit has two sounders of 20 ohms each in series, the line resistance is 20 ohms. The line is operated by two sets of 10 cells each. Each cell has an E.M.F. of 1.5 volts and a resistance of 3 ohms. What current flows through the circuit (a) when the two sets of cells are in series, (b) when the two sets of cells are in parallel?

Fig. 24 is a sketch of the connections showing a double throw switch which permits the series arrangement in position *a* and the parallel arrangement in position *b*.

$R$  = The total battery resistance + the resistances of the sounders and line,

$E$  = the total battery E.M.F.

$I$  = current.

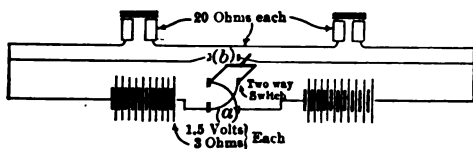


FIG. 24.

Arrangement (a):

$$R = 20 \times 3 + 40 + 20 = 120\Omega,$$

$$E = 20 \times 1.5 = 30 \text{ volts},$$

$$I = \frac{E}{R} = \frac{30}{120} = .25 \text{ amps},$$

arrangement (b):

$$R = \frac{10 \times 3}{2} + 40 + 20 = 75,$$

$$E = 10 \times 1.5 = 15 \text{ volts},$$

$$I = \frac{E}{R} = \frac{15}{75} = .2 \text{ amps}.$$

**Ex. 27.** How many electric heaters of  $10\Omega$  each can be connected in series to a trolley pressure of 550 volts so as to draw 5 amperes?

$N$  = number of heaters,

$R = 10 N$  = total resistance,

$$R = \frac{E}{I} = 10 N = \frac{550}{5} = 11.$$

$\therefore N = 11$ , the number of heaters connected in series.

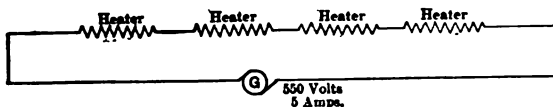


FIG. 25.

**Ex. 28.** The electric heaters on a street car draw 5 amperes at a trolley pressure of 550 volts. There are four heaters in series, Fig. 25. What is the resistance of each heater? How much power does it consume?

**Ex. 29.** The hot resistance of the field coil, Fig. 26, of a shunt dynamo is 10 ohms. How much current is it drawing when the machine is generating 220 volts?

$$I = \frac{E}{R} = \frac{220}{10}.$$

$I = 22$  amperes passing through the field coil.

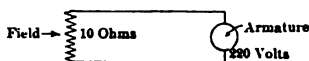


FIG. 26.

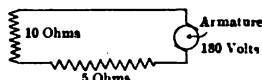


FIG. 27.

**Ex. 30.** When a 5-ohm resistor, Fig. 27, is inserted in series with the field coil of the above machine, the armature pressure drops to 180 volts. What current flows through the field coil?

**Ex. 31.** The open-circuit E.M.F. of a railway storage battery, Fig. 28, is 580 volts. When 500 amps. are drawn from it the E.M.F. falls to 540 volts. What is the resistance of the battery?

$$E_2 \text{ (open circuit)} = 580,$$

$$E_1 \text{ (closed circuit)} = 540,$$

$$E = E_2 - E_1 = 40 = \text{volts drop in the battery}$$

$$R = \frac{E}{I} = \frac{40}{500}.$$

$$R = .08\Omega \text{ resistance of the battery.}$$

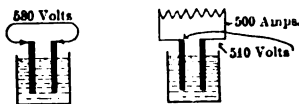


FIG. 28.

The **power** consumed in an electric circuit is the product of its voltage and its amperage measured simultaneously.

$$\text{watts} = \text{volts} \times \text{amperes}, \quad P = EI.$$

**Ex. 32.** A 550-volt trolley circuit is supplying 100 amperes to a trolley car, Fig., 29. The line and track resistance is  $.5\Omega$ .

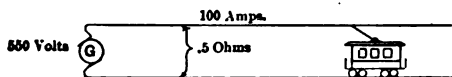


FIG. 29.

What is the pressure available at the car and what power does it consume?

$$E = IR = 100 \times .5,$$

$$E = 50 \text{ volts drop in the line and track,}$$

$$E_1 = 550 - 50 = 500 \text{ volts available at the car,}$$

$$P = E_1 I = 500 \times 100,$$

$$P = 50 \text{ K.W., the power used by the car.}$$

**Ex. 33.** If an electric motor is delivering 5 horse-power at an efficiency of 80 per cent, what current is it drawing from a 220-volt line?

**Ex. 34.** A residence contains 25 incandescent lamps, Fig. 30, which operate on an average of 20 hours per month. Each

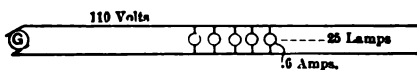


FIG. 30.

lamp takes 0.6 amp. at 110 volts. What is the monthly bill at 11 cents per K.W. hour?

$$I = 25 \times .6 = \text{total current used,}$$

$$P = EI = 110 \times 25 \times .6 = 1650 \text{ watts} = 1.65 \text{ K.W.}$$

$$\text{K.W. hours} = 1.65 \times 20 = 33,$$

$$33 \times .11 = \$3.63, \text{ monthly bill.}$$



**Ex. 38.** A resistance frame, i.e., a subdivided resistor, Fig. is to be made of coils of wire in series of sufficient size to carry amps. How must the resistance be divided to provide taps at voltages of 0.5, 1, 2, 3, 5, 10, 20, 30, 50, 100, 150 volts?

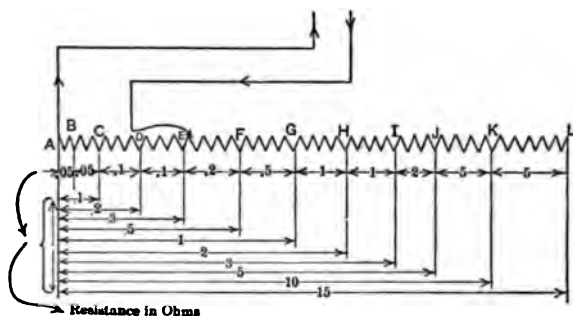


FIG. 33.

When a number of resistors are connected in parallel reciprocal of their combined resistances equals the of the reciprocals of the individual resistances,

$$\frac{1}{R} = \frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} + \dots$$

**Ex. 39.** A current of .6 ampere is supplied to two resistors,  $r_1$  and  $r_2$ , in parallel, Fig. 34. What current flows through each resistor when  $r_1 = 2\Omega$ , and  $r_2 = 3\Omega$ ?

$$\frac{1}{R} = \frac{1}{r_1} + \frac{1}{r_2}$$

**Ex. 40.** What is the joint resistance of  $r_1 = 4\Omega$  and  $r_2 = 6\Omega$  joined in parallel?



FIG. 34.

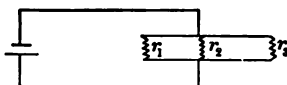


FIG. 35.

**Ex. 41.** What is the joint resistance, Fig. 35. of three parallel resistors when  $r_1 = 5\Omega$ ,  $r_2 = 10\Omega$ , and  $r_3 = 20\Omega$ ?

**Ex. 42.** The sum of the currents in the three branches  $r_1$ ,  $r_2$ ,  $r_3$ , of a divided circuit is 52 amps.  $r_1=4$ ,  $r_2=6$ ,  $r_3=8$  ohms. How much current flows through each branch?

**Ex. 43.** Four resistors  $A$ ,  $B$ ,  $C$ ,  $D$ , are connected in parallel. The resistances are 4, 5, 8, 10 ohms respectively. If the current

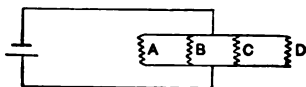


FIG. 36.

through  $A$  alone is 40 amps., how much current will flow through each of the other resistors?

**Ex. 44.** Two resistors,  $A$  and  $B$ , are in parallel and joined to a resistor  $C$  in series, Fig. 37.  $A=40$ ,  $B=60$ ,  $C=80$  ohms.

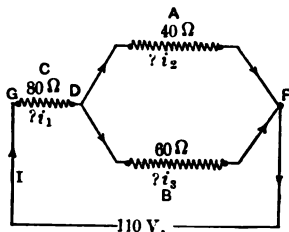


FIG. 37.

A voltage of 110 is connected to the extremities of the combination. What current flows through each resistor?

**Ex. 45.** A generator, Fig. 38, supplies 5.96 amperes to a multiple-series circuit. The brush voltage is 110 volts and the

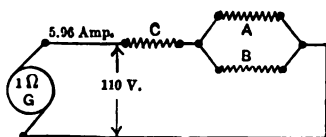


FIG. 38.

resistance of the generator is  $1\Omega$ . What voltage will be shown at the brushes on open circuit?

**Ex. 46.** Two copper wires 1000 ft. each are joined in parallel, Fig. 39. One is a round wire 0.02 in. diameter, the other a square wire 0.02 in. on a side. How will a current of 3 amps. distribute itself in these wires?

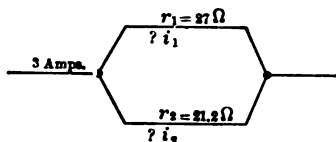


FIG. 39.

**Es. 47.** Determine the power consumed in a circuit, Fig. 40, having  $17.5\Omega$  resistance when an E.M.F. of 110 volts is applied to it?

**Ex. 48.** Compute the K.W. consumed in a circuit, having a resistance  $= 2\Omega$  when 110 volts is applied to it?

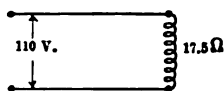


FIG. 40.

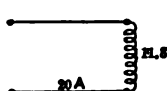


FIG. 41.

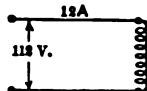


FIG. 42.

**Ex. 49.** Compute the kilowatts consumed in a circuit, Fig. 41, having a resistance  $= 11.8\Omega$  and a current  $= 20$  amps.

**Ex. 50.** Compute the kilowatts consumed in a circuit, Fig. 42, having an E.M.F.  $= 112$  volts and a current  $= 12$  amps.

**Ex. 51.** Galvanometer shunts are resistors which are adjusted to  $\frac{1}{9}$ ,  $\frac{1}{99}$ , and  $\frac{1}{999}$  of the resistance of the galvanometer.

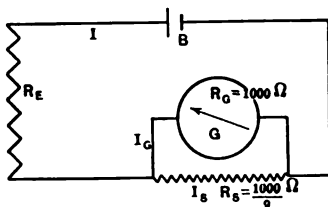


FIG. 43.

Write a formula for the current flowing through a shunted galvanometer circuit containing an external resistance. Designate

the resistance of galvanometer, shunt, external resistance by  $R_G$ ,  $R_S$ , and  $R_E$ , respectively.

Calculate the currents in the main circuit, in the shunt, and in the galvanometer branches when the battery E.M.F. = 2 volts,  $R_G = 100\Omega$ ,  $R_E = 5\Omega$ .

**Ex. 52.** What is the rate of doing work in watts when a current of 50 amps. flows against a resistance of 2 ohms?

The **power** expended in a circuit is directly proportional to the square of the voltage impressed upon the circuit and inversely proportional to the resistance of the circuit.

**Ex. 53.** How many watts are expended in a resistor in overcoming its resistance of 220 ohms when a pressure of 110 volts is applied to the terminals of the resistor?

**Ex. 54.** How many watts are equivalent to 10 horse-power?

**Ex. 55.** How many K.W. are equivalent to 5000 horse-power?

**Ex. 56.** The power in a circuit is equivalent to 5 horse-power. What current will flow to correspond to a 110 volt pressure?

**Ex. 57.** Two car motors consume 75 amperes at 550 volts. What is the equivalent horse-power?

**Ex. 58.** A resistor of 2 ohms is placed in series with two arc lamps so that a current of 9.6 amperes flows when connected with a 110-volt circuit. What is the drop across each lamp?

In making resistors of small resistance it is convenient to cut a piece of material approximately near but slightly in excess of the required value and then to adjust a shunted wire around it.

**Ex. 59.** A piece of wire of  $.102\Omega$  is to be adjusted to make a resistor of  $.1\Omega$ . (a) What resistance of wire is required to shunt it? (b) Suppose an error of 2 per cent is made in the adjustment of the shunt, how will it affect the desired result?

**Ex. 60.** What power in kilowatts will be taken by a 10-horse power motor having an 85 per cent efficiency? At a pressure of 220 volts what amperage is required?

The **power** expended in overcoming the resistance of lamps, resistors, machines, leads, and other devices in a circuit is proportional to their resistance and to the square of the current passing through them.

Consider the proportionality factor equal to one and write the formula.

**Ex. 61.** 1000 kilowatts is to be transmitted 19 miles. We can transmit this at a low pressure and large current or by means of a small current and at a high pressure.

(a) Allow 10 per cent of total power for loss at 1000 volts. Construct a table of values, of current, voltage, and resistance, for transmission at 1000, 5000, 10,000, 50,000, and 100,000 volts.

**Conductance.** The conductance of a wire is the reciprocal of its specific resistance.

**Specific Resistance.** The specific resistance of a material is the resistance of a unit cube of the material. In the C.G.S. system the unit is a centimeter. In the English system the specific resistance is the resistance of a foot of a material having a diameter of one mil.

**Ex. 62.** If the specific resistance of a wire be taken as 0.02 in practice instead of 0.017, what error is made if we call the conductance 60?

**Ex. 63.** A sample of aluminum wire  $\frac{1}{4}$  in. in diameter under test shows a pressure drop of 19.2 millivolts in a length of 3 ft. with a current of 100 amps. What is its specific resistance in C.G.S. units and what is the resistance of a mil-foot in ohms?

**Temperature Coefficient.** The temperature coefficient of a material is the increase in resistance for a temperature rise of one degree centigrade. The increase is expressed in terms of resistance at zero degrees centigrade:

$R_1$  is the resistance at temperature  $T_1$  centigrade;  
 $R_2$  is the resistance at temperature  $T_2$  centigrade;  
 $R_0$  is the resistance at temperature zero centigrade;  
 $C$  is the temperature coefficient;

$$(a) \quad R_1 = R_0(1 + CT_1),$$

$$(b) \quad R_2 = R_0(1 + CT_2).$$

Solve (a) and (b) simultaneously for  $C$  by eliminating  $R_0$ .

**Ex. 64.** The field coils of a street railway motor have a resistance of 0.25 ohm at a temperature of 70° F. After operating for some time the resistance is again measured and is found to be 0.31 ohm. What has been the average rise of temperature of the coils?

**Ex. 65.** The resistance of the filament of an incandescent lamp at a temperature of 20° C. is 50.1 ohms. When operating under normal conditions at a temperature of 2000° a pressure of 110 volts sends a current of 0.37 ampere through the filament. What is the average temperature coefficient of the filament? (This is a tantalum lamp.)

**Ex. 66.** A track is built of 90-lb. rails (90 lbs. per yard). The rails are 30 ft. long, and the lengths are "bonded" together with 30-in. lengths of round copper bonds. Assuming the conductivity of steel to be one-twelfth that of copper, what should be the cross-section of the bonds to give a bond resistance equal to 1.25 times a ft. of rail resistance? (Steel rails weigh 0.280 lb. per cubic inch.)

**Ex. 67.** If the resistance of a mil-foot of copper is 10.8 ohms, what will be that of a wire 10 miles long and 0.25 in. diameter?

**Ex. 68. Variation of Resistance of an Incandescent Lamp with Voltage Applied to it.**

In Figs. 44 and 45,  $L$  = lamp,  $A$  = ammeter,  $V$  = voltmeter,  $R$  = variable resistance,  $s$  = switch.

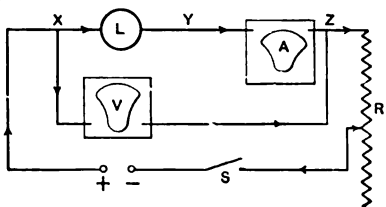


FIG. 44.

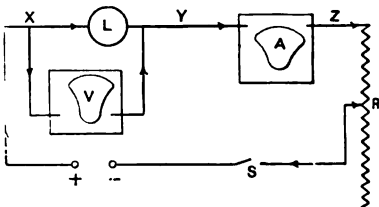


FIG. 45.

Derive the formulas for determining the resistance of  $L$  in both cases and allow for the error due to the use of instruments.

**Ex. 69.** An electric heater constructed of No. 10 iron wire radiates sufficient heat when the wire carries 10 amperes. The heater is placed across 110 volts. What is the value of the hot resistance?

**Ex. 70. Temperature Coefficient.**

$R_1 = \frac{R_2}{1 + a(T_2 - T_1)}$  where  $a = 0.004$  for copper, when temp. is taken centigrade degrees. An annealed copper conductor has a resistance of 15 ohms at a temperature of  $20^\circ \text{C}$ . What will the resistance be at  $50^\circ$  and also at  $8^\circ \text{C}$ ?

**Ex. 71.** The observed resistance of a copper wire is 12.74 ohms at  $85^\circ \text{F}$ . What is the resistance at  $65^\circ \text{F}$ ? The factor  $a = 0.0023$  for temperature  $\text{F}$ .

The unit of **electric energy** is the work done in one second when a current of one ampere flows under a pressure of one volt. This amount of energy is called a **joule** or **watt-second** ( $J$ ).

1 H.P. hour = 1,980,000 ft.-lbs.;

1 K.W. hour = 3,600,000 watt-seconds or joules;

1 joule = .74 ft.-lb.

**Ex. 72.** (a)  $J = I^2 R t$ . Solve for  $IR$ ,  $t$ .

(b) Substitute  $E$  for  $IR$  and solve for  $E$ ,  $I$ ,  $t$ ,  $J$ .

(c) Substitute  $\frac{E^2}{R^2}$  for  $I^2$  and solve for  $J$ ,  $E$ ,  $R$ ,  $t$ .

(d) Substitute  $\frac{Q^2}{t^2}$  for  $I^2$  and solve for  $Q$ ,  $t$ ,  $R$ ,  $J$ .

Interpret each formula.

**Ex. 73.** What is the amount of work done in joules when a current of 15 amps. flows for  $1\frac{1}{2}$  hours against a resistance of 2 ohms?

**Ex. 74.** Required, the amount of work done in joules in 1 hour by a current of 2.5 amperes under an E.M.F. of 20 volts?

**Ex. 75.** What is the amount of work done in 45 minutes in a circuit having 20 ohms resistance, the E.M.F. being 110 volts?

**Ex. 76. Electric Power.**

$$J = IEt, \text{ joules} = \text{amp.} \times \text{volts} \times \text{seconds.}$$

Divide both members by time and call  $\frac{\text{joules}}{\text{seconds}} = \text{watts.}$

Therefore  $\text{watts} = \text{amps.} \times \text{volts.}$

$$W = IE.$$

Interpret each formula.

**Ex. 77.** What power in watts is consumed in a circuit in which 0.65 amp. flows under a pressure of 110 volts?

**Ex. 78.** In Ex. 76 substitute  $E=IR$  and express the value of  $W$ .

Interpret the resulting formula.

**Ex. 79.** Determine the power expended in watts in an electric circuit having a resistance of 175 ohms, through which a current of 6 amps. is flowing.

**Ex. 80.** In Ex. 76 substitute  $I=\frac{E}{R}$  and express the value of  $W$ .

Interpret the resulting formula.

**Ex. 81.** What is the equivalent of 1 ft.-lb. in terms of  $J$ ?

**Ex. 82.** Express in ft.-lbs. the work accomplished in a circuit where a current of 10 amps. flows for 1 hour at 10 volts.

**Ex. 83.** What is the amount of work done in ft.-lbs. by a current of 5 amps. flowing for 20 seconds against a resistance of 4 ohms?

**Ex. 84.** What is the equivalent in ft.-lbs. of 1,356,000,000 ergs. 1 joule = 10,000,000 ergs.

**British Thermal Unit (B.T.U.)** is the quantity of heat which will raise the temperature of one pound of water one degree F. at or near its temperature of maximum density  $39.1^{\circ}$ .

**Ex. 85. Relation between Joules and British Thermal Units.**

1 joule = .0009477 B.T.U. Solve for one B.T.U. and interpret the resulting formula.

**Ex. 86.** Given a circuit having a resistance of 2 ohms through which a current of 10 amps. flows for 1 hour. Determine (a) the work done in joules; (b) the equivalent in ft.-lbs.; (c) the number B.T.U. developed.

**Ex. 87.** How many B.T.U. are developed in an electric circuit having a resistance of 220 ohms through which a current of 0.5 amp. flows for 1 min.?

**Ex. 88.** An electric circuit has a resistance of 5 ohms in which 3 amps. flows for 12 seconds. Determine (a) the work in joules, the equivalent ft.-lbs., and B.T.U.

**Pound Calorie** is the quantity of heat that will raise the temperature of one pound of water  $1^{\circ}$  C.

A **small calorie** ( $H$ ) is the heat required to raise one gram of water from  $0^{\circ}$  to  $1^{\circ}$  C.

**Ex. 89.** Calorie:

$$H = .24I^2Rt = ? \text{ joules.}$$

Solve for  $J$  and interpret.

**Ex. 90.** An insulated wire having a resistance of 10 ohms is entirely immersed in water. How many calories will be expended in heating the water when a current of 10 amps. flows for 1 hour?

**Ex. 91.** One calorie equals how many B.T.U.?

**Ex. 92.** Watt Hours.

Express 1 watt-second in joules, calories, ft.-lbs., B.T.U.?

**Ex. 93.** What is the power in watts in a closed circuit in which electric energy is expended in 1 hour to be equivalent to 33000 ft.-lbs.?

**Ex. 94.** A power station supplies 500 amperes for 10 hours to a factory. The drop in the line is 25 volts. How much energy in joules is lost and what is its equivalent in kilowatt hours? What power is used? What is the efficiency of the transmission?

**Ex. 95.** What is the equivalent in ft.-lbs. spent in 60 minutes for a lamp rated at 55 watts?

**Ex. 96.** The electric energy expended in a circuit in 2 hours is equivalent to 5,000,000 ft.-lbs. The E.M.F. of the circuit is 110 volts, what is the current?

**Ex. 97.** An incandescent lamp of 210 ohms is to be used on a 110-volt circuit. Determine (a) the current required for the lamp; (b) the watts consumed; (c) how many B.T.U. developed per second; (d) how many such lamps could be kept burning by one electric horse power; (e) what is the mechanical equivalent of the heat developed per second in the lamp; (f) for how many lamps would 10 K.W. suffice; (g) how many K.W. are required to operate 15 such lamps?

**Efficiency of Transformation and Transmission.** Efficiency is a ratio between the useful work performed by a machine and the energy put into it, in other words, it is the ratio of output to input.

$$\text{Efficiency} = \frac{\text{output}}{\text{input}}.$$

When this ratio is multiplied by 100 the result is expressed as per cent efficiency.

The efficiency of a motor =  $\frac{\text{electrical output}}{\text{electrical output} + \text{losses}}$ .

The efficiency of a motor =  $\frac{\text{mechanical output}}{\text{mechanical output} + \text{losses}}$ .

The efficiency of a motor =  $\frac{\text{electrical input} - \text{losses}}{\text{electrical input}}$ .

**Ex. 98.** Write the formulas for efficiency, using the following notation:  $W$  = output,  $w$  = losses,  $\eta$  = efficiency.

**Ex. 99.** The resistance of a dynamo armature, Fig. 46, is 0.016 ohm, that of the external circuit 0.757 ohm. The power

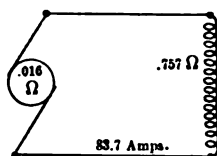


FIG. 46.

required to operate the machine is 7.604 H.P. and the current produced is 83.7 amps. What is the efficiency of the machine?

**Ex. 100.** A circuit, Fig. 47, consists of 100 inc. lamps arranged in 20 groups, each group containing 5 lamps in series. The volts between the main wires at the center of the lamp system is 550

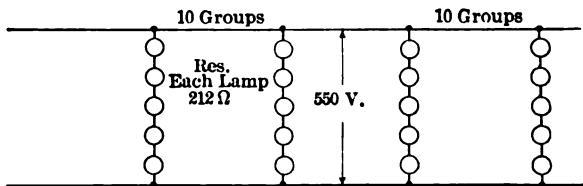


FIG. 47.

and the average resistance of each lamp is 212Ω. (a) Determine the watts used per lamp; (b) allow a line resistance of 8 ohms. What is the efficiency of the transmission?

**Ex. 101.** What is the commercial efficiency of a dynamo when a dynamometer shows that it absorbs 8 horse-power of mechanical energy while furnishing 92 incandescent lamps with 46 amps. at 115 volts?

**Ex. 102.** The resistance of an armature is 0.02 ohm and the int fields have a resistance of 25 ohms. The generator, absorbs horse-power and delivers 50 amps. to the line while the brush M.F. is 118 volts. Calculate the various losses. (Sec Fig. 48A.)

**Ex. 103.** Calculate the efficiency of a long-distance line when amps. at 3600 volts are supplied to it. The resistance of the e is 10.8 ohms.

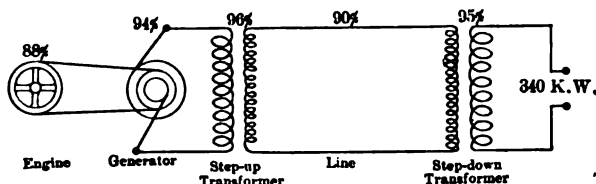


FIG. 48.

**Ex. 104.** A plant, Fig. 48, consists of a generator of 94 per cent efficiency; step-up transformers (lower to higher) 96 per cent; line 90 per cent; lowering or step-down transformers 95 per cent. The engine that drives the dynamo has an efficiency 88 per cent. If 340 K.W. of energy is at the secondary of the p-down transformer, calculate the plant efficiency and horse-power supplied to the engine. What is the station efficiency?

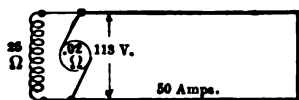


FIG. 48A.

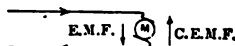


FIG. 49.

**Ex. 105.** What efficiency will be necessary in a stationary tor, Fig. 49, in order that it may develop 450 volts counter M.F. when the applied E.M.F. is 500 volts?

DATA:

Applied E.M.F. = 500 volts,

Counter E.M.F. = 450 volts.

$$\text{Efficiency} = \frac{\text{output}}{\text{input}} = \frac{\text{counter E.M.F.}}{\text{applied E.M.F.}}$$

$$\text{Per cent efficiency} = \text{efficiency} \times 100.$$

Complete the problem.

**Ex. 106.** 50 lamps, Fig. 50, are to be supplied at 110 volts, the hot resistance of each being 220 ohms. The line loss is 5 per cent; the commercial efficiency of the generator is 92 per cent and the engine and belt losses 15 per cent. What H.P. will be necessary to operate the plant?

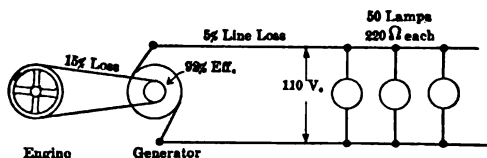


FIG. 50.

**Ex. 107.** How much power is required to furnish 40 arc lamps, Fig. 51, in series, with 7 amps. of current? The resistance of each lamp is 8 ohms and the line 25 ohms. How many watts are necessary per lamp? Assuming a normal candle power of 1200 for each lamp, what is the efficiency in watts per candle?

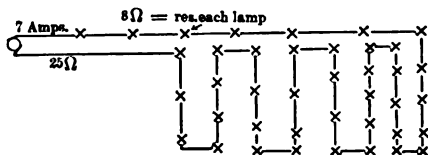


FIG. 51.

**Ex. 108.** An engine on full load indicates 125 horse-power. Assuming engine and belt losses to be 12 per cent and the commercial efficiency of dynamo 85 per cent, line loss 5 per cent. How many 110-volt incandescent lamps constitute the load? What is the voltage at the brushes?

**Net Work in an Electric Circuit. Kirchhoff's First Law.** The sum of the currents flowing to any point is equal to the sum of the currents flowing away from that point.

**Ex. 109.** Construct a formula which shall express the current in the following circuit, Fig. 52, using the following notation.

$E$  = volts at the terminals of a shunt generator, , ,  
 $R_f$  = shunt field resistance,  
 $R_a$  = armature resistance,  
 $R$  = resistance of lamps in the circuit,  
 $I$  = main current,  
 $i_a$  = armature current,  
 $i_f$  = field current.

Determine  $I$ ,  $i_a$  and  $i_f$ , when  $E = 220$  volts,  $R_f = 50\Omega$ ,  $R_a = .5\Omega$ ?

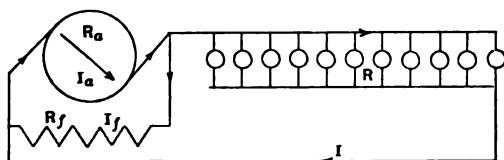


FIG. 52.

**Kirchhoff's Second Law.** In every closed circuit the sum of the products of current and resistance taken round the whole circuit equals the sum of the E.M.F. of the circuit. A back or counter E.M.F. is negative in sign.

**Ex. 110.**

$E$  = E.M.F. of dynamo,  $R_a$  = armature resistance,  
 $E_b$  = E.M.F. of battery,  $R_l$  = resistance of leads,  
 $I$  = current through battery,  $R_b$  = battery resistance.

In Fig. 53 a generator,  $E = 116$  volts, is connected to charge a battery of 50 accumulators having an E.M.F. of 2 volts each.

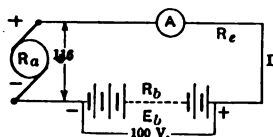


FIG. 53.

The generator resistance  $R_a = 0.1$  ohm, battery resistance  $R_b = 0.18$  and resistance of leads  $R_l = 0.12$ .

Explain the formula  $IR_a + IR_b + IR_l = E - E_b$ .

Determine the current  $I$ .

**Ex. 111.** A generator, Fig. 54, of 135 volts and  $0.015\Omega$  charges storage cells for seven hours. The resistance of leads  $= 0.025\Omega$ , and 53 cells have each  $0.0002\Omega$  internal resistance. At starting the E.M.F. of a cell is 2.1 volts and at the end of the run 2.35

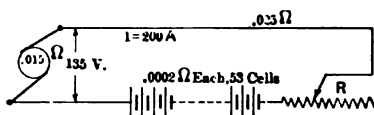


FIG. 54.

volts. What is the necessary regulator resistance to keep current at 200 amps. at starting and stopping?

DATA:

$E$  = E.M.F. generator = 135,  
 $E_1$  = initial E.M.F. battery =  $53 \times 2.1$ ,  
 $E_2$  = final E.M.F. battery =  $53 \times 2.53$ ,  
 $R$  = resistance of regulator,  
 $R_a = 0.015\Omega$   
 $R_b = 53 \times 0.0002$ ,  
 $R_l = 0.025\Omega$ .

$$E - E_1 = I(R_l + R_a + R_l + R_b).$$

$$135 - (2.1 \times 53) = 200(R_l + 0.015 + 0.025 + [53 \times 0.0002]).$$

$R_l = ? \Omega$  regulator resistance at start of charge.

$$E - E_2 = I(R_2 + R_a + R_l + R_b).$$

$R_2 = ? \Omega$ , regulator resistance at end of charge.

**Ex. 112.** What is the monthly cost of operating a 10-H.P. motor 8 hours a day at 10 cents per K.W. hour?

**Ex. 113.** The 0.11 ohm leads from a 50-volt P.D. source are carrying 10 amps. to an arc lamp of 39 volts, B.E.M.F. which has 0.09 ohm resistance in the lamp coil. The carbons have resistances 0.08 and 0.12 ohm and the arc 0.1 ohm. What is the value of the adjustable resistance which keeps the current in the lamp at 10 amps.? See Figs. 55 and 56.

Fig. 55 shows an arc lamp in which  $s$ =switch,  $OK$ =clutch,  $P$ =dash pot,  $T$ =trip,  $M$ =solenoid magnet,  $CC$ =carbons,  $R$ =adjustable resistance.

**Ex. 114.** When a storage battery, Fig. 57, a heavy rheostat, and an armature whose resistance was sought, were joined in series, the current was 15 amps. The voltmeter bridged across the two commutator bars in contact with the brushes showed 0.09 volt. What is the armature resistance?

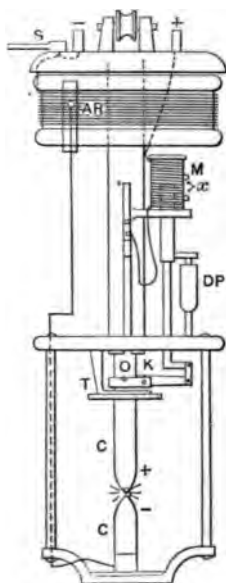


FIG. 55.—Arc Lamp.

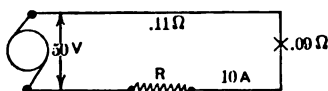


FIG. 56.

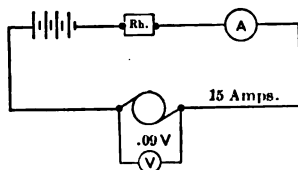


FIG. 57.

A voltmeter may be used to test a joint, switch contact, battery connection, etc. The working formula is

$$R = \frac{V}{I},$$

where  $I$  is the strength of current passing through the connection or joint and  $V$  the voltage drop across it, and  $R$  its corresponding resistance.

**Ex. 115.** A foot of search-light wire has a poor connection. It shows a drop of .1 volt, whereas a foot of regular main shows  $\frac{1}{30}$  volt while carrying 100 amperes. What is the resistance of the joint and the power lost in it?

Insulation resistance may be measured by connecting it to a well-insulated generator or battery in series with a high resistance Weston voltmeter. These are designated by  $R$ ,  $B$ , and  $r$  respectively, and the corresponding voltmeter reading is  $v$ . The voltmeter is then shunted by the switch  $S$ , as shown in the figure (58). Upon closing the

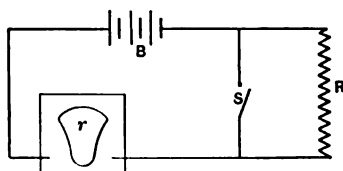


FIG. 58.

switch the voltmeter reading becomes  $V$ . The working formula is

$$R = \frac{V - vr}{v}.$$

**Ex. 116.** The deflection on a 20000-ohm voltmeter indicated 100 volts when connected to a generator. An unknown resistance was inserted in the circuit and the voltmeter then indicated 40 volts. What was the value of the unknown resistance?

**Ex. 117.** The deflection of a voltmeter was 120 when the 150 volt coil was connected to a well-insulated storage battery. The 3 volt coil of the instrument was then connected between a resistance  $R$  and the battery and the corresponding deflection indicated  $\frac{1}{30}$  volt. What was the resistance of  $R$ ?

**Ex. 118.** What current passes through a 1.5-ohm electro-magnet which is connected to a cell having 2 volts E.M.F. and .5 ohm resistance?

**Ex. 119.** A cell of 2 volts and internal resistance 0.5 ohm is joined in series with a number of different resistance spools. The drop over a 0.4 ohm spool is 0.6 volt. What is the current

flowing through the circuit? What part of the total E.M.F. is used in overcoming the resistance of the cell?

**Ex. 120.** A cell with an internal resistance of 2 ohms sends a current of 0.035 amp. through an electromagnet of a bell having resistance of 48 ohms. What is the E.M.F. of the cell?

The diameter of copper wire required for feeders, mains, branches, service wires or inside wiring depends upon the following:

$n$  = number of lamps in parallel;

$I$  = the current required for each lamp;

$l$  = the distance of the lamps from the center of distribution expressed in feet;

$E$  = the total drop in the wires;

$d$  = the diameter of the wire;

$$d = \sqrt{\frac{21.6nIl}{E}}.$$

**Ex. 121.** Compute the diameter and determine the size of copper wire which is required to supply 50 16 c.p. lamps at a distance of 150 ft. Each lamp is to burn at 110 volts and consume 53 watts. The drop in the leads is 2 volts.

**Ex. 122.** Watts per candle.

$$I = \frac{W_1 cp}{E}.$$

$W_1$  = number of watts per candle,  
 $cp$  = candle power of lamps.

Solve for  $cp$  and interpret the formula.

Solve for  $W_1$  and interpret the formula.

**Ex. 123.** Use the preceding formula (Ex. 122) to determine  $W_1$  when  $I = 2.15$ ,  $E = 220$ , and  $cp = 32$ .

**Ex. 124.** Use the formula in Ex. 122 to determine  $W_1$  when  $I = .5$ ,  $E = 110$ , and  $cp = 16$ .

The number of volts drop in lamp wires is given by the formula,

$$v = \frac{x E}{100 - x},$$

where  $v$  = the number of volts drop in the wires;

$E$  = voltage delivered to a lamp;

$x$  = a whole number and is the percentage drop or 100 times the ratio of the drop to the voltage received by the lamps.

**Ex. 125.** The leads to a cluster of 110-volt lamps are figured for a 5 per cent drop. What is the actual volts lost in the leads?

**Ex. 126.** What size of wire will carry 50 amps. 100 ft. to a 110-volt motor with a drop of 2 per cent?

**Ex. 127.** In the preceding formula solve for  $x$ , and express the per cent drop in wires in terms of volts drop and volts delivered.

**Ex. 128.** There is 5 volts drop in the leads connected to a motor running at an E.M.F. of 105 volts. What is the per cent drop?

**Ex. 129.** A group of 110-volt lamps is to be fed by means of a cable whose length is 50 yds. The voltage drop is not to exceed about 2 volts. Allow 55 watts per lamp. What size wire should be used?

**Ex. 130.** A current of 35 amps. is conducted 150 yds. The maximum drop allowed is 3 per cent. Calculate the cable so that 220 volts are delivered to the receiving circuit?

**Ex. 131.** A current of 20 amps. is to be conducted 150 yds. The voltage drop allowed is 6 volts. Determine the size of the cable.

**Ex. 132.** A stationary armature, resistance 0.03 ohm, was accidentally connected with a 110-volt circuit. Estimate the amount of current causing the illumination.

A storage battery, motor, or arc lamp exerts a back pressure when supplied by a generator. If the generator pressure =  $E$  and the back pressure =  $e$ , then the effective pressure =  $E - e$  and therefore the current flowing through the circuit is given by the formula,

$$I = \frac{E - e}{R}.$$

**Ex. 133.** Consider a simple circuit containing a generator of 3 volts and 0.02 ohm, a storage cell of 2 volts and 0.005 ohm,

and leads of 0.1 ohm. What current flows through this circuit and what is the drop at the different points?

**Ex. 134.** What is the E.M.F. of a generator of 0.02 ohm supplying 100 amps. to 54 cells in series, each of which has 2.3 volts back E.M.F. and 0.004 ohm resistance? The leads resistance = 0.03 ohm. What is the voltage at the generator and at the battery terminals?

**Ex. 135.** A 10-mile arc-light circuit of No. 6 B. & S. wire has 10 amperes flowing through it. What horse-power is lost in the circuit?

The diameter  $d$  of a wire required to transmit  $HP$  horse-power over a distance of  $L$  feet with  $v$  volts loss in the wire to a motor requiring  $E$  volts and having an efficiency of  $\eta$  is given by

$$d = \sqrt{\frac{746 HP \ 2 L \ 10.8}{v E \eta}}.$$

Simplify the numeric part of this expression and show how it will have to be modified in order to apply to a 50 per cent overload on the motor.

**Ex. 136.** A 110-volt hoist motor of 15 horse-power is 75 ft. from the main switch. Select the tap wires to allow a drop of 3 volts from the switch to a motor of 90 per cent efficiency.

A constant current motor of  $\eta$  efficiency requires  $E$  volts and  $I$  amperes to give  $HP$  horse-power as follows:

$$E = \frac{746 HP}{I \eta}.$$

Solve for  $\eta$  and interpret the resulting formula.

**Ex. 137.** What current will be delivered to supply a 220-volt motor of 90 per cent efficiency in order to give 15 H.P.?

**Ex. 138.** Make a sketch of the wiring and the distribution of the current in all the buildings of the Institute. Determine the drop at different loads by communicating with the engineer at the generating plant. Work out a complete wiring chart for the system.

**Ex. 139.** In a transmission line three hard-drawn copper wires are used. They have 97 per cent of the conductivity of

pure copper. The wire is 0.182 in. in diameter and there are 99.8 ft. of it in a pound. Determine the following items: (a) tensile strength of copper line; (b) tensile strength of an aluminum line of the same conductivity; (c) feet per pound of this aluminum wire; (d) diameter of the aluminum wire; (e) price per pound of an aluminum wire which shall be equivalent in conductivity to copper.

**Ex. 140.** A wattmeter fluctuates between 7000 and 5000 for 5 minutes. What is the value of the average watt hour?

**Ex. 141.** A bus of copper  $3\frac{1}{2} \times \frac{1}{2}$  in. is 35 ft. in length. (a) What is its resistance at  $20^{\circ}$  C.?  $K=10.4$ . (b) What length of bus will serve as a shunt for a 2000 amp. ammeter requiring 50 millivolts for full-scale deflection.

**Ex. 142.** A 90-lb. steel rail is 30 ft. in length.  $K$  for steel is 8 times that of copper and 1 cu.in. weighs 0.28 lb. (a) What is the resistance of the rail? (b) What is the resistance of a mile of single rail allowing 5 per cent additional for the bonds?

**Ex. 143.** A copper transmission line has a resistance of 4.5 ohms at  $70^{\circ}$  F. What is the range of resistance for temperature varying between  $-25^{\circ}$  F. to  $110^{\circ}$  F.?

**Ex. 144.** A copper field coil has a resistance = 25 ohms at  $70^{\circ}$  F. After running the machine four hours the resistance of the coil increased 4 ohms. What was the corresponding temperature of the coil?

## CHAPTER VIII

### EFFICIENCY OF GENERATORS AND MOTORS

1. **Efficiency** is the ratio of the output to the input of any machine or device.

$$\text{Efficiency} = \frac{\text{output}}{\text{input}}.$$

The **electrical efficiency** ( $\eta_e$ ) of an electric machine is the ratio of the useful energy to the total electric energy ( $W$ ) in its armature. The electric energy in the armature coils consists of the useful energy ( $W_u$ ) plus the energy ( $W_a$ ) loss due to armature resistance and the energy ( $W_f$ ) loss due to the field resistance.

The electrical efficiency of a generator is given in (1).

$$1) \quad \eta_e = \frac{W_u}{W} = \frac{W_u}{W_u + W_a + W_f} = \frac{\text{output}}{\text{output} + \text{copper losses}}.$$

The electrical efficiency of a motor is given in (2),

$$2) \quad \eta_e = \frac{W_u}{W} = \frac{W - (W_a + W_f)}{W} = \frac{\text{input} - \text{copper losses}}{\text{input}}.$$

In the case of a generator  $W_u = EI$ , and in the case of a motor  $W = EI$ , in which  $E$  is the brush voltage.

The electrical efficiency does not include waste due to hysteresis, or eddy current losses, nor friction, but is dependent upon the copper losses only.

**Ex. 1.** Express the  $W$ 's in terms of  $E$ ,  $I$ , and  $R$  and substitute in (1) and (2). Write the formulas for the electrical efficiency of (a) a series-wound generator; (b) a shunt-wound generator;

(c) a compound-wound generator; (d) a series-wound motor; (e) a shunt-wound motor; (f) a compound-wound motor.

Owing to the ambiguity of the word efficiency when not specifically defined there is a widely accepted designation known as the net efficiency.

2. The **commercial or net efficiency** of an electric machine is the ratio of the output to the intake. The intake ( $W_I$ ) of a generator equals the input ( $W$ ) or total energy generated in the armature, plus the energy ( $W_h$ ) losses due to hysteresis, eddy currents ( $W_e$ ) and to friction ( $F$ ). The intake of a motor is the electric energy delivered at its terminals. The output of a generator is the available electrical energy at its terminals. The output of a motor is the available mechanical energy at its shaft. The commercial efficiency ( $\eta_c$ ) is given in (3)

$$(3) \quad \eta_c = \frac{W_u}{W_I} = \frac{W_u}{W + W_h + W_e + F} = \frac{\text{output}}{\text{intake}} \\ = \frac{W_u}{W_u + W_a + W_f + W_h + W_e + F}$$

The commercial efficiency of a motor is given in (4),

$$(4) \quad \eta_c = \frac{W_u}{W_I} = \frac{W_I - (W_a + W_f + W_h + W_e + F)}{W_I} \\ = \frac{\text{intake} - \text{all losses}}{\text{intake}}$$

**Ex. 2.** Express the copper losses in terms of  $E$ ,  $I$ , and  $R$  and substitute in (3) and (4). Write the formulas for the commercial efficiency of (a) a series-wound generator; (b) a shunt-wound generator; (c) a compound-wound generator; (d) a series-wound motor; (e) a shunt-wound motor; (f) a compound-wound motor.

3. The **gross efficiency** ( $\eta_g$ ) of an electric machine is the ratio of the commercial efficiency to the electrical efficiency.

$$(5) \quad \eta_g = \frac{\eta_c}{\eta_e}$$

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**Ex. 3.** Express  $\eta$ , for a generator and for a motor both in terms of  $W$  and also in terms of  $E$ ,  $I$ , and  $R$ .

**Ex. 4.** Determine the electrical and commercial efficiencies of a 150-volt 8000 ampere multipolar shunt generator. The armature resistance equals 105 microhms, and the total exciting current for the field magnets equals 13.9 amperes. The hysteresis loss, eddy current loss, and friction loss are 11.22 K.W., .58 K.W., and 40 K.W. respectively.

**Ex. 5.** Determine the electrical and commercial efficiencies of a 540-volt 3700-ampere multipolar compound generator. The armature resistance equals .00216 ohm; the total exciting current for the shunt fields equals 20.8 amperes; the resistance of the series field equals 147 microhms. All other losses aggregate 39.7 kilowatts.

**Ex. 6.** Determine the electrical and commercial efficiencies of a bipolar low-speed compound 50 K.W. 125-volt generator. The armature resistance equals .0314 ohm; the total current required for the shunt field equals 2.84 amperes; the resistance of the series field equals .00106 ohm. The hysteresis and eddy current losses total 332 watts and the friction losses 2.5 kilowatts.

## CHAPTER IX

### THE ALGEBRA OF THE MAGNETIC CIRCUIT

**1. Comparison of Magnetic and Electric Circuits.** In the magnetic circuit the elements are the **magnetomotive force**, M.M.F. ( $M$ ); the **reluctance** ( $R$ ); a **flux** ( $\Phi$ ). These elements enter into a relation with analogies to the relations of the elements  $E$ ,  $R$ , and Ohm's Law.

$$\begin{array}{c} \text{MAGNETIC} \\ (1) \quad \Phi = \frac{M}{R} \end{array}$$

$$\begin{array}{c} \text{ELECTRIC} \\ (2) \quad I = \frac{E}{R} \end{array}$$

$$(3) \quad \text{Flux} = \frac{\text{magnetomotive force}}{\text{reluctance}} = \frac{\text{magnetic pressure}}{\text{magnetic resistance}}$$

Transform (1) and solve for  $M$  and  $R$  and interpret the resulting equations.

**2.** The intensity of a magnetic field or **magnetic force** ( $H$ ) is compared with the  $PD$  or drop in a circuit, where  $l$  is the length of either the magnetic or electric circuit.  $H$  is called the magnetic drop.

$$\begin{array}{c} \text{MAGNETIC} \\ (4) \quad H = \frac{M}{l} \end{array}$$

$$\begin{array}{c} \text{ELECTRIC} \\ (5) \quad PD = \frac{E}{l} \end{array}$$

Transform (4) solving for  $M$  and  $l$  and interpret the resulting equations.

**3.** The inductance or **flux density** ( $B$ ) is the number of lines of force per unit area and is compared to the

density ( $I_d$ ) per unit area where  $A$  is the area of either the magnetic or electric cross-section.

$$\text{MAGNETIC} \\ (6) \quad B = \frac{\Phi}{A}.$$

$$\text{ELECTRIC} \\ (7) \quad I_d = \frac{I}{A}.$$

Transform (6) solving for  $B$  and  $A$  and interpret (6) and the resulting equations. If  $A$  is expressed in square centimeters, then  $B$  is the density per square centimeter, but if  $A$  is expressed in square inches, then  $B$  is the density per square inch.

4. The **reluctance** ( $R$ ) of a magnetic circuit corresponds to the resistance of an electric circuit and like resistance is dependent upon the length, cross-section and specific nature of the material.

$$\text{MAGNETIC} \\ (8) \quad R = \frac{K_1 l}{A}.$$

$$\text{ELECTRIC} \\ (9) \quad R = \frac{K l}{A}.$$

$$(10) \quad R = \frac{l}{\mu A} = \frac{1}{\mu} \cdot \frac{l}{A}.$$

$$(11) \quad R = \frac{l}{\rho A} = \frac{1}{\rho} \cdot \frac{l}{A}.$$

$K$  is called the **specific resistance** or **resistivity**;  
 $K_1$  is called the **specific reluctance** or **reluctivity**;  
 $\rho$  is called the **specific conductance** or **conductivity**;  
 $\mu$  is called the **specific permeance** or **permeability**.

Transform (8) and (10) and interpret (8) and (10) and the resulting equations.

5. In a magnetic circuit the reciprocal of reluctance is called **permeance** ( $P$ ) and corresponds in the electric circuit to conductance ( $G$ ) which is the reciprocal of resistance:

$$\text{MAGNETIC} \\ (12) \quad P = \frac{1}{R}.$$

$$\text{ELECTRIC} \\ (13) \quad G = \frac{1}{R}.$$

Substitute the value of  $R$  from (10) in (1) and then replace  $\frac{\Phi}{A}$  by  $B$  from (6). Interpret the resulting equation.

**6. The Field within a Coil.** A solenoid is a helix or coil of wire which creates a magnetic field when a current of electricity passes through it. The strength of the magnetic field ( $H$ ) within its interior is directly proportional to the total number ( $N$ ) of turns of wire and to the current passing through it and is inversely proportional to its length measured in centimeters.

$$H \propto \frac{NI}{l} \quad \therefore \quad \frac{H}{H_1} = \frac{NI l_1}{N_1 I_1 l} \quad \text{or} \quad H = \frac{H_1 l_1}{N_1 I_1} \frac{NI}{l}.$$

**7.** The proportionality factor equals  $\frac{4\pi}{10} = 1.26 = \frac{H_1 l_1}{N_1 I_1}$

which is the field intensity in the air within the solenoid when it is wound with 1 turn per centimeter of length and has 1 ampere of current flowing through it. Therefore,

$$(14) \quad H = \frac{4\pi NI}{10l} = \frac{1.26 NI}{l} = 1.26 \frac{NI}{l}.$$

The product  $NI$  is termed the **ampere-turns**. Interpret (14) in terms of ampere-turns. The quantity  $\frac{NI}{l}$  is called the ampere-turns per centimeter of length. (14) may be rewritten (15), in which the product  $Hl$  is called the magnetomotive force and is abbreviated by ( $M$ ) according to (4).

$$(15) \quad Hl = 1.26 NI = M.$$

**8.** Iron or any other magnetic material, when inserted within the solenoid, has the effect of increasing the number of lines of force from ( $H$ ) per square centimeter to a flux density of  $B$  lines per square centimeter, i.e.,  $B$  gaussess. The

ratio of  $B$  to  $H$  is called the **permeability** of the magnetic material and is designated by  $(\mu)$ .

$$(16) \quad \mu = \frac{B}{H},$$

solve (16) for  $H$  and for  $B$ .

$H$  is called the **magnetizing force** and is the field intensity in air, i.e., the number of lines of force per square centimeter in air. What is the distinction between magnetizing force ( $H$ ) and magnetomotive force ( $M$ ). The permeability of a magnetic substance is a variable quantity, i.e., for each definite value of  $B$  there is a definite value of  $\mu$ . These values are determined in Ex. 10.

There is no electric analogue for equation (16).

9. The **reluctance** ( $R$ ) of a **compound circuit** such as the magnetic circuit of a generator may be obtained by adding the individual reluctances in the path of the flux.  $R_1, R_2, R_3$  correspond respectively to the reluctances of the magnet frame, air-gap and core of the armature. Reluctances in series (17a) are united like series resistances and reluctances in parallel (17b) are united like resistances in parallel.

$$(17a) \quad R = R_1 + R_2 + R_3,$$

$$(17b) \quad \frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3},$$

show that  $\Phi = \frac{M}{R_1 + R_2 + R_3}$  and interpret.

10. The **total ampere-turns** ( $NI$ ) required for the magnetic circuit is the sum of the  $\overline{NI_1}, \overline{NI_2}, \overline{NI_3}$ , which represent the ampere-turns required to overcome the reluctance of the magnet frame, the armature core and the air gap spaces respectively,  $\overline{NI_4}$  represents the ampere-turns required to compensate for the armature reaction.

$$(18) \quad NI = \overline{NI_1} + \overline{NI_2} + \overline{NI_3} + \overline{NI_4}.$$

**Ex. 1.** Show that (14) becomes (19) when  $H'$  is given in lines per square inch and  $l'$  in inches.

$$(19) \quad NI = \frac{10}{4\pi \cdot 2.54} H' l' = .3133 H' l'.$$

One **maxwell** means one line of force; one **gauss** equals one maxwell or one line of force per square centimeter.

**Ex. 2.** (a) What is the meaning 2.5 kilomaxwells; (b) What is the meaning of 3.5 megamaxwells; (c) What is the meaning of 2500 gaussess; (d) How many gaussess are there in 2500 maxwells per square inch?

**Ex. 3.** In a magnetic circuit the cross-section is  $\frac{1}{2}$  in.  $\times$   $\frac{1}{2}$  in., and the magnetic density is 50000 lines of force per square inch. How many lines of force thread the circuit?

$$\text{The area} = \frac{1}{2} \times \frac{1}{2} = .375 \text{ sq.in.}$$

$$\Phi = BA = 50000 \times .375 = \underline{18750} \text{ lines thread the circuit.}$$

**Ex. 4.** The cross-section of a magnetic circuit is circular and 1.5 cm. in diameter. The magnetic density is 3000 lines per square centimeter. What is the total number of lines threading the circuit?

**Ex. 5.** How many maxwells per square centimeter are there in a round bar magnet  $\frac{1}{2}$  in. in diameter when 4500 lines of force pass through it?

**Ex. 6.** How many gaussess are there in a bar magnet 2 cm.  $\times$  .75 cm. when 9000 lines of force pass through it?

**Ex. 7.** The magnetic density in a bar magnet  $\frac{1}{2}$  in.  $\times$   $\frac{1}{2}$  in. is 40000 lines of force per square inch. What total number of lines thread the magnet?

**Ex. 8.** The permeability of a piece of iron is 850, when the magnet density is 59500 lines of force per square inch. What is the field density required to produce that magnetic density?

**Ex. 9.** The magnetizing force acting on a piece of iron is 600, and the magnetic density produced is 54300 lines of force per square inch. What is the permeability at that stage of magnetization?

**Ex. 10.** Prepare a table of values between  $B$ ,  $H$ , and  $\mu$ , using (16) in connection with (20), (21), (22), (23), and (24). Assume values of  $B$  beginning at 2000 and extending to 25000 in increments (increases) of 500 in each step.

Permeability for low combined cast iron:

$$(20) \quad \mu = 380 - \frac{26(4000 - B)^2}{10^6}.$$

Permeability for high combined carbon cast iron:

$$(21) \quad \mu = 200 - \frac{6.2(4000 - B)^2}{10^6}.$$

Permeability for malleable cast iron:

$$(22) \quad \mu = 700 - \frac{15(6000 - B)^2}{10^6}.$$

Permeability for cast steel:

$$(23) \quad \mu = 1200 - \frac{19(6500 - B)^2}{10^6}.$$

Permeability for wrought iron:

$$(24) \quad \mu = 2800 - \frac{3.2(7500 - B)^2}{10^6}.$$

**Ex. 11.** A bar magnet of low combined cast iron has a cross-section of  $1\frac{1}{4}$  ins. and has a mean length of 18.25 ins. How many ampere-turns are required to force 7000 lines through the magnet if the air-gap between the magnet and its armature is negligible? How will the result alter if the air-gap is  $\frac{1}{4}$  in. in length? Use formula (19) in Ex. 1 for English units.

The pull ( $F$ ) in pounds of an electromagnet is given in (25) where  $B$  is the magnetic density per square inch and  $A$  the polar area in square inches.

$$(25) \quad F = \frac{B^2 A}{72134000}.$$

**Ex. 12.** What is the pull on the magnet in Ex. 10 when the current is .25 ampere?

**11.** In calculating magnetic leakage, the total number of lines of force ( $\Phi$ ) is expressed in terms of the useful lines ( $\Phi_u$ ), the stray lines ( $\Phi_s$ ), the percentage of leakage by ( $p$ ) and the coefficient or allowance for leakage by ( $\lambda$ ).

$$(26) \quad \Phi = \Phi_s + \Phi_u.$$

$$(27) \quad p = \frac{100\Phi_s}{\Phi}.$$

$$(28) \quad \lambda = \frac{\Phi}{\Phi_u}.$$

Transform (26), (27), (28) and interpret the resulting equations.

**Ex. 13.** Assuming that the magnetic leakage in an electromagnet is 25 per cent, and that there are 75000 useful lines of force, how many lines of force are provided by the magnetizing coil?

**Ex. 14.** 100000 lines of force are produced by the magnetizing coils of an electromagnet of which 84000 are useful. What is the percentage leakage? What is the coefficient of allowance?

**Ex. 15.** In an electromagnet there are 27000 stray lines of force and 63000 useful lines. What is the percentage of leakage?

**Ex. 16.** The magnetic leakage in an electromagnet is 15 per cent and there are 110000 useful lines of force. How many lines are produced in the magnetizing coils?

**Ex. 17.** The magnetic leakage in an electromagnet is 35 per cent. There are 60000 lines produced by the magnetizing coils. How many lines are useful?

**12. Leakage Permeance** between two surfaces. From (10) and (12) we have the law for the permeance of an air-gap expressed in C.G.S. units in (29). Since the permeability of air equals 1 we may write:

$$(29) \quad P = \frac{1}{R} = \frac{\text{mean area of exposed surfaces}}{\text{mean length of path between surfaces}}.$$

In the case of two parallel surfaces, Fig. 59, of approximate equal area the permeance is given in (30) for C.G.S. units,

$$(30) \quad P = \frac{A_1 + A_2}{2d}.$$

Write the formula for the permeance when English units are to be substituted.

When the two rectangular surfaces lie in the same plane with their edges parallel as shown in Fig. 60 the leakage paths will be semicircular for a small separation of the

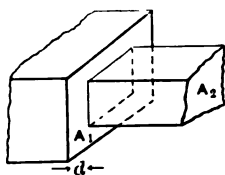


FIG. 59.

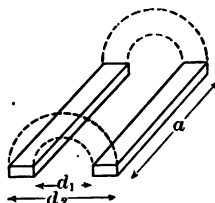


FIG. 60.

edges and the permeance is given in (31) and (31a) for C.G.S. units, (31) is an approximation,

$$(31) \quad P = \frac{a}{\frac{2d_1}{d_2 - d_1} + \frac{\pi}{2}} \quad (31a) \quad P = \frac{a}{\pi} \log_e \frac{d_2}{d_1}.$$

**Ex. 18.** Determine the value of  $P$  when  $a = 8$  ins.,  $d_2 = 3$  ins.,  $d_1 = 1$  in.

Substitute in (31), (31a) and (31b). When the separation between the edges is considered, the path of the flux is made of arcs joined by straight lines and (31a) is modified into (31b),

$$(31b) \quad P = \frac{a}{\pi} \log_e \frac{1}{2} \left\{ \frac{\pi d_2}{d_1} + 2 - \pi \right\}.$$

When the two surfaces are at right angles as shown in Fig. 61, which represents the rotation of the left-hand plane upward with a radius  $d_1$ , then the permeance formula is given in (32) and (32a), (32) is an approximation,

$$(32) \quad P = \frac{(d_2 - d_1)2a}{(d_2 + d_1)\pi}, \quad (32a) \quad P = \frac{a}{\pi} \log_e \frac{1}{4} \left\{ \frac{\pi d_2}{d_1} + 2 - \pi \right\}.$$

13. Fig. 62 illustrates the path of the flux in a generator where  $\Phi$  is the total flux in the air-gap. The mean

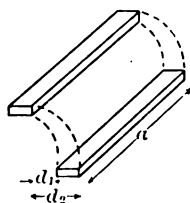


FIG. 61.

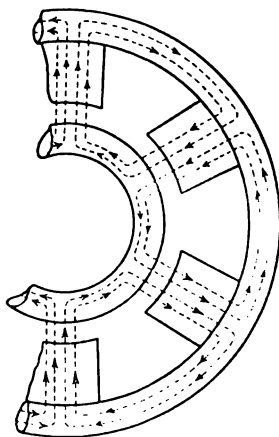


FIG. 62.—Generator Flux.

length of the magnetic circuit in the frame, poles, air-gaps, and core are designated by  $l_f$ ,  $2l_p$ ,  $2l_a$  and  $l_c$  respectively. The corresponding cross-sections are  $A_f$ ,  $A_p$ ,  $A_a$  and  $A_c$  and the respective flux densities are  $B_f$ ,  $B_p$ ,  $B_a$  and  $B_c$ .

Assuming no leakage interpret (33) and (34)  $IN_f$ ,  $IN_p$ ,  $IN_a$ ,  $IN_c$  are the respective ampere-turns per unit length.

$$(33) \quad \Phi = A_f B_f = A_p B_p = A_a B_a = A_c B_c,$$

$$(34) \quad IN = \overline{IN}_f l_f + 2\overline{IN}_p l_p + 2\overline{IN}_a l_a + \overline{IN}_c l_c.$$

**14. The Energy Loss Due to Hysteresis.** The retention or lagging of part of the magnetic field in the iron of armature cores and alternating current devices is called hysteresis. A definite expenditure of energy ( $P_h$ ) in ergs, is required to reverse or reestablish a varying flux through a magnetic circuit.  $P_h$  is directly proportional to the  $n$ th power of the maximum value of the flux density ( $B_{\max}$ ), also to the volume ( $V$ ) of magnetic material in cubic centimeters, and to the number of cyclic changes( $f$ ) or complete reversals per second.

$$P_h \propto V f B_{\max}^n,$$

$$(35) \quad P_h = K V f B_{\max}^n \text{ ergs.}$$

Dr. Steinmetz and his associates use 1.6 for the average value of the exponent  $n$ . The proportionality factor  $K$  is called the hysteretic resistance and its average value for sheet iron is .0035.

**Ex. 20.** Show that (35) becomes (36) when the  $P_h$  expresses the energy lost in ergs for 1 cu.cm. per 1 cycle per second.

$$(36) \quad P_h = K B_{\max}^{1.6} \text{ ergs.}$$

Show that (35) reduces to (37) when  $P_h$  expresses the energy in watts,  $V$  the mass in cubic feet,  $B_{\max}$  the flux density per square inch, assuming  $K = .0035$ .

$$(37) \quad P_h = 5.19 \times 10^{-7} f V B_{\max}^{1.6}.$$

Calculate the values of  $P_h$  from (36), assuming  $K = .0035$ , and the values of  $B_{\max}$  ranging from 1000 to 12000 lines per square centimeter in gradations of 1000. Use a slide rule as a check and tabulate the values for  $B_{\max}$ ,  $B_{\max}^{1.6}$ , and  $P_h$ .

**Ex. 21.** Calculate the values of  $P_h$ , from (37), assuming  $f=1$ , and  $V=1$ , and values of  $B_{\max}$  ranging from 10000 to 100000 lines per square inch, in gradations of 10000.

**Ex. 22.** What is the hysteresis energy loss in a machine containing 3000 cu.cm. of iron? The number of cycles equals 60,  $K = .0035$ , and  $B_{\max} = 10000$ .

**Ex. 23.** The magnetic material in a machine has a cross-section of 10 sq.in. and a mean length of 30 ins. There are 45000

lines per sq.in. and  $K = .0035$ . What is the value of the hysteresis loss for 25 and also for 60 cycles?

**15. The Energy Loss Due to Eddy Currents.** The motion of the core of an armature through a magnetic field induces local or eddy currents within it. The energy loss ( $P_e$ ) in ergs, due to eddy currents is directly proportional to the square of the maximum value of the flux density ( $B_{\max}$ ) per square centimeter, to the square of the frequency ( $f$ ) in cycles per second, and to the mass in cubic centimeters.

$$P_e \propto f^2 V B^2,$$

(38)

$$P_e = K f^2 V B^2.$$

$K$  is determined by the formula (39) where  $t$  is the thickness of the material in centimeters and  $c$  is the electric conductivity in mhos. For iron  $c = 100000$  mhos and for copper  $c = 700,000$  mhos.

$$(39) \quad k = \frac{\pi^2}{6} \times 10^{-9} t^2 c = 1.645 \times 10^{-9} t^2 c.$$

**Ex. 24.** Substitute the value of  $K$  from (39) in (38), and interpret the resulting equation. (b) How will the resulting equation alter when  $P_e$  is expressed in watts,  $V$  in cubic feet,  $t$  in inches, and ( $B_{\max}$ ) in lines per square inch? (c) How will the last equation read when the thickness is expressed in mils?

**Ex. 25.** Calculate the eddy current loss for the core of a generator giving 60 cycles per second. The volume of iron is 11 cu.ft. The thickness of lamina = 0.1 in.

**Ex. 26.** A magnetic circuit consists of a wrought-iron bent bar, 250 cm. length and 50 sq.cm. in cross-section, with an air-gap of .5 cm. length. How many ampere-turns are necessary to set up (a) 50000 lines of force, (b) 100000 lines, (c) 500000 lines, (d) 1000000 lines?

**Ex. 27.** A magnetic circuit consists of 150 cm. of cast steel, with 60 sq.cm. in cross-section; 135 cm. of sheet steel, with 55 sq.cm. cross-section; and 1 cm. of air with 55 sq.cm. of cross-section. There are 1500 turns of wire wound upon this circuit. How many amperes will be necessary to set up 600000 lines?

**Ex. 23.** Construct a table of hysteresis loss for transformers used on 25-, 60-, and 133-cycle circuits. The range of core weights to extend from 50 to 300 lbs. in gradations of 25 lbs.

**16.** The **average E.M.F.** which is induced in an inductor, i.e., a moving wire or electric conductor, is proportional to the flux density ( $H$ ), and to the length ( $l$ ) and velocity ( $v$ ) of the inductor,

$$\text{E.M.F.} \propto Hlv.$$

In the C.G.S. system the proportionality factor is one and therefore,

$$(40) \quad \text{E.M.F.} = Hlv.$$

The average E.M.F. is equal to the total flux ( $\Phi$ ) divided by the time ( $t$ ) in which it is cut.

$$(41) \quad \text{E.M.F.} = \frac{\Phi}{t} (\text{abvolts}) = \frac{\Phi}{t 10^8} \text{ volts.}$$

Transform and interpret equations (40) and (41).

*Observation.* One volt is induced in an inductor by the cutting of 100000000 lines of force per second. One inductor of an armature cuts  $2\Phi$  lines of force in one revolution for each pair of poles.

**17.** The **force** tending to push a conductor aside in a magnetic field varies as the product of the strength of the field, the length of the conductor and the current flowing through the conductor.

$$(42) \quad F = I l H \text{ dynes.}$$

Interpret this equation for a generator and for a motor.

**18.** The **E.M.F. generated** in any **direct current armature** is directly proportional to the product of the total flux ( $\Phi$ ) which is cut, the total number ( $N$ ) of conductors on the periphery of the armature, the number ( $p$ ) of poles and the number of revolutions per second ( $n$ ) and inversely proportional to the number ( $q$ ) of parallel paths. The

proportionality factor is  $10^{-8}$  when the result is expressed in volts.

$$(43) \quad \text{E.M.F.} = \frac{\Phi N p n}{q 10^8} \text{ (volts).}$$

**Ex. 29.** In the air-gap of a generator a flux of 2000000 maxwells exists under each pole. At a speed of 1200 R.P.M. what E.M.F. is generated in each conductor of the armature?

**Ex. 30.** If the poles cover 70 per cent of the armature surface of the above generator and the armature is 1 ft. long, 1 ft. in diameter and the field strength 5000 gaussess at the armature surface, what average E.M.F. is generated in each conductor?

19. The work ( $W$ ) done by the armature of a generator or motor may be expressed electrically by (44) and mechanically by (45).

$$(44) \quad W = EI \text{ watts.}$$

$$(45) \quad W = \frac{2\pi m T 746}{33000} = .142 mT,$$

where  $m$  = number revolutions per minute,

$T$  = torque in foot-pounds,

Equate (44) and (45) and obtain (46).

$$(46) \quad T = \frac{7.042 EI}{m} \text{ ft.-lbs.}$$

In (46) substitute the value of  $E$  from (43) and interpret.

**Torque** is the product of the peripheral force ( $F$ ) of the armature multiplied by the mean  $R$  radius of the armature winding.

$$(47) \quad T = FR = \frac{RI\ell H}{10}.$$

Solve for  $F$  and  $R$  and interpret.

The torque or turning moment necessary to cause a generator armature to rotate, and the torque produced by a motor armature are of precisely the same nature. Each

depends upon the strength of the field, the armature current, the length of coil and its displacement from the axis of rotation.

The peripheral force divided by the number of effective conductors ( $N_c$ ) gives the peripheral force ( $F_c$ ) acting on each armature conductor.

$$(48) \quad F_c = \frac{F}{N_c}.$$

**Ex. 31.** In the two preceding problems (29), (30), calculate the force in pounds upon each conductor tending to resist the motion of the armature when the current in each conductor is 25 amps.

**20. Flux Density in Armature.** For slotted cores the flux density will be greater in the teeth.

$$(49) \quad \frac{\Phi_t}{\Phi_a} = \frac{B_t}{B_a} = \frac{fb_t\mu}{b_t + b_s + B_tf + b_sf}$$

Interpret (49), simplify (49) and reinterpret.

$B_a$  = the apparent flux density in the teeth in lines per square inch,

$B_t$  = the actual flux density in the teeth in lines per square inch,

$b_t$  = the mean width of teeth in inches,

$b_s$  = the mean width of slot in inches,

$\mu$  = the permeability of the tooth corresponding to  $B_t$ ,

$f$  = the ratio of the net length to the gross length of the armature core.

**Ex. 32.** The total ampere-turns in any D.C. armature winding equals one-half the product of the number of conductors times the current per conductor.

Write this as a formula.

**Ex. 33.** Compute the force acting on a conductor of 25 cm. length. The current passing through the conductor is 5 abamps and the field strength is 3000 gauss.

**Ex. 34.** A circular wire of 50 cm. radius rotates 25 times per second in a magnetic field of 1000 gaussses. What voltage is generated in the wire?

**Ex. 35.** Determine the E.M.F. of a 10-pole generator making 25 revolutions per second. The flux equals 2500000.

**Ex. 36.** A 2-pole machine has a flux of 3000000 lines per pole. There are 200 conductors and 2 parallel paths. The machine rotates at 20 revolutions per second. What voltage is developed?

**Ex. 37.** A 110-volt motor of a motor-generator set takes 100 amperes and transmits with 15 per cent loss to a generator of 84 per cent efficiency. The generator shows 20 volts. What current is it giving off?

**Ex. 38.** A compound generator at full load, which is 50 amperes at 110 volts, requires 11000 ampere-turns and at no load 8500 ampere-turns. How many turns are there in the series field?

## CHAPTER X

### WINDING CALCULATIONS

1. THE dimensions of an insulated wire are,  $d$  the diameter of the bare wire, and  $d_1$  the outside diameter over insulation and lead casing. Designate the thickness of insulation by  $t$ , then  $d_1 = d + 2t$ .

$l$  = length and  $b$  = depth of cross-section of space to be filled with wire.

2. **Square Winding.** Assume  $l$  and  $b$  to be exact multiples of  $d_1$ . The space may be considered as divided into

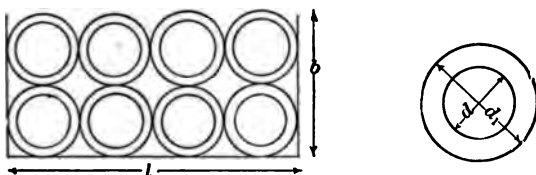


FIG. 63.—Square Winding.

squares. Each square will circumscribe a circle. The circles will be tangential and their centers will be in horizontal and vertical alignment as in Fig. 63. The number of turns of wire is given in (1).

$$(1) \quad N = \frac{lb}{d_1^2}.$$

How many wires are there per layer? How many layers are there?

The ratio of the cross-section of metal to the square space which is required is called the space factor  $\sigma$ .

Solve (5) for  $d$  and substitute for  $R$  from (4) simplify and interpret these equations.

**Ex. 7.** A spool of wire is rewound with wire having twice the number of circular mils. Select any convenient size for the first wire. What will be the relative resistance of the two windings?

**6. Volumes of Winding Space.** The volume ( $V$ ) of any winding space is determined by multiplying the transverse sectional area of the winding space by its length ( $l$ ). The area of an ellipse equals  $\frac{\pi}{4}$  times the product of its long and short diameters.

**Ex. 8.** Verify and interpret the following formulas for the volumes of the winding space corresponding to the sections shown in Fig. 65.

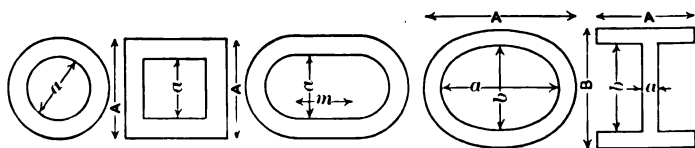


FIG. 65.—Winding Space Sections.

$$(6) \quad V = \frac{\pi l}{4}(A^2 - a^2), \quad \text{Circular core}$$

$$(7) \quad V = l(A^2 - a^2), \quad \text{Square core}$$

$$(8) \quad V = l \left\{ \frac{\pi}{4}(A^2 - a^2) + m(A - a) \right\} \quad \text{Link core}$$

$$(9) \quad V = \frac{\pi l}{4}(AB - ab), \quad \text{Elliptic core}$$

$$(10) \quad V = b \{ (l + \pi)(A - a) + 2a \}. \quad \text{Rectangular core}$$

**7. The heating of magnets** is given by (11).

$$(11) \quad T = K \frac{P}{A}.$$

Interpret (11) according to the following notation:

$T$  = use in temperature centigrade,

$P$  = power in watts dissipated in the coil,

$A$  = outside cylindrical surface of the coil in square inches,

$K$  = temperature rise in centigrade per watt per square inch of outside cylindrical surface,

$K = 130$  for open electromagnets,

$K = 95$  for iron-clad electromagnets,

$K = 70$  for field magnets open,

$K = 140$  for field magnets closed.

**Ex. 9.** Determine the rise in temperature of the magnet specified in Ex. 11, Chapter IX.

**Ex. 1.** Show that for square winding

$$(2) \quad \sigma = 0.7854 \frac{(d)^2}{(d_1)^2}.$$

**Ex. 2.** Show that the total cross-section of wire is  $\frac{lb\sigma}{d_1}$ . Simplify by substituting for  $\sigma$  from (2).

**3. Stagger or Imbedded Winding.** The space may be considered as divided into hexagons. Each hexagon will circumscribe a circle. The circles will be tangential. Their center lines will be horizontal and will also incline  $60^\circ$  out of vertical alignment.

**Ex. 3.** In Fig. 64 determine the value of  $q$ , which is the vertical distance between horizontal center lines.

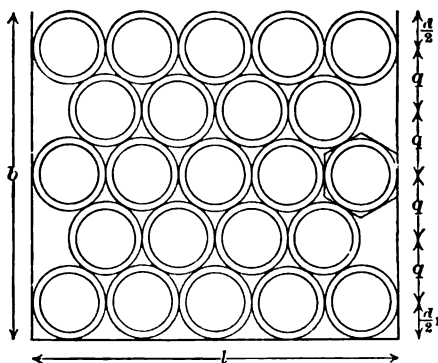


FIG. 64.—Imbedded Winding.

**Ex. 4.** In this example consider  $l$  an exact multiple of  $d_1$ . Then in one row there will be  $\frac{l}{d_1}$  circles, suppose we say  $\frac{l}{d_1} = x$ .

How many circles will there be in two rows?

How many circles will there be in three rows?

How many circles will there be in four rows?

How many circles will there be in  $n$  rows?

What is the depth of the rectangle enclosing one row of circles?

What is the depth of the rectangle enclosing two rows of circles?

What is the depth of the rectangle enclosing three rows of circles?

What is the depth of the rectangle enclosing  $n$  rows of circles?

**Ex. 5.** (a) What is the metallic cross-section for  $n$  rows of wire of outside diameter  $d_1$ , having  $x$  wires in the lower row?

(b) What is the area of the enclosing rectangle for  $n$  rows?

(c) Determine  $\sigma$  where  $\sigma = \frac{(a)}{(b)}$ .

(d) Calculate  $\sigma$  for one, two, three, four layers of winding.

**Ex. 6.** Determine  $\sigma$  for a winding of rectangular bar of dimensions  $a$  and  $b$  and insulation thickness  $t$  where the winding spaces are filled by the material.

The values of  $\sigma$  in Ex. 1-5 are for ideal conditions which can be only approximated in practice. The value of  $\sigma$  is more often expressed in terms of  $d$ , i.e., the bare diameter of the wire.

#### 4. Number of Turns.

$$(3) \quad N = \frac{\sigma lb}{A_c},$$

where  $A_c$  = area of one conductor (bare) in centimeters.

Solve for  $\sigma lb A_c$  and interpret.

#### 5. Diameter of Wire in Terms of Resistance and Winding Volume.

(4)  $W = I^2 R$  represents the permissible loss in watts.

$$(5) \quad R = \frac{VK}{d^2 d_1^2}.$$

$V$  = volume of winding space,

$K$  = a constant varying with allowance rise in the temperature of coil,

$K = 0.8484$  for  $15^\circ \text{C. rise}$ ,

$K = 0.9001$  for  $30^\circ \text{C. rise}$ ,

$K = 1.405$  for  $60^\circ \text{C. rise}$ .

Solve (5) for  $d$  and substitute for  $R$  from (4) simplify and interpret these equations.

**Ex. 7.** A spool of wire is rewound with wire having twice the number of circular mils. Select any convenient size for the first wire. What will be the relative resistance of the two windings?

**6. Volumes of Winding Space.** The volume ( $V$ ) of any winding space is determined by multiplying the transverse sectional area of the winding space by its length ( $l$ ). The area of an ellipse equals  $\frac{\pi}{4}$  times the product of its long and short diameters.

**Ex. 8.** Verify and interpret the following formulas for the volumes of the winding space corresponding to the sections shown in Fig. 65.

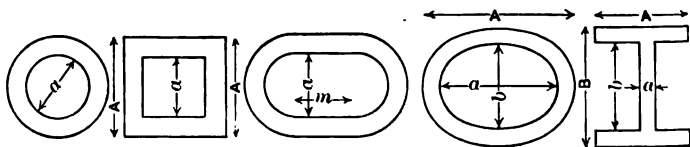


FIG. 65.—Winding Space Sections.

$$(6) \quad V = \frac{\pi l}{4}(A^2 - a^2), \quad \text{Circular core}$$

$$(7) \quad V = l(A^2 - a^2), \quad \text{Square core}$$

$$(8) \quad V = l \left\{ \frac{\pi}{4}(A^2 - a^2) + m(A - a) \right\} \quad \text{Link core}$$

$$(9) \quad V = \frac{\pi l}{4}(AB - ab), \quad \text{Elliptic core}$$

$$(10) \quad V = b \{ (l + \pi)(A - a) + 2a \}. \quad \text{Rectangular core}$$

**7. The heating of magnets is given by (11).**

$$(11) \quad T = K \frac{P}{A}.$$

Interpret (11) according to the following notation:

$T$  = use in temperature centigrade,

$P$  = power in watts dissipated in the coil,

$A$  = outside cylindrical surface of the coil in square inches,

$K$  = temperature rise in centigrade per watt per square inch of outside cylindrical surface,

$K = 130$  for open electromagnets,

$K = 95$  for iron-clad electromagnets,

$K = 70$  for field magnets open,

$K = 140$  for field magnets closed.

**Ex. 9.** Determine the rise in temperature of the magnetified in Ex. 11, Chapter IX.

## CHAPTER XI

### FORMULAS OF MENSURATION

1. THE formulas of this chapter are the familiar formulas of mensuration supplemented by practical formulas for approximations.

The area of any figure is its ratio to a unit of area. The area of a surface or a cross-section of a solid may be measured directly or indirectly by comparing it with any shaped surface or cross-section. The unit of area takes its name from the shape of the unit figure. Thus unit areas which are square are called **square units**. A unit area which is a circle is called a **circular unit**. A unit area which is a triangle is a **triangular unit**. All such units of area have their linear dimensions equal to one unit. A square unit area having a side equal to 1 mil is a square mil. A circular unit area having a diameter of 1 mil is a circular mil.

The volume of a figure is its ratio to a unit of volume. The volume of a solid or geometric figure of three dimensions may be measured directly or indirectly by comparing it with any shaped solid or geometric figure of three dimensions. The unit of volume takes its name from the shape of the unit figure. Thus unit volumes which are cubes are called **cubic units**. A unit volume which is a sphere is called a **spheric unit**. A unit volume which is a cylinder, 1 foot in length and 1 circular mil in cross-section is called a **circular mil foot**. A board foot is a rectangular prism, 1 ft. in length and having a rectangular cross-section of 12 sq.ins. A cube having its edges equal to 1 cm. is a cubic centimeter.

**Area of a Parallelogram.** The area ( $A$ ) of a parallelogram equals the product of its base ( $b$ ) times its altitude. Formulate this law and solve for  $b$  and  $h$  and interpret resulting equations.

Fig. 66  $K$  is the diagonal,  $\alpha$  the angle opposite the

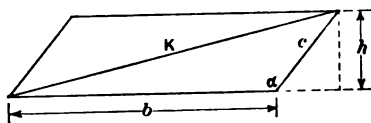


FIG. 66.—A Parallelogram.

$a$  and  $c$  the side adjacent to the base. Express the altitude and the diagonal in terms of  $b$ ,  $c$  and  $\alpha$ .

How do these various formulas simplify when the figure is a rectangle and also when it becomes a square?

What is the relation of the opposite angles of the parallelogram?

What is the sum of the angles of the parallelogram?

**Area of a Triangle.** What relation does a triangle have to a parallelogram which has an equal base and an altitude. Formulate the law for the area of a triangle in terms of  $b$  and  $h$ , solve and interpret the resulting equation for  $b$  and  $h$ .

Fig. 67  $\alpha$  is the angle between the base and the adjacent side  $c$ . Express the area and altitude of the triangle in terms of  $b$ ,  $\alpha$ .

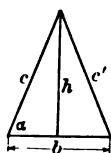


FIG. 67.

How do these formulas simplify when the triangle becomes a right triangle, an isosceles triangle, and an equilateral triangle?

A general formula for the area of a triangle is given in which  $b$ ,  $c$  and  $c'$  are the three respective sides and  $s$  is an abbreviation for half the sum of the sides.

$$A = \sqrt{s(s-b)(s-c)(s-c')}$$

$$s = \frac{b+c+c'}{2}.$$

Substitute  $\frac{bh}{2}$  for  $A$  and solve for  $b$  and  $h$ .

**4. Area of a Trapezoid.** Construct a trapezoid, designating its parallel sides, i.e., bases by  $(b)$  and  $(e)$  and the altitude by  $(h)$ , the diagonals by  $(k_1)$  and  $(k_2)$  and the angles opposite the diagonals by  $(\alpha)$  and  $(\beta)$  respectively.

Write all the possible formulas for the area and the diagonals in terms of the mentioned parts.

**5. Area of a Regular Polygon.** What is the magnitude of the central angle which is subtended by half the side?

In Fig. 68 the side is designated by  $b$ , the apothem by  $a$  and the radius of the circumscribing circle by  $r$ .

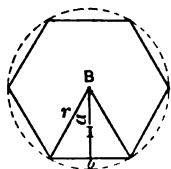


FIG. 68.—A Regular Polygon.

A regular polygon is divided into  $n$  equal isosceles triangles ( $I$ ) and therefore its area is computed by multiplying the area of ( $I$ ) by  $n$ , which is also the number of sides. Formulate the law for the area of a polygon in terms of  $a, b$ ,

and  $n$ , also in terms of  $b, r, n$ , and the central angle subtended by the half side. In the above equations substitute  $(p)$  the perimeter, for  $bn$  and interpret the resulting equations.

**6. Area of a Circle.** The circumference ( $c$ ) of a circle equals  $\pi$  times its diameter ( $d$ ). The value of  $\pi$  is an unending decimal and is approximately represented by 3.1416 and by  $\frac{22}{7}$ .

The area of a circle equals one-half the product of its circumference times its radius ( $r$ ). Formulate the law for the area of a circle. Substitute for  $c$  in terms of  $d$  and solve for  $d$  and interpret. Substitute for  $d$  in terms of  $r$  and interpret the equations of the area and of the circumference in terms of  $r$ .

A central angle of  $1^\circ$  intercepts an arc equal to  $\frac{1}{180}$  of the circumference. What part of the circumference will

is intercepted by a central angle of  $\theta^\circ$ ? A central angle of  $57.3^\circ$  (approximate) will intercept an arc of  $\frac{57.3}{360} = \frac{1}{2\pi}$  of the circumference. The central angle of  $57.3^\circ$  is called a radian of angle and its intercepted arc is called a radian of arc. Show that a radian of arc equals the radius of the circle? The length of an arc ( $L$ ) equals the radius times the number of radians ( $\theta$ ) in the arc.  $L = r\theta$ .

**7. Area of an Annulus or Circular Ring.** An annulus is the figure formed by a circle interior to another circle. The area of an annulus is the difference in areas between the outer and inner circle. In Fig. 69  $d_1$  and  $d_2$  are the respective diameters. Formulate the law for the area of an annulus in terms of the diameters and also in terms of the radii.

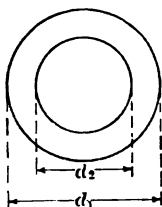


FIG. 69.—An Annulus.

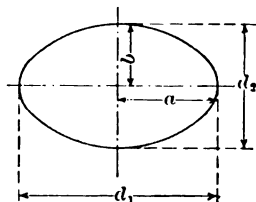


FIG. 70.—An Ellipse.

**8. Area of an Ellipse.** The area of an ellipse equals  $\frac{\pi}{4}$  times the product of the major diameter ( $d_1$ ) times the minor diameter ( $d_2$ ). Formulate the law for the area of an ellipse.

In Fig. 70 half the major and minor diameters, i.e., the semimajor and semiminor diameters, are designated by ( $a$ ) and ( $b$ ) respectively. Formulate the area of an ellipse in terms of  $a$  and  $b$ .

**9. "The squaring of the circle"** is a problem handed down from the classic days of Greece. It was an attempt to determine by the aid of the straightedge and compass a square exactly equal in area to a circle. Until the sig-

nificance of  $\pi$  was appreciated this problem is unsettled.

The length of an arc of a circle is given in (3)  $L$ =length of the arc,  $c_1$  represents the length of the subtended chord, and  $c_2$  the length of the chord subtended by half the arc. This is known as H approximation.

$$(3) \quad L = \frac{8c_2 - c_1}{3}.$$

Interpret (3).

Another approximate formula is given in (4)  $c_4$  is the length of the chord subtended by one-fourth the arc.

$$(4) \quad L = \frac{1}{45}(c_1 - 40c_2 + 256c_4).$$

Interpret (4).

**Ex. 1.** Construct arcs of  $30^\circ$ ,  $60^\circ$ ,  $90^\circ$ ,  $120^\circ$ ,  $150^\circ$ ,  $270^\circ$  with a radius of 10 ins. Measure the values of  $c_1$ ,  $c_2$ , and  $c_4$  and substitute in (3) and (4). Compare the results of (4) by computing the arcs in terms of their respective angles. See Table VIII for the values of chords for unit radius.

**10.** The rectification of a circular arc is the determination of the length of a straight line which shall be equal to the given arc. In Fig. 71 the rectification of the circular arc  $AVB$  is represented by  $FS$ .

The construction suggested by Ceradini is as follows:  $OV$  is a radius constructed perpendicular to the horizontal diameter  $AB$ ; the tangents  $VC$  and  $BC$  from  $V$  and  $B$  respectively, intersect at  $C$ ; the line  $AC$  is constructed equal to  $AB$  and terminate at  $F$ . With  $V$  as a center strike an arc equal to the radius and intersecting the circumference at  $D$ ; draw the line  $VS$  through  $V$  and  $D$  intersecting  $CB$  at  $S$ ; join  $F$  and  $S$ . A proportional part of the semicircumference can be obtained by dividing  $FS$  proportionately.

A second method of rectification given by Rankine is presented in Fig. 72. An arc  $AV$  with center  $O$

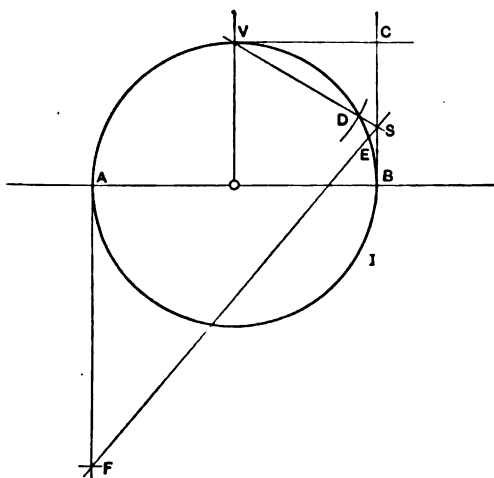


FIG. 71.—Rectification of an Arc.

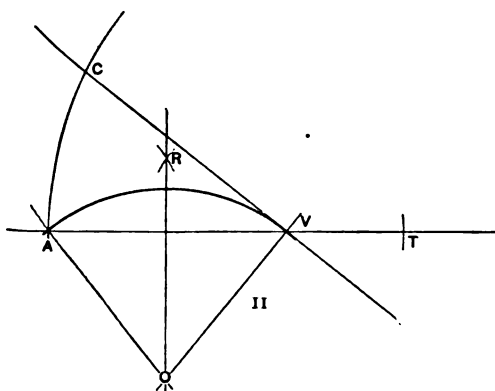


FIG. 72.—Rankine's Method.

to be rectified. Draw the chord  $AV$  and bisect it. Extend  $AV$  to  $T$ , making  $AT = \frac{3}{4}AV$ . Draw  $CV$  tangent

to the arc and therefore perpendicular to  $OV$  at  $V$ . With  $T$  as a center strike the arc  $AC$ . Then arc  $AV = CV$ .

A third method of rectifying an arc  $AC$  which was suggested by Snell is as follows: Construct the semicircle  $ACB$  with diameter  $AB$ , and the tangent  $AE$  at  $A$ . Extend the diameter through  $B$  to  $D$  making  $BD = \frac{1}{2} AB$ . A secant passing through  $D$  and  $C$  cuts the tangent  $AE$  at  $E$ . Then  $AE = \text{arc } AC$ .

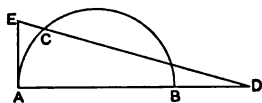


FIG. 73.—Snell's Method.

The secant cuts the semicircle at the second point  $C'$ . If the above construction is modified so that  $C'D = \frac{1}{2} AB$  the approximation is slightly in excess whereas the former is slightly deficient.

It may be necessary to determine the length of the arc of a circle which shall be equal to a given line. The method is presented in Fig.

74. The given line  $BX$  is a portion of a tangent to the circle at  $B$ , the point which marks the beginning of the equal arc. Locate  $L$  so that  $BL = \frac{1}{4} BX$ . From  $L$  as a center strike an arc of radius  $LX$  intersecting the circle at  $A$ . Then arc  $AB = BX$ .

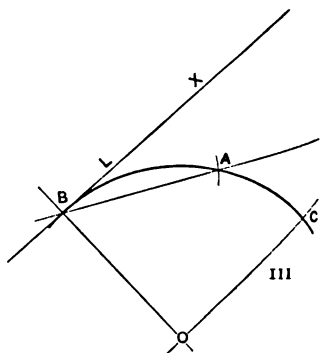


FIG. 74.—Circularization of a Line.

11. The approximate perimeter of an ellipse is given in terms of the semimajor diameter ( $a$ ) and semiminor diameter ( $b$ ) in equations (5), (6) and (7).

$$(5) \quad L = \pi(a+b).$$

$$(6) \quad L = \pi\sqrt{a^2 + b^2}.$$

$$(7) \quad L = \frac{\pi}{2}(a+b+\sqrt{a^2+b^2}).$$

transform (5) and (6) and solve for  $a$  and  $b$ .

will give an answer .5 per cent too small and (6) will give an answer .5 per cent too large, whereas (7) will give an answer within  $\frac{1}{100}$  per cent of the truth. Substitute  $a=4$ ,  $b=3$  in (5), (6) and (7).

The **sector of a circle** is the area bounded by two radii and an arc. Express the area of the sector  $AOD$  of Fig. 75, in terms of  $r$  and  $b$ .  $b$  represents the arc  $ACD$  and  $r$  represents the radius. The area of a sector equals one-half the product of the radius times the square of the sine of the central angle. Substitute the value of  $b$  for the arc and the central angle  $\phi$  according to 5. Interpret the following relation.

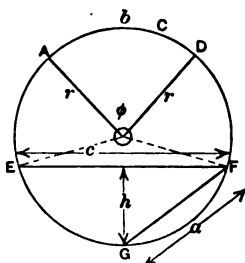


FIG. 75.

$$\frac{\text{Area of sector}}{\text{Area of circle}} = \frac{\phi}{360} = \frac{\text{arc } b}{\text{circumference } 2\pi r}.$$

A **segment of a circle** is the area bounded by a chord and its intercepted arc. The sector  $EOF$  of Fig. 75 is a sector of a circle with a chord  $EF$  designated by  $c$  and a height or rise designated by  $h$ . Express the area of segment  $EGF$  by subtracting the area of the triangle  $EOF$  from the area of the sector  $EOF$ . Show that the length of the chord  $c$  can be expressed by (8).

$$c^2 = 4h(2r - h).$$

Show that (9) expresses the length of the chord  $a$ , which subtends the central angle  $\theta$ .

$$a = 2r \sin \frac{\theta}{2}.$$

The approximate formula for the area of a segment is given in (10) and (11), and the exact formula in (12).

$$(10) \quad A = \frac{h^3}{2c} + \frac{2ch}{3}.$$

$$(11) \quad A = \frac{2}{3} \sqrt{h(c^2 + \frac{8}{3}h^2)}.$$

$$(12) \quad A = \frac{r^2}{2}(\theta - \sin \theta).$$

**Ex. 2.** Compare the results of (10) and (11) with (12), for a circle of 10 ins. diameter, in which the chord subtends a central angle of  $60^\circ$ . Transform (11) and solve for  $c$ .

**14.** The area of the segment of an ellipse whose chord  $c$  is perpendicular to one of its diameters ( $d$ ) equals the product of its major and minor diameters times the area ( $A_c$ ) of the segment of a circle of diameter ( $d$ ) and whose chord equals  $c$ . Formulate this law.

**Ex. 3.** Show that (13) expresses the diameter ( $d$ ) of a circle whose area equals that of an ellipse of diameter  $d_1$  and  $d_2$ .

$$(13) \quad d = \sqrt{d_1 d_2}.$$

Interpret (13), transform and solve for  $d_1$  and  $d_2$ .

**15.** The area of the segment of a parabola Fig. 76, equals two-thirds the area of the circumscribing rectangle. Formulate this law in terms of  $h$  the height of the parabola, and  $a$  the chord of the parabola. The height of the parabola is called its ordinate and half the chord is called its abscissa. The mid point of the base of the rectangle is tangent to the parabola at a point called its vertex.

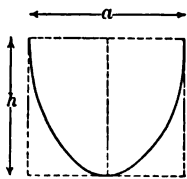


FIG. 76.—A Parabola.

16. The approximate length of the parabola is given by (14).

$$(14) \quad L = 2\sqrt{\frac{a^2}{4} + \frac{4h^2}{3}}.$$

Interpret (14), transform and solve for  $a$  and  $h$ .

17. The approximate area  $A$  of the segment of an hyperbola is given in (15) and the approximate length ( $L$ ) which is measured from the vertex is given in (16).

The semimajor and semiminor diameters of the hyperbola are represented by  $a$  and  $b$  respectively and the ordinate and abscissa of the curve are represented by  $y$  and  $x$  respectively. See Fig. 135.

$$(15) \quad A = .16 \frac{b}{a} x (\sqrt{49ax + 35x^2} + \frac{4}{3} \sqrt{ax}).$$

$$(16) \quad L = y \left( 1 + \frac{a^2 x}{(1.5a + 2.1x)b^2 + .9a^2 x} \right).$$

Transform and solve for  $y$  and  $x$ .

Simplify (15) and (16) by factoring and by the application of the axiom of fractions.

18. Area of an Irregular Figure. A figure bounded by straight lines may be decomposed into triangles, rectangles, and trapezoids. The sum of the areas of the simpler constituent figures equals the area of the entire figure.

When the boundaries include arcs of circles the figure will include sectors and segments.

19. When the outline is irregular or bounded by curves the area may be determined by one of the following methods:

- (a) by using a planimeter,
- (b) by using squared paper,
- (c) by weighing,
- (d) by mean ordinate rule.

Very close approximations may be obtained by other methods more commonly known as:

- (e) mid ordinate rule,
- (f) trapezoidal rule,
- (g) Simpson's one-third rule,
- (h) Simpson's three-eighths rule,
- (i) Durand's rule,
- (j) Weddle's rule.

(a) *The Planimeter.* The planimeter is an instrument which enables us to measure the area of a figure in square inches or any other units of area. It consists primarily of two rigid arms  $AB$  and  $BC$  pivoted at  $B$ .

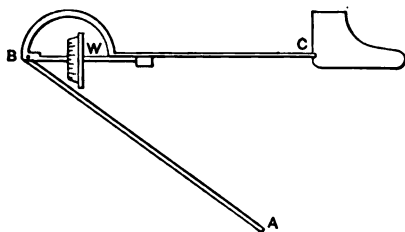


FIG. 77.—A Planimeter.

The extremity  $A$  of arm  $AB$  is fixed by a needle point to the table. The tracing point  $C$  of arm  $BC$  is carried around the outline of the figure. This motion causes rotation of the wheel  $W$  which is graduated on its rim. The difference of the readings of the wheel before and after using the instrument, when multiplied by a suitable constant gives the area of the figure. The constant ( $K$ ) of the instrument may be determined by tracing the outline of known area  $A$ . If the wheel readings before and after tracing are  $d_1$  and  $d_2$  respectively then,

$$K = \frac{d_2 - d_1}{A}.$$

The **horse-power** of an **engine** may be determined from the indicator diagram which shows the performance of an

ngine during a single stroke. The vertical scale of the diagram in pounds per inch is the calibrated test number of the spring used in the indicator. The horizontal scale of the diagram per inch is the ratio of the length of the stroke in feet to the length of the card in inches. What is the value in foot-pounds of each square inch of the indicator diagram. The total area of the indicator diagram multiplied by the effective area of the piston in square inches times the number of strokes per minute divided by 33000 gives the horse-power of the engine. The planimeter can be used to determine the area of the indicator diagram.

(b) *Use of Squared Paper.* The figure may be drawn upon cross-section paper. This is usually ruled horizontally and vertically with 10 or 20 divisions per square inch. The number of full squares enclosed by the figure is counted and with a little care and practice the remaining fractional squares can be closely approximated.

(c) *Weighing of Areas.* After the figure has been traced upon heavy cardboard or thin sheet metal of uniform thickness its outline is cut. The figure is then weighed and its weight compared to the weight of similar material of known area.

(d) *Mean Ordinate Rule.* The area of the irregular, Fig. 78, is equivalent to a rectangle, whose base is  $AB$

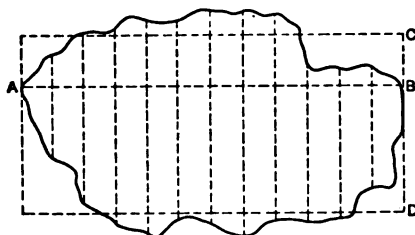


FIG. 78.—Area by Mean Ordinate Rule.

which is also the length of the figure, and whose altitude  $CD$  is the mean of all the ordinates, i.e., perpendiculars

drawn to  $AB$  within the figure. The mean of the ordinates extending above  $AB$  is  $CB$  whereas the mean of the ordinates extending below  $AB$  is  $BD$ . The product of the mean ordinate  $CD$  of the whole figure multiplied by its length  $AB$  is the area of the figure.

How may we determine the mean ordinate of a figure from a knowledge of the value of its area.

The mean effective pressure of an engine is determined from its indicator diagram, by dividing the area of the diagram by its length. The horse-power of an engine is given by (17), in which  $P$  is the mean effective pressure per square inch,  $A$  the effective area of the piston in square inches,  $L$  the length of the stroke in feet, and  $N$  the number of strokes per minute.

$$(17) \quad \text{Horse-power} = \frac{PLAN}{33000}.$$

(e) *Mid Ordinate Rule.* The base  $AB$  of the irregular Fig. 79, is divided evenly into equal spaces. Ordinates

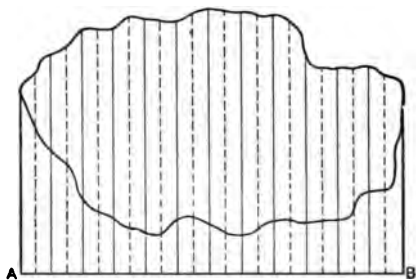


FIG. 79.—Area by Mid Ordinate Rule.

which are represented by dotted lines are erected at the mid points of the spaces.

The sum of these ordinates divided by the number erected gives an approximation to the mean ordinate of the figure, the accuracy of which increases with the number

of measured ordinates. It is sufficient to measure only the segments of the ordinates which lie within the figure.

(f) *Trapezoidal Rule.* The base of a figure is divided into any number of equal divisions. In Fig. 80, the base or line of greatest length is divided into twelve equal divisions. At these points ordinates are erected dividing the figure into trapezoids (approximate) of equal width  $S$ . The area of each trapezoid equals the product of  $S$  times the half sum of its bounding ordinates.

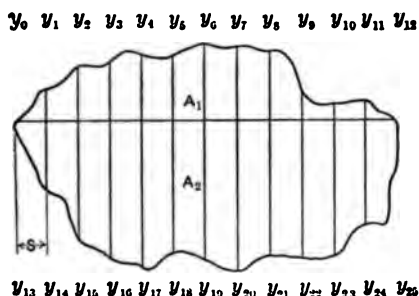


FIG. 80.—Area by Trapezoidal Rule.

The ordinates measured above the base are designated by  $y_0, y_1, y_2, \dots, y_{12}$  and those measured below the base by  $y_{13}, y_{14}, \dots, y_{25}$ .

Show that the total area of the Fig. 80 equals  $S$  times the sum made up by taking one-half the outside ordinates, to which is added the sum of the intermediate ordinates. The two segments of an ordinate need not be measured separately.

For a figure divided into seven equal parts the area is expressed in (18).

$$(18) \quad \text{Area} = S \left\{ \frac{1}{2}(y_0 + y_6) + y_1 + y_2 + y_3 + y_4 + y_5 \right\}$$

(g) *Simpson's One-third Rule.* A closer approximation for the area than can be obtained by the trapezoidal rule is given in (19) and is known as the one-third rule for an

ordinates are used and designated by the  $y$  :

$$(20) \quad \text{Area} = \frac{S}{3} \{y_0 + y_8 + 4(y_1 + y_3 + y_5 + y_7) +$$

For the special case of seven ordinates rule is more accurate. The one-third rule when the curve approaches a parabola.

(h) *Simpson's Three-eighths Rule.* For 8 of ordinates a close approximation is given in terms of the sum of the end ordinates;  $M$  represents the sum of the two middle ordinates;  $R$  represents the remaining ordinates.

$$(21) \quad \text{Area} = \frac{3S}{8} (E + 2M + 3R).$$

(i) *Durand's Rule.* A close approximation (22).  $E$  represents the sum of the end ordinates;  $P$  represents the sum of the penultimates (the second to last ordinate);  $R$  represents the remaining

$$(22) \quad \text{Area} = S \{ .4 E + 1.1 P + R \}$$

(j) *Weddle's Rule.* When the boundary (23) is especially applicable  $R$  represents

**20. The Cycloid.** The cycloid is a curve generated, i.e., traced by a point on a circle which is rolled on a straight line called the base. The length of the base equals the circumference of the generating circle. The length of the cycloid equals four times the diameter of the generating circle. The area between the cycloid and the base equals three times the area of the generating circle.

**21. The Helix.** The helix or curve of a screw thread is generated by advancing and rotating a point on a cylinder. The length of a helix is given in (24).  $C$  represents the circumference of the cylinder;  $P$  represents the pitch of the screw or the axial advance of the point in one revolution;  $N$  represents the number of advances or the number of revolutions about the axis.

$$(24) \quad \text{Length} = N\sqrt{C^2 + P^2}.$$

Transform and solve for  $N$ ,  $C$ , and  $P$ .

**22. The Spiral.** The spiral is a curve which is generated by a point rotating about a pole, i.e., a fixed center, so that its polar distance is continuously increased. A flat spring is a plane spiral in which the polar distance is increased uniformly. Its length is given in (25) in which  $D_1$  and  $D_2$  are the greater and lesser diameters and  $N$  the number of convolutions.

$$(25) \quad \text{Length} = \pi N \left( \frac{D_1 + D_2}{2} \right).$$

Transform and solve for  $N$ ,  $D_1$  and  $D_2$ .

**23. The Conical Spiral.** When the spiral motion is traced on a cone it is called a conical spiral. The length of a conical spiral is given in (26) in which  $D_1$  and  $D_2$  are the greater and lesser diameters,  $N$  the number of convolutions and  $h$  the height of the polar axis.

$$(26) \quad \text{Length} = \sqrt{\left\{ \pi N \left( \frac{D_1 + D_2}{2} \right) \right\}^2 + h^2}.$$

Transform (26) solving for  $N$ ,  $h$ ,  $D_1$  and  $D_2$ .

24. The volume of any figure is its ratio to a unit volume. The **unit of volume** may be any figure of volume but is generally considered as a **cube**. The twelve edges of the unit cube are 1 unit in length. The six faces or bounding surfaces are 1 sq.in. in area. At each corner there are three mutually perpendicular edges which extend in the three distinct directions right-left, up-down, and front-rear. The dimensions of the cube are the lengths of three edges measured in the three directions. The measure of the extension of a figure of volume in these three directions are variously termed the length, breadth, thickness, height, width and depth.

In Fig. 81, *I* is the unit cube. The bottom surface on

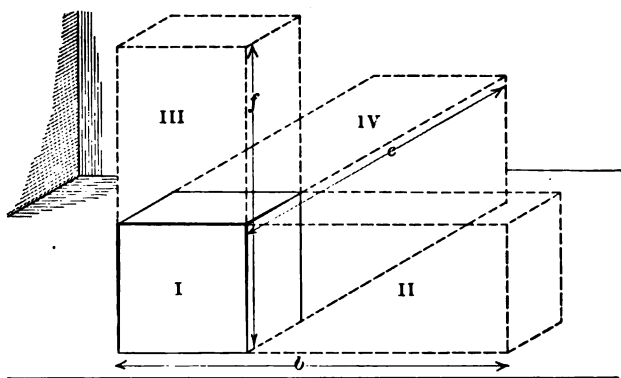


FIG 81.—The Cube and its Extensions.

which the cube is supposed to rest is called its lower **base** to distinguish it from the top or upper base which is parallel and equal to it. The vertical faces which include the front, rear, and the two end surfaces are called by the collective name of **lateral surface**.

If the length of an edge of the cube is **one inch**, the volume of the cube is **one cubic inch (cu.in.)**. If the length of the edge of a cube is **one centimeter**, then the vol<sup>m</sup>

the cube is one cubic centimeter (cu.cm.). The cubic unit takes its name from the linear unit of its edge.

A new solid called a **rectangular prism** is formed by extending any one of the three sets of the four parallel edges of cube I Fig. 81, providing the three pairs of opposite faces retain their parallelism. The vertical extension produces III, which has a base 1 unit square and a height of  $f$  units. III will have  $f$  times the volume of I. In II the extension has been to the right side and in IV the extension has been to the rear. The volume of II will equal  $b$  times the volume of I and the volume of IV will equal  $c$  times the volume of I.

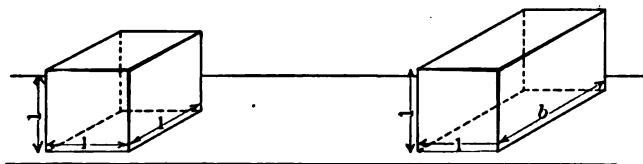


FIG. 82.—The Rectangular Prism.

In Fig. 82 we have a rectangular prism built by extending the cube to the rear.

The dimensions of the cube are  $1 \times 1 \times 1$ , and the dimensions of the prism are  $1 \times 1 \times b$ . The volume of the prism equals  $b$  times the volume of the cube.

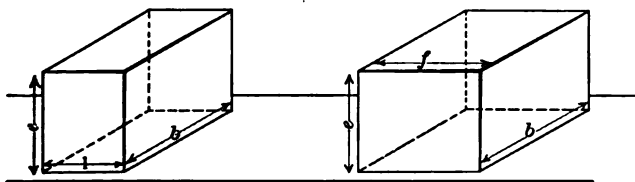


FIG. 83.—Extensions of a Prism.

If the prism of Fig. 82 is extended in the vertical shown in the left of Fig. 83, its dimensions the latter has  $e$  times the volume of the

prism in Fig. 82, and therefore  $eb$  times the volume of the unit cube. The extension of the left prism of Fig. 83 produces a new prism shown in the right of Fig. 83. The dimensions of the right-hand figure are  $e \times f \times b$ , and therefore it has  $f$  times the volume of the left-hand figure, and therefore  $e \times f \times b$  times the volume of the unit cube. But  $e \times f \times b$  is the product of its three dimensions.

*Observation.* The numeric value of the volume of a rectangular prism is the product of the numeric values of its three dimensions.  $\text{Volume} = \text{length} \times \text{breadth} \times \text{thickness} = \text{height} \times \text{width} \times \text{depth} = \text{area of a base} \times \text{altitude}$ .

**25. Volume of a Cube.** A cube is also a rectangular prism in which the faces are equal squares and the edges are of equal dimensions. The volume ( $V$ ) of a cube equals the cube of the length of its edge ( $e$ ). Formulate this law and solve for  $e$  and interpret the resulting equation.

Fig. 84 represents a rectangular prism  $ABCDEFGH$  which has been cut through the opposite edges,  $HD$  and

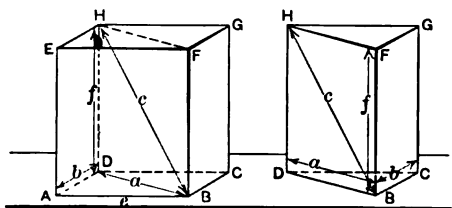


FIG. 84.—Bisection of a Cube.

$FB$ , dividing the upper and lower bases into two equal areas, by lines  $HF$  and  $DB$  respectively. The square prism  $ABCDEFGH$  is thus divided into two equal triangular prisms  $BCDHFG$  and  $ABDHEF$ . The triangular prism  $BCDHFG$  is reproduced to the right of the square prism, in order that it may be shown that both the square and the triangular prisms have equal dimensions. The volume of the triangular prism  $BCDHFG$  equals one-half the volume of the square prism  $ABCDEFGH$ . Formulate

the law for the volume of a triangular prism in terms of its three dimensions and also in terms of the area of its base and altitude.

The diagonal of a square prism is represented by  $c$  in Fig. 84. It is also the diagonal of the parallelogram  $DBFH$ . Show that  $c$  is expressed by (27).

$$(27) \quad c = \sqrt{e^2 + b^2 + f^2}.$$

Interpret (27) and solve for  $e$ ,  $b$ , and  $f$ .

**26. Volume of an Oblique Prism.** An oblique prism differs from the rectangular prism in so far as the lateral faces are parallelograms, some of which are oblique to the bases. In Fig. 85,  $E_1F_1G_1H_1ABCD$  is an oblique square prism stand-

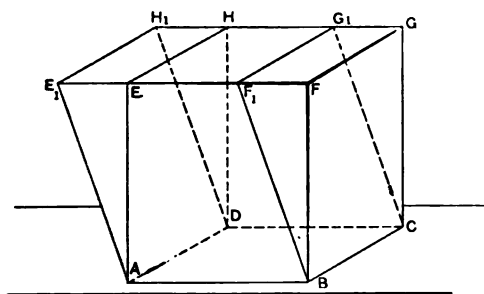


FIG. 85.—An Oblique Prism.

ing on the same base  $ABCD$  as the rectangular square prism  $EFGHABCD$ . It may be imagined that the upper base has been slid along in its own plane so that  $FG$  takes the new position  $F_1G_1$  and  $EH$  takes its new position  $E_1H_1$ . The altitude of both prisms is the same, as the altitude is the perpendicular distance between the bases. Both prisms have the same dimensions. Both the square and oblique prisms are composed of one of the two equal triangular prisms  $F_1FBCG_1G$  and  $E_1EADH_1H$ , and both have in common the solid figure  $ABCDHEF_1G_1$ . Therefore by sum of

parts axiom, addition axiom and equality axiom we conclude that the volumes of the two square prisms are equal.

A prism has an upper and a lower base which are parallel polygons of equal area and similar in shape. The sides or lateral faces of all prisms are parallelograms. The different prisms take their descriptive name from the shape of their bases. A right, i.e., a rectangular prism is one having its lateral edges perpendicular to the bases.

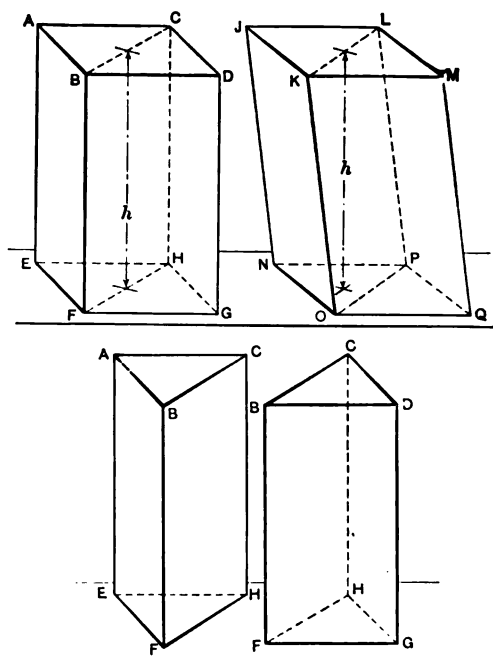


FIG. 86.—Quadrangular Prisms.

27. The quadrangular prisms in Fig. 86, have an equal altitude  $h$  and equal areas for their bases. The two prisms are divided into two sets of equal triangular prisms by sections passed through the diagonals of their bases. The

triangular prisms of both sets are respectively equal. Right and oblique triangular prisms of equal altitude and equal bases have equal volumes. Make a drawing to show the two triangular prisms which constitute the quadrangular prism  $NOQPJKML$ .

28. Fig. 87 illustrates a right and an oblique triangular prism. They have an equal altitude  $h$  which is the perpendicular distance between the planes of their respective

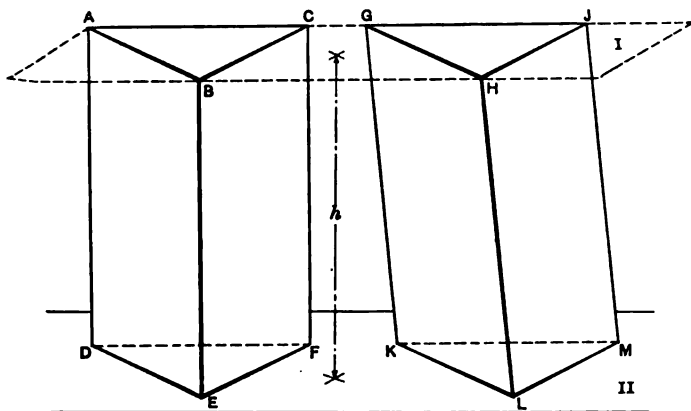


FIG. 87.—Triangular Prisms.

bases. Formulate the law for the ratio of the volumes of two triangular prisms of equal altitude in terms of their basal areas and also in terms of their basal dimensions.

29. Fig. 88 shows two pentagonal prisms with equal and similar bases and equal altitudes. The bases are divisible into three mutually equal triangles. The right prism is divided into three triangular prisms shown in the lower view. The sum of the volumes of the triangular prisms equals the volume of the pentagonal prism. The area of each triangular prism is the product of its base times altitude. Therefore the sum of the areas of the triangular bases times their common altitude equals the volume of the pentag-

onal prism. Substitute for the sum of the areas of the bases of the triangular prisms the area of the base of the pentagonal prisms. Put this proof in algebraic form designating the area by  $A$ ,  $A_1$ ,  $A_2$ ,  $A_3$  for the respective pentagonal and triangular prism. Make a drawing of the triangular

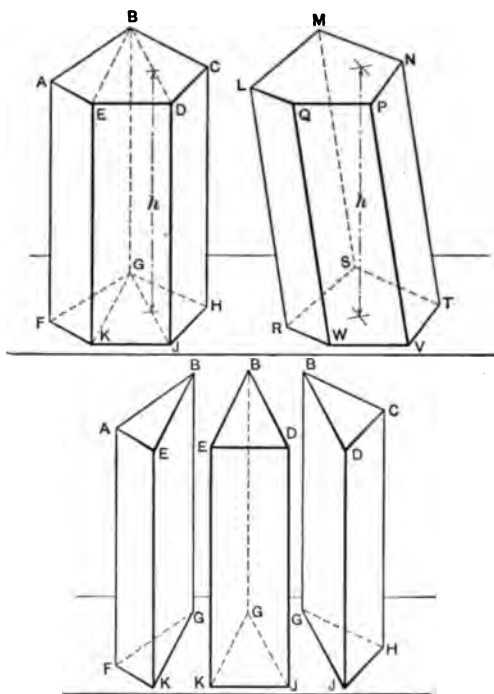


FIG. 88.—Pentagonal Prisms.

prisms which are obtained by decomposing the oblique prism in Fig. 88.

30. Fig. 89 represents **right** and **oblique hexagonal prisms**, with the component triangular prisms of the right prism in the lower view. The two hexagonal prisms have an equal volume because they have an equal

altitude and are decomposable into triangular prisms of equal volume. The triangular prisms may be further decomposed into other prisms of the same height, and these may be united to form a new total prism, differing in shape from the original figure. Therefore two prisms

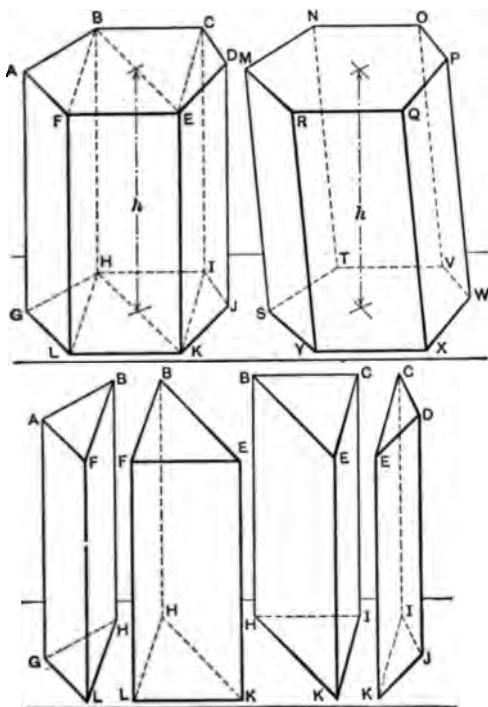


FIG. 89.—Hexagonal Prisms.

have an equal volume when they have equal altitudes with bases of equal areas. The bases need not be similar figures. For prisms of equal altitude the sum of two or more triangular prisms, is to the total prism or to any other sum of triangular prisms, as the sums of the respective areas of the bases. Build new hexagonal prisms from the triangular prisms.

If the bases of the total prisms are regular polygons of  $n$  sides then sections which are constructed through the axis and lateral edges will divide the total prism into  $n$  equal triangular prisms. If  $a$  and  $p$  represent the apothem and perimeter of the base of a regular prism then its volume is given in (28).

$$(28) \quad \text{Volume} = \frac{aph}{2} = \text{area of base} \times \text{altitude},$$

31. Fig. 90 represents a **right** and an **oblique** cylinder. The lateral surface of a cylinder spreads into a parallelo-

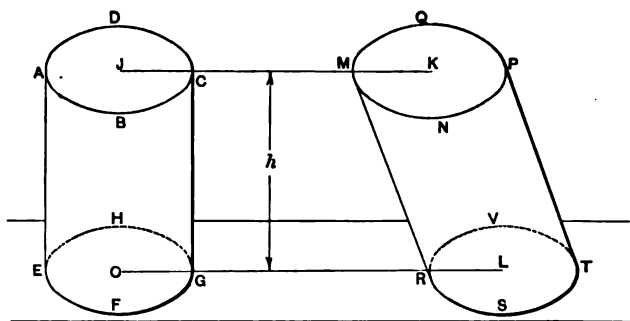


FIG. 90.—Right and Oblique Cylinders.

gram and its two bases are equal parallel figures. Therefore a cylinder may come under the classification of prisms, and accordingly the volume of a cylinder equals its base times its altitude. Suppose regular prisms were inscribed in these cylinders and the number of sides of the bases continually increased. Then the area of the polygons  $A_p$ , representing the bases of the prism, would gradually approach the limiting (lim) value which is the area  $A_c$  of the bases of the cylinder. The volumes  $V_p$ , of the inscribed prisms would gradually approach their limiting value which is the volume  $V_c$  of the cylinder.

$$\begin{aligned} \therefore (29) \quad \lim V_p &= V_c \\ (30) \quad \lim A_p &= A_c \end{aligned} \left. \vphantom{\begin{aligned} \lim V_p &= V_c \\ \lim A_p &= A_c \end{aligned}} \right\} \text{construction of the inscribed prisms.}$$

$$(31) \quad \text{but } V_p = hA_p, \text{ --- volume of a prism.}$$

$V_p$  and  $hA_p$  are variables, i.e., quantities continuously changing and approaching some assigned limit. Two variable quantities which are always equal approach equal limits. This is known as the Fundamental Theorem on Limits.

$$\therefore (32) \quad \lim V_p = h \lim A_p \quad \text{theorem on limits.}$$

$$\therefore (33) \quad V_c = hA_c \quad \text{substitution (29)(30) in (32).}$$

*Observation.* The interpretation of (33) states that the volume of any cylinder, like the volume of any prism, is the product of the area of its base times its altitude.

Write a formula for the volume of a cylinder in terms of its altitude, perimeter and the radius of its base.

A cylinder is any prismatic figure, in which the perimeter of the base is a closed curve. An element is a line in the lateral surface which joins corresponding points in the two bases.

**32.** The total surface of any prismatic figure equals twice the area of the base plus the lateral area. Formulate this law and then substitute the value of the basal and lateral areas in terms of the perimeter of the base. In the case of the cylinder what is the total area expressed in terms of the radius of the base and the altitude?

**33. The Volume of a Pyramid.** A pyramid has one base and its lateral edges converge to a point called the apex.

Fig. 91 represents a cube with its four diagonals 17, 28, 35 and 46 intersecting in a common point. The point of intersection is the common apex of six equal pyramids. Each pyramid has one of the faces of the cube for its base

and its altitude is one-half the length of an edge of the cube. The volume of the pyramid is one-sixth of the volume of the cube of twice the altitude. Therefore the volume of a pyramid is one-third of the volume of a prism built on the same base and having an equal altitude. Therefore the

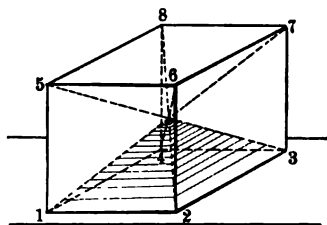


FIG. 91.—Decomposition of a Cube.

volume of a pyramid is expressed in (34) where  $h$  is the altitude and  $A$  the area of the base.

$$(34) \text{ Volume of pyramid} = \frac{hA}{3} = \frac{\text{altitude} \times \text{basal area}}{3}.$$

Pyramids are equal when they have equal bases and equal altitudes.

34. Fig. 92 represents a cube 12345678 which has been bisected, i.e., divided into two triangular prisms by the cut-

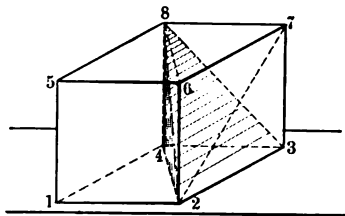


FIG. 92.—Decomposition of Prism.

ting plane 4268. In the triangular prism 423867 the diagonals 82, 27 and 38 have been drawn through the three lateral faces 4268, 6237 and 4378 respectively. Cutting

planes passed through each pair of diagonals will divide the triangular prism into three equal pyramids 423-8, 867-2 and 873-2 as shown separately in Fig. 93. Pyramids 423-8 and 867-2 have the equal bases 423 and 867 and the equal altitudes 48 and 62. Pyramid 423-8 may be designated 438-2. Pyramids 438-2 and 873-2 have the equal bases 438 and 837 and the equal altitudes 23. Therefore

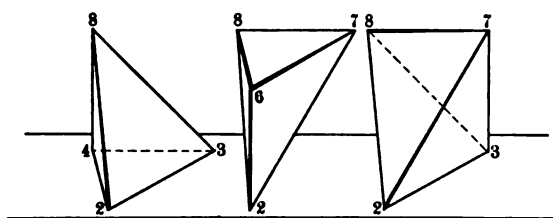


FIG. 93.—Equivalent Pyramids.

the three pyramids are equivalent and the volume of each is equal to one-third of the volume of the triangular prism.

*Observation. The volume of a pyramid is unaffected by change in the shape of the base and the position of its apex provided the product of its basal area times its altitude remains constant.*

**35. Volume of a Cone.** A cone may be regarded as a pyramid with a curved figure for its base. The cone has one-third the volume of a cylinder which has an equal base and altitude.

$$(35) \quad \text{Volume of cone} = \frac{Ah}{3} = \frac{\text{basal area} \times \text{altitude}}{3}.$$

A line joining any point in the perimeter of the base with the apex is called an element.

*Relations of Prismatic and Pyramidal Pairs.* Two prisms, two cylinders, two pyramids, two cones, a prism and a cylinder or a pyramid and a cone, have their volumes in the same ratio (36) as the products of their respective bases and altitudes.

$$(36) \quad \frac{V_1}{V_2} = \frac{A_1 h_1}{A_2 h_2}.$$

$$(37) \quad \frac{V_1}{V_2} = \frac{A_1}{A_2}.$$

$$(38) \quad \frac{V_1}{V_2} = \frac{h_1}{h_2}.$$

$$(39) \quad \frac{h_1}{h_2} = \frac{A_2}{A_1}.$$

Show that for the above pairs of figures, (37) is true when they have equal altitudes; (38) is true when they have equal bases; (39) is true when they have equal volumes.

**36. The total surface of a pyramidal figure** equals the area of its base plus the lateral area. Express the area of a regular pyramid in terms of the perimeter and apothem of the base and in terms of the slant height. Derive another expression in which the slant height is replaced by its equivalent in terms of the altitude of the pyramid and the apothem of the base. Express the total area of a right circular cone in terms of the radius of the base and the altitude of the cone. A right pyramid and a right cone have the position of the apex vertically above or below the center of the base. Any other position of the apex gives an oblique pyramidal figure. A circular cone is a pyramidal figure with a circular base.

**37. Plane Sections.** In plane geometry lines may be either parallel or non-parallel. In geometry of space a third line (gouge) may be represented fulfilling neither of these familiar relations of the plane.

Three points or their equivalents a point and a line or two intersecting lines or two parallel lines determine a plane. Two parallel planes have no point in common. Two non-parallel planes have a line of intersection.

Parallel planes cut a prism, or a cylinder, or a pyramid, or a cone, or a sphere in sections which are similar figures.

The parallel sections of cylindrical figures are equal.

Pyramidal figures having equal altitudes and equivalent bases have equivalent sections at equal distances from their apexes. Figures are equivalent when they have equal areas or volumes.

**38. Frustum of a Pyramid or Cone.** The frustum of a pyramid or a cone is the figure intercepted between the base and a parallel section. The volume is expressed in (40) in which  $A_1$  and  $A_2$  are the lower and upper bases respectively

$$(40) \quad V = \frac{h}{3}(A_1 + A_2 + \sqrt{A_1 A_2}).$$

For the case of the frustum of a right-circular cone show that this simplifies to

$$(41) \quad V = \frac{\pi h}{3}(r_1^2 + r_2^2 + r_1 r_2).$$

In Fig. 94, I represents the frustum of a triangular pyramid, II being the part which has been removed. The

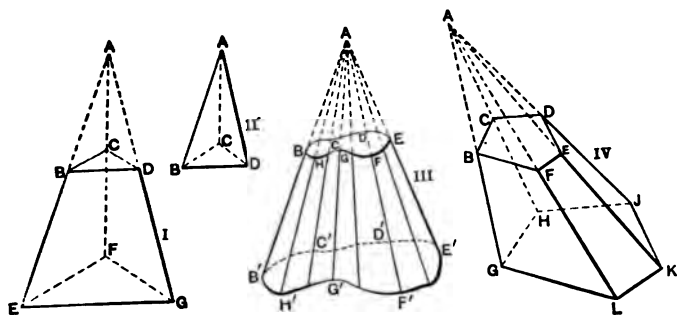


FIG. 94.—Frustums.

volume of the frustum is the difference between the volume of pyramid  $A-EFG$  and the volume of pyramid  $A-BCD$ . The base  $EFG$  is designated by  $A_1$  and the base  $BCD$  by  $A_2$ , and the altitude, i.e., the perpendicular distance between the bases is designated by  $h$ . The conic frustum III and the pentagonal frustum IV in Fig. 94 are equivalent. Their altitudes and their respective bases are equal in areas.

**Ex. 5.** Determine the formula for the *total surface* of the *frustum* of a regular pyramid in terms of its basal areas, basal

perimeters, and the slant height. Write the corresponding formula for the total area of the frustum of a right circular cone.

**39. Volume of a Prismoid.** A prismoid is a figure in which any two parallel plane figures are joined by quadrilaterals and triangles. The volume is expressed in (42) in which  $A_l$ ,  $A_u$  and  $A_m$  represent the lower, upper and mid sectional areas respectively.  $h$  represents the altitude of the figure. The majority of prismoids have quadrangular bases.

$$(42) \quad \text{Volume of prismoid} = \frac{h}{6}(A_l + A_u + 4A_m).$$

Deduce the formulas for the volume of a prism, a cylinder, a pyramid, and a cone from (40).

**40. Volume of a Wedge.** A wedge may be considered as a solid figure with a trapezoidal base.

The usual upper base is replaced by a line parallel to the base. Each extremity of the line is joined to two adjacent vertices of the base by the lateral edges. Therefore a wedge has five faces, two of which are triangular and three of which are quadrangular.

Deduce the volume of the wedge from the prismoid formula. Derive the volume of a wedge by dividing it into a triangular prism and a triangular pyramid.

**41. Ungula of a Solid.** A figure is said to be **truncated** when it is cut by a section which is not parallel to the base of the figure. The portion of the figure between the base and non-parallel section is called an **ungula**. A section of a prism or cylinder which is inclined at an angle  $\theta$  with the base, has an area equal to the product of the base times the secant of the angle.

The volume of a cylindric ungula shown in view II of Fig. 14 is equal to the product of the area of the base times the perpendicular distance between the base and the center of gravity of the non-parallel section.

A cylindric ungula is represented by  $A_1FJ$  in Fig. 99. The volume of the cylindric ungula which is above the section  $A_1A_2$  is given in (41).  $A_1F=l$  the length of the ungula;  $FJ=b$  the height of the base;  $c$ =the chord of the base;  $D$ =the diameter of the cylinder; and  $A$ =the area of the segment  $FJ$ , which corresponds to the base of the ungula. The base of the ungula is a segment of the circular end of the cylinder.

$$(43) \quad \text{Volume of ungula} = \frac{l}{2b} \left\{ \frac{c^3}{6} - A(D-2b) \right\}.$$

**42. Cylindric Surface.** A straight line (named the **element**) when moved parallel to itself, so that one of its extremities follows the perimeter of a plane figure generates a cylindric surface.

The **development** or **plane spread** of a cylindric surface is a rectangle.

$p$ =perimeter of the base of a cylinder or prism,  
 $H$ =length of the element.

$$(44) \quad A = pH = \text{area of cylindric surface.}$$

Interpret (44).

**43. Surface of a Cone.** A straight line (named the **element**) when moved so as to pass through a fixed point, and have one of its extremities follow the perimeter of a plane figure generates a conic surface.

The development of a conic surface is a circular sector. The development of a pyramidal surface is a sector of a polygon.

$p$ =perimeter of the base of a cone or pyramid,  
 $s$ =the length of the element.

$$(45) \quad A = \frac{ps}{2} = \text{area of the surface of a cone.}$$

Interpret 45.

**44. Figures of Revolution.** Any plane figure may be rotated about any line in its plane and thereby generate a figure of revolution.

Sections of figures of revolution perpendicular to the axis of rotation are either circles or figures formed by concentric circles. This principle is applied to the lathe.

A **rectangle** whose side is an axis of revolution generates a **cylinder**.

A **right triangle** whose side is an axis of revolution generates a **cone**.

A **circle** whose diameter is an axis of revolution generates a **sphere**.

A **parabola** whose diameter is an axis of revolution generates a **paraboloid**.

An **ellipse** whose diameter is an axis of revolution generates a **spheroid** or an **ellipsoid**.

An **hyperbola** whose diameter is an axis of revolution generates an **hyperboloid**.

Show that there are two varieties for each of the above figures of revolution.

Two such rotations may be compounded and this principle may be used upon the lathe to produce gun stocks, lasts, etc.

That portion of the perimeter of the rotating figure which is not perpendicular to the axis generates a surface of revolution. The line generates the surface of revolution. The surface generates the solid of revolution.

**45. Guldinus' Theorems.** (I) The area of a surface traced out by the revolution of a curve about an axis in its plane, is equal to the product of the perimeter of the curve times the distance moved through by its center of gravity.

(II) The volume generated by the revolution of such a curve is the product of the area enclosed by the curve times the distance moved through by the center of area or center of gravity.

**46. Solid Ring. Circular Section.** A circular disc centered at  $C$ , Fig. 95, rotates about the axis  $MN$  and generates a solid circular ring.

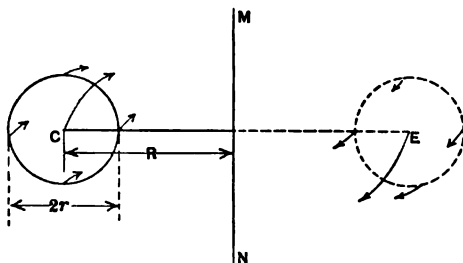


FIG. 95.—Section of a Circular Ring.

$A$  = the area of the surface of the ring and  $V$  the volume of the ring.

$r$  = radius of cross-section.

$R$  = the mean radius of rotation.

$$46) \quad A = 2\pi r \times 2\pi R, \quad (48) \quad V = \pi r^2 \times 2\pi R,$$

$$47) \quad A = 4\pi^2 r R, \quad (49) \quad V = 2\pi^2 r^2 R.$$

Interpret (47) and (49).

**47. Solid Ring. Rectangular Section.** A rectangle centered at  $C$ , Fig. 96, rotates about the axis  $MN$  and generates a solid rectangular ring.

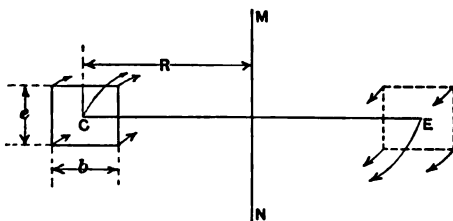


FIG. 96.—Section of Rectangular Ring.

Determine the area and volume of a solid ring of rectangular section.

**Ex. 6.** Sphere. Determine the surface and volume of a sphere by Guldinus' Theorems. The distance of the center of gravity of a semicircle measured from the axis equals  $\frac{4r}{3\pi}$ .

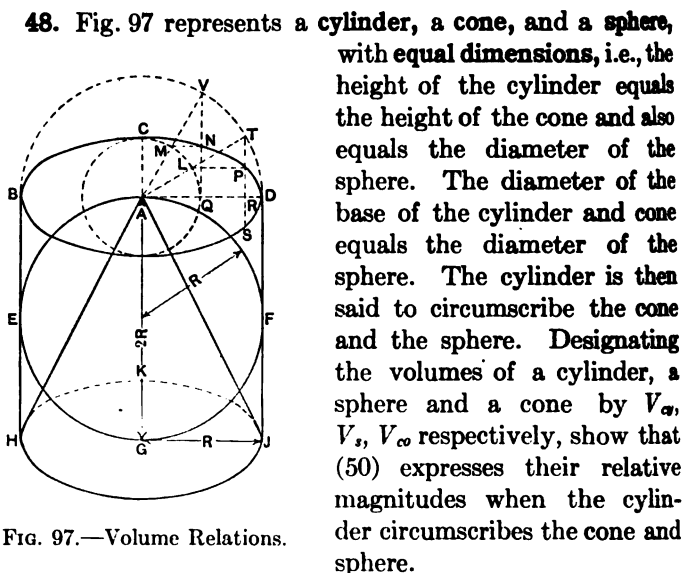


FIG. 97.—Volume Relations.

$$(50) \quad V_c : V_s : V_{co} = 3 : 2 : 1$$

49. When a cylinder or cone is represented in perspective as shown in Fig. 97, the bases are represented as ellipses. The construction of the ellipse is shown at the upper base of the figure. The center  $A$  of the upper base is also the center of a minor circle  $CMLQ$ , whose radius is the semiminor axis of the desired ellipse, and  $A$  is also the center of a major circle  $BVT D$  whose radius is the semimajor axis of the desired ellipse.

From  $A$  draw radial lines  $AV$  and  $AT$ , intersecting the minor circle at  $M$  and  $L$  respectively, and the major circle at  $V$  and  $T$  respectively. The intersection  $N$  of a horizontal

line through  $M$  with a vertical line through  $V$  is a point on the ellipse. The intersection  $P$  of a horizontal line through  $L$  with a vertical line through  $T$  is a point on the ellipse. The points  $C, N, P, D$ , determine a quadrant of the ellipse. The other quadrants are symmetrical to the major and minor axes. The distance  $RS = RP$ ,  $PS$  being perpendicular to  $AD$ .

**Ex. 7.** Show that the surface of a sphere equals four times the area of a section passing through the center of a sphere.

**50. Segment of a Sphere.** The segment of a sphere is the volume cut off by any section. A line drawn perpendicular to the section, i.e., the base of the segment and extending to the surface is the altitude of the segment. Formula (51) expresses the volume of a spherical segment in which  $r$  is the radius of its base. (52) expresses the area of the spherical segment in which  $R$  is the radius of the sphere.

$$(51) \quad \text{Volume of spherical segment} = \frac{\pi}{2} h \left( r^2 + \frac{h^2}{3} \right).$$

$$(52) \quad \text{Area of spherical segment} = 2\pi R h.$$

Interpret (51) and (52). Solve (51) for  $r$ .

**51. The volume of a spherical zone** is given in (53) in which  $r_1$  and  $r_2$  are the radii of the respective bases and  $h$  is the altitude of the zone.

$$(53) \quad \text{Volume of spherical zone} = \frac{\pi}{2} h \left( r_1^2 + r_2^2 + \frac{h^2}{3} \right).$$

Show that (53) may be derived from the formula of Ex. 6, and formula (51). Interpret (53) and solve for  $r_1$  and  $r_2$ .

**52. Oblate and Prolate Spheroids.** When an ellipse is rotated about its minor axis it generates an oblate spheroid, and when it is rotated about its major axis it generates a prolate spheroid. Derive the formulas for the surface and

volumes of these figures of revolution by Guldinus' Theorems.

- The center of gravity of a semicircle equals  $\frac{4}{3\pi}$  times the semi-diameter on which it is measured from the center.

**53. The volume of a segment of a spheroid** whose base is perpendicular to one of the axes, is expressed in (54) when the base is circular, and in (55) when the base is elliptical.  $D_1$  is the diameter of the polar axis, i.e., the axis of revolution and  $D_2$  is the equatorial diameter, i.e., the diameter perpendicular to the axis of rotation.  $h$  represents the altitude of the segment.

$$(54) \quad V = \pi \left( \frac{D_1}{2} - \frac{h}{3} \right) \left( \frac{D_2 h}{D_1} \right)^2$$

$$(55) \quad V = \pi \left( \frac{D_2}{2} - \frac{h}{3} \right) \frac{D_1 h^2}{D_2}.$$

**54. Volume of a Paraboloid.** The volume of a paraboloid equals one-half the volume of a circumscribing cylinder. Formulate the law and interpret the formula. Write the formula for the frustum of a paraboloid.

**55. Volume of an Hyperboloid.** By means of the prismoidal formula (42) show that the volume of an hyperboloid is expressed in (56).  $r_1$  is the radius of the base;  $r_2$  is the radius of the middle section;  $h$  is the altitude of the paraboloid.

$$(56) \quad \text{Volume of hyperboloid} = \frac{\pi}{6} (r_1^2 + 4r_2^2) h.$$

Interpret (56) and solve for  $r_1$  and  $r_2$ .

**56. Regular Solids.** A regular polyhedron, i.e., regular solid has its faces and edges equal and its plane and solid angles equal. There are only five regular solids.

The **tetrahedron** is a regular triangular pyramid whose sides are equilateral triangles. It has six edges and four *faces*.

The **hexahedron** is a cube. It has twelve edges and six faces.

The **octahedron** is a solid which is bounded by equilateral triangles. It has twenty edges and eight faces.

The **dodecahedron** is a solid which is bounded by regular pentagons. There are twenty edges and twelve faces.

The **icosahedron** is a solid bounded by equilateral triangles. There are thirty edges and twenty faces.

With the exception of the tetrahedron the other polyhedrons have their opposite faces and edges in parallel pairs.

**Ex. 8.** Construct the development or spread for the five polyhedrons. Determine the volume of a tetrahedron with a unit edge.

**57. Volume of an Irregular Solid.** Any rule which is applicable to the determination of an irregular area may be applied to the determination of a volume. A base line is drawn in the base of the solid so as to represent the horizontal projection of the greatest dimension of the solid. The base line is then divided into equal spaces and at the points of division of the base line a series of parallel vertical planes are extended through the solid. The planes cut a series of parallel sections of more or less irregularity through the solid. Their area is determined by any of the rules for area. The mean of these areas multiplied by the length of the base line gives the volume of the solid. If the magnitudes of the measured areas are scaled and laid off as ordinates at the corresponding points of division on the base line, a new irregular figure is formed whose area expresses the magnitude of the volume of the solid.

**Ex. 9a. Area-linear Error.**

An error of 5 lbs. in the weight of 1 mile of Cu wire causes what per cent error in the calculated value of the diameter?

**Ex. 9b. Linear-square Error.**

The length of a seconds pendulum is increased by  $\frac{1}{10000}$  part. How many seconds will the clock lose in a day?

The time of a complete oscillation  $T = 2\pi\sqrt{\frac{l}{g}}$ , where  $l$  is the length, and  $g$  is constant for gravity.

**Ex. 9c. Linear-volume Error.**

The radius of a sphere is found by measurement to be 5 ins. What error will be caused in the calculated value of the volume by an error of 1 per cent in the measured value of the radius?

**Ex. 10. Mixed Error.**

The value  $M$  of the magnetic moment of a magnet is calculated from the two formulas

$$(a) \quad T = \sqrt{\frac{L}{MH}}$$

$$(b) \quad \frac{M}{H} = \frac{(d^2 - l^2)^2}{d}$$

Eliminate  $H$ .

$L$ ,  $d$ ,  $l$ , and  $T$  are observed quantities. An error of 2 per cent excess is made in observing  $T$  and all other readings are correct to 0.1 per cent. What is the approximate per cent error in the calculated value of  $M$ ?

**58. The Binomial Theorem.** The binomial theorem gives the following formula for expanding a binomial to any power. An **expansion** is the complete or partial statement of terms which result when an indicated operation has been performed.

$$(a+b)^n = a^n + na^{n-1}b + \frac{n(n-1)}{2!}a^{n-2}b^2 + \frac{(n)(n-1)(n-2)}{3!}$$

$$a^{n-3}b^3 + \dots \frac{n(n-1) \dots (n-K+1)a^{n-K}b^K}{K!} + \dots b^n.$$

$a$  and  $b$  represent any two quantities and  $n$  is any exponent which may be integral or fractional and positive or negative. In like manner  $a$  and  $b$  may have opposite signs.

**Ex. 11.** Expand  $(a+b)^n$  for each of the following values of  $n$ , viz., 1, 2, 3, 4, 5, 6, 7.

**Ex. 12. Approximations.**

If we allow the approximation  $a^2 + 2ab$  for  $(a+b)^2$ , what is the per cent error (a) when  $a=b=1$ ; (b) when  $a=3$ , and  $b=2$ ;

c) when  $a=10$  and  $b=1$ ; (d) when  $a=100$  and  $b=.1$ . What are the conditions for least error?

**Ex. 13.**  $(1+x)(1+y)=?$  If we allow the approximation  $+x+y$ ,

- (a) What is the per cent error when  $x=1$ ;  $y=1$ ?
- (b) What is the per cent error when  $x=0.1$ ;  $y=0.1$ ?
- (c) What is the per cent error when  $x=100$ ;  $y=100$ ?

What are the conditions for least error?

**Ex. 14.**  $(1+x)(1+y)(1+z)=?$  If we allow the approximation  $+x+y+z$ ,

- (a) What is the per cent error when  $x=1=y=z$ ?
- (b) What is the per cent error when  $x=1=y$ ;  $z=.1$ ?
- (c) What is the per cent error when  $x=1$ ;  $y=.1=z$ ?
- (d) What is the per cent error when  $x=1$ ;  $y=.1$ ;  $z=.01$ ?

What are the conditions for least error?

**Ex. 15.**  $(1+x)^n=?$  If we allow the approximation  $1+nx$ ,

- (a) What is the per cent error when  $x=1$ ;  $n=1$ ?
- (b) What is the per cent error when  $x=1$ ;  $n=100$ ?
- (c) What is the per cent error when  $x=100$ ;  $n=1$ ?
- (d) What is the per cent error when  $x=5$ ;  $n=5$ ?

What are the conditions for least error?

**Ex. 16. Ship Area.**

The semiordinates of the load water plane of a vessel are 2, 3.6, 7.4, 10, 11, 10.7, 9.3, 6.5, 2 ft., respectively, and they are 15 ft. apart. What is the area? Compute by different methods and tabulate.

**Ex. 17. Area Curve.**

The ordinates of a curved figure in inches are 2.6, 3.5, 3.66, 3.63, 3.37, 2.85, 2.4, 2.1, 1.89, 1.74, 1.6, 1.38, .49. The common intervals =  $\frac{1}{2}$  in. Determine the area below the curve.

**Ex. 18. Area Indicator Diagram.**

The length of an indicator diagram is 4 ins., the end ordinates are 1 and .22, and the other ordinates following the first are 1, .82, .71, .55, .45, .38, .33, .29, .26 in. respectively. The scale of pressure is 60 lbs. equals 1 in. Determine the mean pressure (a) by the planimeter; (b) by Simpson's rule; (c) by mid ordinate rule; (d) by counting squares.

**Ex. 19.** The half ordinates of the midship section of a vessel are 12.8, 12.9, 13, 13, 13, 12.9, 12.6, 12, 10.5, 6, 1.5 ft. respectively. The common distance between the ordinates is 18 ins. What is the area?

**Ex. 20.** The base of a prism is a triangle whose sides are 27, 25, 14, respectively and whose height is 10 ft. What is its volume?

**Ex. 21. Volume of Stream.**

A section of a stream is 10 ft. wide and 10 ins. deep. The mean flow of the water through the section is 3 miles an hour. (a) How many gallons flow through the section per hour? (b) per day?

**Ex. 22. Solid vs. hollow pillars.**

What is the weight of a hollow steel pillar 10 ft. long, whose external diameter is 8 ins., and internal diameter 4 ins.? What is the diameter of a solid pillar of the same weight and length? One cubic foot steel weighs 490 lbs.

**Ex. 23. Weight of Cable.**

A single-core electric cable consists of a cylindric copper wire surrounded by a coating of insulation and an outer coating of lead. The area of the cross-section of the copper is .25 sq.in. The thickness of insulation is .11 in., and the thickness of the outer covering is .11 in. What is the diameter and weight of the cable?

1 cu.in. copper weighs .32 lb.

1 cu.in. lead weighs .41 lb.

1 cu.in. insulation weighs .034 lb.

**Ex. 24. Brass Ball.**

A hollow sphere of brass is found to weigh 50 lbs. Its external diameter is 10 ins. What is the internal diameter?

1 cu.in. brass weighs .3 lb.

**Ex. 25. Railway Embankment.**

A railway embankment is 12 ft. high. The top is 28 ft. wide and sides slope 1 : 2 to the horizontal. How many cubic yards of earth are required to continue the embankment 1000 ft.?

**Ex. 26.** A bus of copper bar  $3\frac{1}{2}$  ins. by  $\frac{3}{4}$  in. is 35 ft. in length. What is its resistance? What length of the bar will serve as an ammeter shunt to give 45 millivolts for a current of 2000 amp. What is the total weight of the bus?

**Ex. 27.** Determine the number of cubic yards in a bank of earth on a horizontal rectangular base 60 ft. by 20 ft. The four sides of the bank slope to a ridge at an angle of  $40^\circ$  to the horizontal.

**Ex. 28. Storage of Water.**

A cylindrical vessel 16 ft. diameter, 20 ft. long is filled with water. What is the weight of water in tons?

One cubic foot of water weighs 62.5 lbs.

**Ex. 29.** A prism has a base of 50.32 sq.ins. What is the area of a section inclined  $20^\circ$  to the horizontal?

**Ex. 30.** The base of a right cone is an ellipse whose axes are 21 ft. and 14 ft. respectively. The altitude is 12 ft. What is its volume?

**Ex. 31.** A right circular cone is divided by two planes parallel to the base. These planes are equidistant from the base and apex. Compare the three volumes of the divided solid. The cone has a radius of 28 ins. and an altitude of 30 ins. The sections are separated by 12 ins.

**Ex. 32. Comparison of Surfaces.**

Compare the surface of a cube, cylinder, sphere, and cone, the volume in each case being 1 cu.ft. The altitudes of cones and cylinders are to be considered equal to the diameters of their bases.

**Ex. 33. Comparison of Volumes.**

A cone, cylinder, sphere, cube, have equal surfaces. Determine their volumes. Cones and cylinders are to have equal altitudes and diameters.

**Ex. 34.** Compare the surface and volume of a sphere and cube of equal dimensions.

**Ex. 35. Funnel.**

A tin funnel is 3.5 ins. diameter and has a slant height of 7 ins. How much material is required in the manufacture of its conic surface allowing 3 per cent for overlapping.

**Ex. 36.** Determine the volume of an incandescent lamp bulb from its measurements. Check the result by filling the bulb with a measured and weighed powder. Check also by immersing the bulb in a graduated beaker containing water.

**Ex. 37. Cylindrical Boiler.**

Determine the volume of water in a horizontal cylindrical boiler of length  $L$ , diameter  $D$ , when the height of the water is  $h$ . In the answer express the value of the angle as an anti-function.

**Ex. 38. Governor Ball.**

A sphere of radius  $R$  is pierced by a cylindrical hole through the center.  $r$  is the radius of the cylinder. What is the volume of the ball?

**Ex. 39.** What would be the equivalent circle for a lake whose area is 3 acres? How would you determine the area of the lake from the contour on a map? What is the equivalent number of sections (land measure)? How would you determine the area and volume of a lake from a topographic map?

**Ex. 40.** A running track is in the form of a link enclosing a football field with 10 ft. margin and has circular ends. Can such a track be constructed 4 laps to the mile?

**Ex. 41.** With a radius of 3 ins. construct an arc subtending a chord of 3 ins. What is the length of the arc? Determine the length accurately and compare the result with that obtained by "hods.

**Ex. 42.** A dome is in the form of a segment of a sphere of radius 100 ft. The height of the dome is 80 ft. How many cubic yards of cement 1 in. thick will be required to cover it?

**Ex. 43.** A basin is in the form of a zone of a sphere. It is 8 ins. deep. The bottom is a circle of 4 ins. diameter. The top is a circle of 15 ins. diameter. How much water will it hold at each inch level?

**Ex. 44.** What is the thickness of lead pipe 2.5 ins. internal diameter if it weighs 7.862 lbs. per linear foot?

**Ex. 45.** What is the weight of an iron pipe 12 ft. long, 14 ins. of external diameter, 13 ins. internal diameter given the weight of 1 cu.in. of iron = 0.27 lb.

**Ex. 46.** What is the weight of a cast-iron bar 1 in.  $\times$  1 in.  $\times$  1 yd.? What error would be made by calling this weight 10 lbs.?

**Ex. 47.** Water flows at the rate of 4.96 ft. per sec. through a cylindrical pipe 11 ins. in diameter. What is the supply in gallons per minute? 7.5 gals. = 1 cu.ft.

**Ex. 48.** The outside diameter of a road roller is 3 ft. and its outside width 4 ft. The metal is 2 ins. thick on the surface. It is closed at the ends which are 1 in. thick. The axle and handle weigh 11 lbs. and the metal of which the roller is made weighs 437 lbs. per cubic foot. What is the weight of the outfit?

**Ex. 49.** How many bricks will be required to build a foundation 4 ft.  $\times$  5 ft.  $\times$  4 ft.? Each brick measures 9 ins.  $\times$  3 ins.  $\times$  4.5 ins., including mortar.

**Ex. 50.** A solid sphere has a cylindrical hole drilled centrally through it. If  $2b$  be the length of the bore, show that the remaining volume is equal to that of a sphere of radius  $b$ .

**Ex. 51. Application of the Pythagorean Theorem.**

On a piece of cross-section paper construct a right triangle whose sides are 3, 4, and hypotenuse 5. Build a square on each side and on the hypotenuse and count the area. It will be found that the sum of the areas on the two sides equals the area on the hypotenuse. Try this again for a 9-12-13-right triangle.

But these facts are also substantiated by the theorem of the algebraic squares. Construct a right triangle with sides equal to one. What is the length of the hypotenuse?

**Ex. 52.** Construct lines corresponding to  $\sqrt{3}$ ,  $\sqrt{5}$ ,  $\sqrt{7}$ ,  $\sqrt{8}$ ,  $\sqrt{10}$ ,  $\sqrt{11}$  and prove.

Show how these lines could be constructed by using a theorem on the circle.

**Ex. 53.** What is the range of area of the earth's surface which may be illuminated by a searchlight mounted in the top of a 550-ft. tower.

**Ex. 54.** A circle 25 ins. in diameter is to be divided into four equal areas by concentric circles. What are their diameters?

**Ex. 55.** Fig. 98 represents the cross-section of a drum of a boiler. The drum is 40 ins. in diameter and its cylindric portion

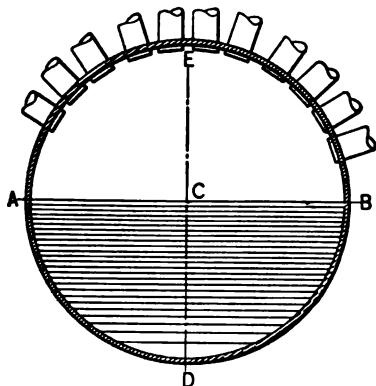


FIG. 98.—Cross-section of a Steam Drum.

is 11 ft. 10 ins. in length. The ends of the drum are spherical segments with a 40-in. radius and extend 7 ins. beyond the end of the cylindric portion. Half the contents of the drum is evaporated in 25 min. Determine the level of the liquid at the end of each minute assuming the same rate of evaporation to continue. What error will be made if the volume of the spherical ends is considered negligible?

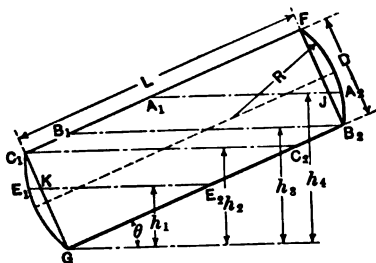


FIG. 99.—Cylindric Tank.

**Ex. 56.** Fig. 99 represents a cylindric vessel of length  $L$  and diameter  $D$ . The ends are spherical segments of radius  $R$  and

height  $a$ . The vessel is tipped at an angle  $\theta$  when measured to a horizontal plane. The height of the liquid above the plane is designated by  $h_1, h_2, h_3, h_4$ . Determine the volume of the liquid for the levels  $A_1A_2, B_1B_2, C_1C_2$ , and  $E_1E_2$ .

**Ex. 57.** Fig. 100 represents two views of a commutator bar. Determine the volume and weight of 432 bars like  $ABCEGKLN$  allowing .32 lb. per cubic inch. The lower edge of the bar ends along the line  $KG$ . The inside diameter of the commutator has a diameter = 1 ft. 10 $\frac{7}{8}$  ins. Determine the thickness of the mica insulation. The cross-section of a bar is a trapezoid. The bar has

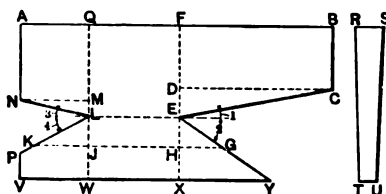


FIG. 100.—A Commutator Bar.

the following dimensions:  $AB = 8\frac{1}{8}$  ins.;  $NM = 2.55$  ins.;  $DC = 3.11$  ins.;  $KJ = 2.00$  ins. =  $HG$ ;  $FQ = JH = 3.15$  ins.;  $FH = 2\frac{1}{4}$  ins.;  $FD = 1.56$  ins.;  $DE = .16$  in.;  $EH = 1.15$  ins.;  $QM = 1.59$  ins.;  $ML = .13$  in.;  $LJ = 1.15$  ins. The thickness at  $B = .1769$  in.; the thickness at  $G = .1351$  in.; angle 1 =  $3^\circ$ ; angle 2 =  $4^\circ$ .

**Ex. 58.** Determine the volume of 288 bars like  $ABCEYVPLN$  in Fig. 100. The inside diameter of the commutator is  $11\frac{1}{8}$  ins. The section of a commutator bar is trapezoidal. Determine the thickness of the mica insulation. The dimensions of the bar are as follows:  $AB = 10\frac{1}{2}$  ins.;  $AQ = 1.36$  ins.;  $QF = 6.65$  ins.;  $FB = 2.96$  ins.;  $FX = 2\frac{1}{2}$  ins.;  $FD = 1.20$  ins.;  $DE = .15$  in.;  $EX = 1.18$  ins.;  $QM = 1.28$  ins.;  $ML = .07$  in.;  $LO = .78$  in.;  $OW = .40$  in.; the thickness at  $B = .1528$  in.; the thickness at  $Y = .0976$  in.; angles 1 =  $3^\circ$ ; angles 2 =  $4^\circ$ .

## CHAPTER XII

### THE QUADRATIC EQUATION

1. A **FORMULA** or equation may contain a number of letters or letters and numeric quantities. When the given equation is cleared of fractions and radicals we can determine the degree of the equation for each of its **elements**, i.e., its letters. Each element may be considered an **unknown quantity**, i.e., a letter whose value is to be expressed numerically and literally in terms of the other elements of the formula.

If an element enters a formula with no **exponent** other than **one** then the formula is a **simple equation**, while considering that element only. It is called a **first degree** or **linear equation** in other parts of this text.

A **quadratic formula** is an equation which contains the **second power** of an unknown quantity and is therefore called a **second degree** equation. Such an equation usually contains a term with the **first power** of the unknown and an **absolute term**, i.e., a term without the unknown quantity. Either of the latter may be missing from a quadratic equation.

$$(1) \quad t = \frac{abu + agv}{u^2 + v^2}.$$

Eq. (1) is a quadratic equation for each of the elements,  $u$  and  $v$  but a simple equation for each of the elements  $t$ ,  $a$ ,  $b$ , and  $g$ .

Quadratic equations are solved by **factoring** when the formula contains one element only. A method known as **completing the trinomial square** is resorted to in other cases but this method may be abbreviated by the use of a **working formula**. Graphic methods may also be used for the solution of a quadratic equation.

**2. Solving a Quadratic Equation by Factoring.** Multiply the coefficient of the second power of the unknown by the absolute term. Resolve this product into two factors, whose sum equals the coefficient of the first power of the unknown. Replace the latter by its equal and factor by grouping.

$$(1) \quad 5p^2 + 18p + 16 = 0;$$

$$5 \times 16 = 80, \quad 80 = 10 \times 8, \quad 10 + 8 = 18, \quad 18p = 10p + 8p$$

$$(2) \quad 5p^2 + 10p + 8p + 16 = 0 \quad \text{subs. for } 18p \text{ in (1)}$$

$$(3) \quad 5p(p+2) + 8(p+2) = 0 \quad \text{factoring in groups}$$

$$(4) \quad (5p+8)(p+2) = 0 \quad \text{factoring H.C.F.}$$

$$(5) \quad (5p+8) = 0 \quad \text{div. (4) by } (p+2)$$

$$(6) \quad p = -\frac{8}{5} \quad \text{trans. and div. in (5)}$$

$$(7) \quad p+2 = 0 \quad \text{div. (4) by } (5p+8)$$

$$(8) \quad p = -2 \quad \text{trans. in (7)}$$

Steps (5) and (7) could be omitted if the solutions of  $p$  expressed in (6) and (8) are written by inspection of (4). The two values of  $p$  in (6) and (8) are written from the factors of (4) by changing the signs of the numeric terms and dividing the latter by the corresponding coefficient of  $p$ .

Solve the following examples by factoring:

**Ex. 1.**  $5p^2 + 24p + 16 = 0.$

**Ex. 2.**  $2a^2 + 4a + 16 = 0.$       2 " "

**Ex. 3.**  $6x^2 - 13x - 63 = 0.$

**Ex. 4.**  $c^2 + 3.7c + 3 = 0.$

**Ex. 5.**  $12k^2 - 26k + 10 = 0.$

As a check on the work we should substitute the value of the unknown in the original equation.

**3. Solving a Quadratic Equation by Completing the Square.** This method may be summarized by five distinct steps:

(I) Transpose the first and second powers of the unknowns to the left member and the knowns to the right member and collect.

(II) Divide the equation by the coefficient of the second power of the unknown.

(III) Complete the trinomial square in the left member by adding to the equation the square of half the coefficient of the first power of the unknown.

(IV) Perform the square root upon the equation, indicating the root in the right member with the ambiguous sign ( $\pm$ ) which is read plus or minus.

(V) Transpose and simplify the solution.

(2)	$5p^2 + 24p + 16 = 0$	The given equation
-----	-----------------------	--------------------

(I)	$5p^2 + 24p = -16$	Trans. in (2)
-----	--------------------	---------------

(II)	$p^2 + \frac{24}{5}p = -\frac{16}{5}$	Div. (I) by 5
------	---------------------------------------	---------------

(III)	$p^2 + \frac{24}{5}p + \left(\frac{12}{5}\right)^2 = \frac{144}{25} - \frac{16}{5} = \frac{64}{25}$	Add $\left(\frac{12}{5}\right)^2$ to (II)
-------	---	---

$$(IV) \quad p + \frac{12}{5} = \pm \sqrt{\frac{64}{25}} = \pm \frac{8}{5} \quad \text{Root of (III)}$$

$$(V) \quad p = -\frac{12}{5} \pm \frac{8}{5} = \frac{-12 \pm 8}{5} \quad \text{Trans. (IV)}$$

$$(VI) \quad \left\{ \begin{array}{l} \therefore p = \frac{-12+8}{5} = -\frac{4}{5} \\ \text{and } p = \frac{-12-8}{5} = -\frac{4}{5} \end{array} \right\} \quad \text{simplifying (V)}$$

**Ex. 6.** Check the results of Ex. 1, 2, 3, 4, and 5 by the method of completing the trinomial square.

*Observation.* If the value of the unknown is substituted, it should reduce the original equation to zero when simplified. Those values of the unknown which reduce an equation to zero form its factors when connected to the unknown by a minus sign.

4. Every equation of the second degree may be considered as a special form of (3) by substituting definite literal or numeric values for  $a$ ,  $b$  and  $c$ , and the corresponding unknown to replace  $x$ .

$$(3) \quad ax^2 + bx + c = 0.$$

Any equation derived from (3) will also validate the like values substituted for  $a$ ,  $b$  and  $c$ . Therefore, solving (3) by the method of completing the square, we obtain (7) which is a working formula for all quadratic equations.

$$(4) \quad x^2 + \frac{b}{a}x = -\frac{c}{a};$$

$$(5) \quad x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 = \frac{b^2}{4a^2} - \frac{c}{a} = \frac{b^2 - 4ac}{4a^2};$$

$$(6) \quad x + \frac{b}{2a} = \pm \frac{\sqrt{b^2 - 4ac}}{2a}.$$

$$(7) \quad x = \frac{-b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Given any equation, whose coefficient of the (unknown squared) is  $a$ , coefficient of (unknown) is  $b$ , and absolute term is  $c$ , then the unknown is given in the formula (7) in terms of the same constants  $a$ ,  $b$ , and  $c$ . The formula should not be used until the student is thoroughly familiar with its derivation so that he may apply it skillfully.

5. When the roots, i.e., the values of the unknown of an equation are given, and we wish to construct the quadratic equation, we may use the principle stated in the observation under (3). Suppose the roots are  $m$  and  $n$ , then

$$(8) \quad (x-m)=0 \text{ and } (x-n)=0;$$

$$(9) \quad (x-m)(x-n)=0;$$

$$(10) \quad x^2-(m+n)x+mn=0;$$

but

$$(4) \quad x^2+\frac{bx}{a}+\frac{c}{a}=0.$$

Comparing (10) with (4) we observe,

$$(11) \quad m+n=-\frac{b}{a} \qquad (12) \quad mn=\frac{c}{a}.$$

The sum of the roots equals the negative ratio of the coefficients  $b$  to  $a$ .

The product of the roots equals the ratio of the absolute term  $c$  to  $a$ .

6. **The Graphic Construction of the Roots of a Quadratic Equation.** The roots of a quadratic equation may be constructed graphically upon a sheet of cross-section paper. Lay off  $AB=c$ ; on a perpendicular to  $AB$  from  $B$  lay off  $BC=b$ ; on a parallel to  $AB$  from  $C$  lay off  $CD=a$ . These lines are laid off cyclically in a counter-clockwise direction and have the same sense (positive) as shown in the circle.

A similar circle for counter-clockwise or negative sense will aid in laying off negative values for  $a$ ,  $b$  and  $c$ .

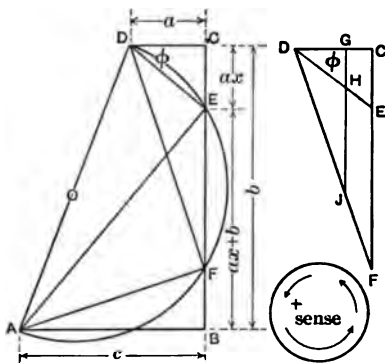


FIG. 101.—Graphic Construction of the Roots of a Quadratic Equation.

Join  $D$  with  $A$ , and bisect  $DA$  at  $O$ ; with  $O$  as center and  $OD=OA$ =radius, draw a semicircle intersecting  $CD$  at  $E$  and  $F$  respectively.  $DEA=90^\circ$  and  $DFA=90^\circ$  (angles inscribed in a semicircle).

On  $DC$  determine a unit's distance  $S=DG$ ; draw  $\perp DC$ ; then  $GH$  and  $GJ$  are the two values of  $x$  which satisfy the equation, and in this case they are both negative answers. The proof follows:

$$\left. \begin{aligned} (12) \quad \frac{GH}{1} &= -x = -\tan \phi = -\tan EDC \\ (13) \quad \tan EDC &= -\frac{CE}{DC} = -\frac{CE}{a} = x. \end{aligned} \right\} \begin{array}{l} \text{Cons. } GH \text{ and def.} \\ \text{tangent.} \end{array}$$

$$(14) \quad CE = -ax. \quad \text{Mult. in (13)}$$

$$(15) \quad BE = BC - CE = b - (-ax) = b + ax \quad \text{Sum of parts } Ax$$

$$(16) \quad \tan AEB = \tan EDC = +x$$

$$= -\frac{BA}{BE} = -\frac{c}{BE} \quad \text{Def. tangent}$$

(17)  $x(BE) = -c$

Mult. in (16)

(18)  $x(ax+b) = -c$

Sub. in (17) from (15)

(19)  $ax^2+bx = -c$

(18)

(20)  $ax^2+bx+c=0$

Trans. (19)

Therefore  $x$  satisfies the general quadratic equation (3).

Apply this method to examples (1) . . . (5)

**Ex. 7.**  $E^2 = E_1^2 + E_2^2$ .

Solve for  $E$ ,  $E_1$ , and  $E_2$ .

**Ex. 8.**  $\frac{Ex_2}{r_2^2+x_2^2} = \frac{E}{x_1}$ .

Solve for  $r_2$  and  $x_2$ .

**Ex. 9.**  $\omega = \sqrt{\frac{1}{LC}}$ .

Solve for  $L$  and  $C$ .

**Ex. 10.**  $E = I\sqrt{R^2 + \omega^2 L^2}$ .

Solve for  $R$ ,  $\omega$ , and  $L$ .

**Ex. 11.**  $i = \frac{E}{\sqrt{R^2 + \omega^2 L^2}} \sin(\omega t - \theta)$ .

Solve for  $R$  and  $L$ .

7. In removing a radical from an equation the latter should be transformed so as to place the radical alone in one member. Then the equation should be raised to a power corresponding to the index of the root.

**Ex. 12.**  $I_2^2 = I_1^2 + \frac{2P}{R} + I_1^2$ .

Solve for  $I_2$ ,  $I_1$ ,  $I_1$ .

**Ex. 13.**  $I = \frac{E}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}}$ .

Solve for  $R$ ,  $\omega$ ,  $L$ ,  $C$ .

In order to solve for  $\omega$  proceed as follows: Square members of the equation and get the ( ) in the left member alone. Take the root of both members and clear of fractions. Solve by five-step method.

*Observation. When an unknown quantity is involved under several indicated signs of operations it can be determined therefrom only by performing the indicated operation on quantity or by performing the inverse operation upon entire equation.*

$$\text{Ex. 14. } E = r_1 L_1 + j \omega L_1 I_1 + \frac{\omega^2 M^2 I_1}{r_2 + j \omega L_2}.$$

Solve for  $M$  and  $\omega$ , substitute  $j^2 = -1$ .

$$\text{Ex. 15. } E_1 = \frac{E(r_1^2 + x_1^2)}{(r_1 + r_2)^2 + (x_1 + x_2)^2}.$$

Solve for  $r_1, r_2, x_1, x_2$ .

$$\text{Ex. 16. } g_1 = \frac{r_1}{r_1^2 + x_1^2}.$$

Solve for  $r_1$  and  $x_1$ .

$$\text{Ex. 17. } b_1 = \frac{x_1}{r_1^2 + x_1^2}.$$

Solve for  $r_1$  and  $x_1$ .

$$\text{Ex. 18. } I_2 = \frac{r_2 E}{r_2^2 + x_2^2} - j \frac{x_2 E}{r_2^2 + x_2^2}.$$

Solve for  $r_2$  and  $x_2$ .

$$\text{Ex. 19. } \frac{I^2 R}{r} = \frac{I^2 (Rr - Xx)}{r^2 + x^2}.$$

Solve for  $r$  and  $x$ .

$$\text{Ex. 20. } P = \frac{R(r^2 + x^2)}{r(Rr - Xx)} W.$$

Solve for  $r$  and  $x$ .

**Ex. 21.**  $r = \left(\frac{N'}{N}\right)^2 R.$

Solve for  $N'$  and  $N$ .

**Ex. 22.**  $x = \left(\frac{N'}{N}\right)^2 X.$

Solve for  $N'$  and  $N$ .

**Ex. 23.**  $W_h = aVfB^{1.6}.$

Solve for  $B$ .

**Ex. 24.**  $W_c = R'(I')^2 + RI^2.$

**Ex. 25.**  $\frac{E_1}{I_1} = r_1 + \frac{r_2 \omega^2 M^2}{r_2^2 + \omega^2 L_2^2} + j \left\{ \omega L_1 - \frac{\omega^2 L_2 M^2}{r_2^2 + \omega^2 L_2^2} \right\}.$

Solve for  $L_2$ ,  $M$ ,  $r_2$ .

**Ex. 26.**  $R' = \frac{1}{2} \left(\frac{N'}{N}\right)^2 R.$

Solve for  $N'$  and  $N$ .

**Ex. 27.**  $P = \frac{4\pi(N')^2}{l} \left( \frac{X}{3} + \frac{Y}{3} + g \right).$

Solve for  $N'$ .

**Ex. 28.**  $I' = -B \left( aY_1 + \frac{Y}{a} \right).$

Solve for  $a$ .

**Ex. 29.**  $\frac{E}{E'} = \frac{-a - ar_2 Y}{a^2 + a^2 ZY_1 + ZY'}$

Solve for  $a$ .

**Ex. 30.**  $s = \frac{agu - abv}{u^2 + v^2}.$

Solve for  $u$  and  $v$ .

**Ex. 31.**  $t = \frac{abu + agv}{u^2 + v^2}.$

Solve for  $u$  and  $v$ .

$$\text{Ex. 32. } I^2 = \frac{A^2 + B^2 + 2AB \cos \phi}{R^2 + \omega^2 L^2}.$$

Solve for  $A$ ,  $B$ ,  $R$ ,  $\omega$ , and  $L$ .

8. An equation is **homogeneous** for two or more of its elements if they are interchangeable without altering the value of the equation. If the value of one element is known its homogeneous element may be obtained by an interchange of letters.

$$\text{Ex. 33. } P = \frac{B^2 R + ABR \cos \phi + AB \omega L \sin \phi}{R^2 + \omega^2 L^2}.$$

Solve for  $B$ ,  $R$ ,  $\omega$ ,  $L$ .

$$\text{Ex. 34. } t^2 = \frac{2\pi^2 K(R^2 + \omega^2 L^2)}{pABL}.$$

Solve for  $t$ ,  $R$ ,  $\omega$ ,  $L$ .

$$\text{Ex. 35. } A^2 = (B + RI)^2 + \omega^2 L^2 I^2.$$

Solve for  $A$ ,  $\omega$ ,  $L$ ,  $I$ .

$$\text{Ex. 36. } \frac{P}{P'} = \frac{AB \cos (\phi - \theta) + B^2 \cos \theta}{AB \cos (\phi + \theta) + A^2 \cos \theta}.$$

Solve for  $A$  and  $B$ .

$$\text{Ex. 37. } B = \sqrt{A^2 + E^2 - 2AE \cos \theta}.$$

Solve for  $A$  and  $E$ .

$$\text{Ex. 38. } P''_{\max} = \frac{B^2 \cos \theta - AB}{\sqrt{R^2 + \omega^2 L^2}}.$$

Solve for  $R$ ,  $\omega$ ,  $L$ ,  $B$ .

9. In the solution of any given equation involving long or **cumbersome phrases** or **expressions**, substitute a single abbreviation for the latter. In the final form of transformation reinstate the substituted value.

$$\text{Ex. 39. } \frac{AB}{\sqrt{R^2 + \omega^2 L^2}} = \frac{B^2 \cos \theta}{\sqrt{R^2 + \omega^2 L^2}}.$$

Solve for  $B$ .

$$\text{Ex. 40. } A^2 = \left( RI + \frac{P''}{I} \right)^2 + (X - \omega LI)^2.$$

Solve for  $R$ ,  $P''$ ,  $X$ ,  $\omega$ ,  $L$ .

$$\text{Ex. 41. } T = \frac{a(n - n')}{1 + b(n - n')^2}.$$

Solve for  $n$  and  $n'$ .

**10. Parenthetical quantities** may be solved as a single unknown in equations like (41) and the values of the enclosed letters determined by subsequent transposition.

$$\text{Ex. 42. } I'' = \sqrt{\frac{a^2 s^2 (E')^2}{u^2 + v^2}}.$$

Solve for  $a$ ,  $s$ ,  $E'$ ,  $u$ , and  $v$ .

$$\text{Ex. 43. } I' = \frac{\left( \frac{1}{ar_1} + \frac{aY}{s} \right) E'}{\frac{a}{s} + \frac{Z}{ar_2} + \frac{aY_1 Z}{s}}.$$

Solve for  $a$ .

$$\text{Ex. 44. } u = a^2 r_2 + s r_2 + a^2 r_2 (g_1 r_1 + b_1 x).$$

Solve for  $a$ .

$$\text{Ex. 45. } P = \frac{s(1-s)c(E')^2}{d + hs + ks^2}.$$

Solve for  $s$ .

$$\text{Ex. 46. } T = \frac{sc(E')^2}{d + hs + KS^2}.$$

Solve for  $S$ .

$$\text{Ex. 47. } T = \frac{qr_2 s (E')^2}{r_2^2 + 2r_1 r_2 s + (r_1^2 + x^2) s^2}.$$

Solve for  $r_2$ ,  $s$ ,  $r_1$ ,  $E'$ .

**Ex. 48.** 
$$T_{\max} = \frac{\pm q(E')^2}{2(r_1 \pm \sqrt{r_1^2 + x^2})}.$$

Solve for  $r_1$ .

**Ex. 49.** 
$$T_0 = \frac{2r_2(E')^2}{r_2^2 + 2r_1r_2 + r_1^2 + x^2}.$$

Solve for  $r_1, r_2$ .

**Ex. 50.** 
$$r = \frac{\sqrt{E_0 - (E \sin \theta + xI)^2} - E \cos \theta}{I}.$$

Solve for  $E, x, I$ .

**Ex. 51.** In (50) substitute  $\cos \theta = \sqrt{1 - \sin^2 \theta}$  and solve for  $\sin \theta$ .

**Ex. 52.** Make a complete list of all quadratic equations occurring in the preceding chapters, solve and interpret them.

**Ex. 53.** Make a complete list of all quadratic and biquadratic equations occurring in the following chapters; solve and interpret them.

**Ex. 54.** In Ex. 33 substitute  $\omega L = 0$  and solve for  $B$  and  $I$ .

**Ex. 55.** In Ex. 49 substitute  $r_1 = r_2$  and solve for  $r_2$ .

**Ex. 56.** In Ex. 50 substitute  $\theta = 0^\circ$  and solve for  $E$ .

**Ex. 57.** In Ex. 50 substitute  $\theta = 90^\circ$  and solve for  $E$ .

## CHAPTER XIII

### ELEMENTS OF THE STRENGTH OF MATERIALS

elements of the strength of materials comprehend king formulas and their interpretations when applied design of structural material. One purpose is the nation of the proper sizes and forms to withstand loads and, secondly, to determine the loads which applied safely to constructive material of known ons. A third purpose is the acquisition of sufficient lge for the use of a handbook of materials.

**stress per Unit of Area.**—The total load or weight ( $P$ ) ids upon a body, is the product of the **area** ( $A$ ) in inches of the **resisting surface** times the **intensity** of i.e., the stress  $S$  per unit of area, as expressed in (1)  
This is the fundamental formula for direct stresses, **tension** and **compression**.

$$\text{Load} = \text{unit stress} \times \text{area.}$$

$$P = SA.$$

e (2) for  $S$  and  $A$ , and interpret the resulting equa-

.. Determine the value of  $S$ , when  $P = 20000$  lbs. and in.

1. Determine the allowable load for a cross-section of ., when the unit stress is 2500 lbs. per square inch.

1. What cross-section is required to sustain a load of s., when the allowable unit stress = 8000 lbs. per square

**Ex. 4.** What is the dimension of the side of a square oak timber which is to carry a compression load of 75000 lbs., allowing a unit stress of 1500 lbs. per square inch. What will be the corresponding cross-sections for a unit stress of one-fifth of the ultimate strength, when square bars of iron, wrought iron, and steel are substituted.

**2. The strength of any material** is determined by placing a sample of the material in a testing machine in which it may be subjected to a direct stress of tension or compression. Under these respective stresses the sample is elongated or shortened, and the deformation is proportional to the stress until the elastic limit is reached. The **elastic limit** is a unit stress beyond which the material shows a marked permanent set, i.e., deformation. The **ultimate strength** of the sample is the highest value of the unit stress just before the sample ruptures.

The **modulus or coefficient of elasticity** ( $E$ ) is the ratio of the unit stress to the unit deformation ( $s$ ) as expressed in (3).

$$(3) \quad E = \frac{S}{s}.$$

The unit deformation ( $s$ ) is the ratio of the total deformation ( $e$ ) of a bar to its length ( $l$ ) as expressed in (4).

$$(4) \quad s = \frac{e}{l}.$$

Substitute in (3) the value of  $S$  from (2) and the value of  $s$  from (4), and simplify. The values of the coefficients of elasticity are practically equal for both tension and compression.

The materials of construction may be ruptured not only by tension and compression but also by shearing, i.e., by transverse breaking or cutting along a section. A shear results when the difference in the forces on the two sides of a section becomes excessive.

The materials which enter into the construction of buildings, bridges, and machines are subject to various combinations of the elementary stresses of tension, compression, and shear. These stresses may be studied by their effects, in direct deformation by compression and elongation, and in indirect deformation by bending and twisting.

The values of the strength of materials vary through a wide range, depending upon the quality of the manufactured product. For any specific product the student should consult a manufacturer's handbook. Table X is provided as a guide and represents a compilation from various sources. The average values are those which have been accepted in practice.

**3. A beam** is a bar of rigid material which is supported in some manner, at one or more points. The sum of the **reactions**, i.e., the forces acting upward at the supports equals the sum of the forces acting downward upon the beam.

**Ex. 5.** Make a drawing of a beam supported at two points  $R_1$  and  $R_2$  and separated by a distance  $d$  feet. At a point  $P$  on the beam distant  $x$  feet from  $R_1$ , represent a concentrated load, i.e., a force acting downward. The beam may be regarded as a balanced lever in which either of the supports acts as a fulcrum. Under this consideration the product of the load times its lever arm equals the reaction times its lever arm. If  $R_1$  and  $R_2$  also designate the respective reactions and  $P$  the load, then (5) expresses the equilibrium about  $R_1$  as a fulcrum, and (6) expresses the equilibrium about  $R_2$  as a fulcrum.

$$(5) \quad Px = R_2d.$$

$$(6) \quad P(d-x) = R_1d.$$

$$(7) \quad P = R_1 + R_2.$$

(7) results by adding (3) and (4) and dividing by  $d$ . Interpret (7).

**4. The product of a force or load times its lever arm is called a moment.** The weight of a beam or a uniformly distributed load is equivalent to a concentrated load at the

TABLE X. STRENGTH OF MATERIALS

Material.	Elastic Limit, Multiply by 10 <sup>3</sup> .			Ultimate Strength, Multiply by 10 <sup>3</sup> .			Modulus of Elas- ticity, Multiply by 10 <sup>6</sup> .		Deform- ation, Ins. per Lin. In.	Weight in Lbs. per Cu. Ft.	Specie Gravity
	Ten- sion.	Com- pres- sion.	Shear.	Ten- sion.	Com- pres- sion.	Shear.	Tension or Com- pression.	Shear.			
Cast iron.....	.....	.....	.....	15-25	70-110	13-26	15-20	5-8	.....	.....	7.1-7.5
Cast iron, average.....	6	20	.....	20	90	20	17	6	.005	450	7.2
Steel.....	36-80	34-80	20-40	56-120	.....	43-85	.....	.....	.....	.....	7.26-7.86
Steel, average.....	50	50	35	100	150	70	30	9	.25	490	7.8
Steel rivets.....	32	.....	26	54	.....	44	.....	.....	.....	.....	.....
Wrought iron.....	22-30	24-28	10-12	42-56	.....	32-40	.....	.....	.....	.....	.....
Wrought iron, average..	25	25	11	55	55	36	28	9	.2	480	7.7
Timber.....	3	3	.....	10	8	6 long. 3 trans.	.....	.....	.015	25-48	.....
Brick.....	.....	.....	.....	.....	3	.....	.....	.....	.....	125	2
Stone.....	.....	.....	.....	.....	8	.....	.....	.....	.....	160	2.6

center of weight. The sum of the loads equals the sum of the reactions. If loads are distributed along a beam, then the sum of the moments about one support equals the moment of the reaction from the other support.

**Ex. 6.** A beam whose weight is 35 lbs. per linear foot rests upon supports 18 ft. apart. A weight of 400 lbs. is placed at 5 ft. from the left end, and a second weight of 600 lbs. at 7 ft. from the right end. Compute the reactions due to the separate loads and also due to the total load.

**Ex. 7.** Compute the bending moments of a 10-ft. girder weighing 100 lbs. per linear yard, with a load of 150 lbs. at the center. Draw the diagram of bending moments and repeat after shifting the load 2 and 4 ft. from the center. The bending moments at the different sections may be represented by a continuous series of vertical lines drawn to scale.

5. Any point on a beam may be considered as a fulcrum. The algebraic sum of the load moments and the reaction moments on either side of a section, is called the **bending moment** ( $M$ ) of the section at the point considered. The forces tending to cause rotation in clockwise direction are positive, while those tending to cause counter-clockwise rotation are negative.

**Ex. 8.** Show that (8) expresses the bending moment of a uniformly loaded beam of length  $l$ , where  $x$  is the distance of the section from one of the supports, and  $w$  is the load per foot of length.

$$(8) \quad M = \frac{1}{2}wx(l-x).$$

What is the value of the bending moment for a mid section, also for a section over either support. Does this tendency to rotation indicate the bending of the beam downward?

A parabola whose chord is  $l$  in length and whose height is  $\frac{1}{8}wl^2$  serves as a diagram for the graphic representation of the varying bending moment. The bending moment at any section is the ordinate of the parabola.

6. A **cantilever** is a beam which is free at one end. In Fig. 102 the free end is at the left whereas the right end is

supported by a brick wall. There is no bending at the left end and since the moments are always negative the tendency to rotation causes the bending of the beam at the end. (7) is the equation of the bending moment for a cantilever of length  $x$ .  $P$  is the concentrated load at the

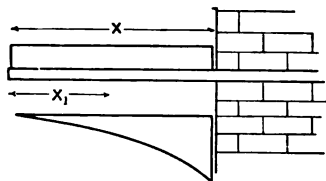


FIG. 102.—A Cantilever.

left end, and  $w$  the uniform load per foot of length,  $x_1$  is the distance of the cross-section from the left end.

$$(9) \quad M = -x_1 \left( P + \frac{wx_1}{2} \right).$$

(9) reduces to (10) which expresses the bending moment at the support for a cantilever with a uniform load only.

$$(10) \quad M = -\frac{wx_1^2}{2}.$$

A parabola, shown in Fig. 20, whose half chord is  $x$  in length, and whose height is  $\frac{wx^2}{2}$ , is a diagram which represents the varying bending moment. The bending moment is greatest at the wall and is represented as the largest ordinate at the left end of the moment diagram.

**7. Center of Gravity.** The centroid, i.e., the center of gravity of a mass or center of area of a plane figure, may be determined experimentally, graphically or by computation. In all cases it is the point about which the material of the figure is equably distributed. The sum of the moments of

the gravitational forces acting on the elements of the figure on one side of the center of gravity, equals the sum of the moments of the gravitational forces acting on the other side.

**Ex. 9.** Make a duplicate of Fig. 103 on a heavy piece of cardboard. Place the figure on a knife edge or table edge, and when balanced draw a line on the cardboard above the balancing edge. Balance the cardboard in a new position, and draw a line over the balancing edge. The intersection of these two lines is the centroid of the figure. Repeat for a new position as a check.

For figures of symmetry the centroid lies at the intersection of the axes of symmetry.

**Ex. 10.** Make a tracing of Fig. 103 on cross-section paper so that the line  $XX'$ , i.e., the axis of reference, is horizontal. The

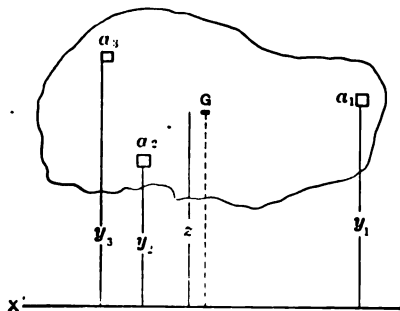


FIG. 103.—The Centroid of a Figure.

horizontal ruling of the paper divides the figure into thin **elements**, i.e., slices of approximately rectangular areas. The moment of any one of these rectangles about the axis  $XX'$ , is its area times its mean distance. The sum of the moments of the rectangular areas equals the moment of the whole area. Designating the elementary areas by  $b_1, b_2, b_3, \dots$ , and their moment arms by  $y_1, y_2, y_3, \dots$ , respectively, and the whole area by  $A$  and its moment arm by  $z$ , (11) states that the sum of the moments of the elementary areas equals the moment of the whole area.

$$(11) \quad b_1 y_1 + b_2 y_2 + b_3 y_3 + \dots = Az.$$

Abbreviate  $b_1y_1 + b_2y_2 + b_3y_3 + \dots$  by  $\Sigma by$ , in which  $\Sigma$  is a symbol of summation of all products of a  $b$  term times a  $y$  term. Therefore (11) may be written (12).

$$(12) \quad \Sigma by = Az.$$

$$(13) \quad \therefore z = \frac{\Sigma by}{A}.$$

8. The interpretation of (13) states that the moment arm ( $z$ ), of the whole figure ( $A$ ) about the axis  $XX'$ , is the ratio of the sum of the moments of the elementary areas to the whole area. If the entire area were concentrated at the distance  $z$  from  $XX'$  the moment of the figure would remain unchanged in value. The centroid of Fig. 103 is located at a distance  $z$  from the  $XX'$  axis. The distance of the centroid from a vertical axis of reference may be obtained in a similar manner by dividing the figure into vertical strips. The ratio of the sum of the moments of the new areas about the vertical axis of reference, to the whole area is the corresponding distance of the centroid from the vertical axis. Draw two lines parallel to the two axes at distances to correspond to the  $z$  values. Their intersection locates the centroid.

Make the necessary measurements for the area and moment arm of each strip both for the vertical and horizontal axes of reference, and compute the position of the centroid of the figure. The accuracy of the result in Ex. 10, will increase as the widths of the areas decrease. A more accurate value would be obtained if the figure were divided into elementary squares  $a_1, a_2, a_3, \dots$ , and then the ratio of the sum of the moments of the elementary figure, to the whole area will give the axial distance of the centroid. Suppose Fig. 103 is rotated about the axis  $XX'$  generating a ring of volume ( $V$ ), then by the Theorem of Guldinus we obtain (14).

$$(14) \quad V = 2\pi zA;$$

$z$  is the axial distance of the centroid of the cross-sectional area  $A$ .

Each unit of area  $a_1, a_2, a_3$ , etc., in its rotation generates a part of the ring at the respective axial distance of  $y_1, y_2, y_3$ , etc. Therefore the volume ( $V$ ) equals the sum of the volumes of the elementary rings. In (15),  $2\pi$  may be written outside the summation symbol because it is a common factor in every term.

$$(15) \quad V = \Sigma 2\pi a y = 2\pi \Sigma a y,$$

and

$$(16) \quad 2\pi \Sigma a y = 2\pi z A.$$

$$(17) \quad \therefore z = \frac{\Sigma a y}{A}.$$

The interpretation of (17) is identical with the interpretation of (13).

**Ex. 11.** Construct the following figures on cross-section cardboard, and determine center of gravity of each: (a) a semicircle; (b) a T section; (c) an I section; (d) a channel; (e) a rail section.

**9.** The second moment of an elementary mass or an elementary area, is the respective mass or area multiplied by the square of its distance from an axis of reference. The sum of the second moments of the elementary masses or areas of a figure, is called the **moment of inertia** ( $I$ ) of the mass or area. In Fig. 103 the elementary areas are squares. If  $a_1, a_2, a_3, \dots$ , are the elementary areas and  $y_1, y_2, y_3, \dots$ , their respective axial distances then the moment of inertia ( $I$ ) is expressed in (18).

$$(18) \quad I = \Sigma a y^2.$$

**Ex. 12.** Determine the moments of inertia for the figures used in Ex. 10.

10. The ratio of the moment of inertia of a mass or an area, to its respective mass or area, expresses the square of the axial distance ( $K$ ) at which the material could be concentrated in order to retain the same value for its moment of inertia.  $K$  is also called the **radius of gyration** as expressed in (17) when  $I$  is the least moment of inertia of the figure.

$$(19) \quad K = \sqrt{\frac{I}{A}}.$$

Solve (19) for  $I$  and interpret.

11. The elements entering into a moment are the area and the arm, and since an area varies as the square of a linear unit, and the arm varies as the first power of a linear unit, then the product varies as the third power of a linear unit. If the linear unit is one inch then the unit of a moment is a **cubic inch**. The moment of inertia is the product of an area times the square of the arm and therefore a moment of inertia is measured in **biquadratic inches**, i.e., the fourth power of a linear unit.

12. The fibers of a loaded beam are subjected to tension on one side and compression on the other side. There is a **neutral axis**, i.e., a line in the section of the beam, at which the stress is zero. The neutral axis passes through the center of gravity of the section and is the axis about which the sum of the moments of the internal forces equals zero. The stress ( $S$ ) in any fiber varies as its distance ( $c$ ) from the neutral axis  $S \propto c$ .

$$(20) \quad \therefore \frac{S}{S_1} = \frac{c}{y}.$$

In (20)  $S$  is the stress in the most remote fiber, and  $c$  is its corresponding distance from the neutral axis.  $S_1$  is the stress in a fiber at a distance  $y$  from the neutral axis.

The algebraic sum of the moments of the internal horizontal stresses about any point in the section is called

the **resisting moment** ( $R$ ). If the bending moment be computed, about the same point in the section, then the bending moment equals the resisting moment (21).

$$(21) \quad R = M.$$

The unit stress may be calculated by dividing the stress ( $S$ ) in the outside fiber by its distance  $c$  from the neutral axis. Solving (20) we obtain (22) the expression for the stress  $S_1$  of a unit area at the distance  $y_1$ .

$$(22) \quad S_1 = \frac{S}{c} y_1.$$

Interpret (22) in terms of unit stress.

If the fiber has an area  $a$  then (22) becomes (23).

$$(23) \quad aS_1 = \frac{S}{c} ay_1.$$

Interpret (23).

$$(24) \quad \therefore R = \sum \frac{aS_1 y_1}{c} = \frac{S}{c} \sum ay_1^2. \quad \text{Def. of } R$$

$\frac{S}{c}$  is a constant factor in all the terms of the summation and therefore may be written preceding the summation symbol.

But

$$(25) \quad \sum ay_1^2 = I. \quad \text{Def. of } I$$

$$(26) \quad \therefore R = \frac{S}{c} I. \quad \text{Subs. in (24)}$$

$$(27) \quad \therefore M = \frac{S}{c} I. \quad \text{Subs. in (26) from (21)}$$

Interpret (24) and (27); solve (27) for  $S$ ,  $c$ , and  $I$  and interpret the resulting equations.

The moments of inertia for standard sections are given in manufacturer's tables.

The moment of inertia ( $I$ ) of a rectangular cross-section of breadth ( $b$ ) and depth ( $d$ ) is expressed in (28) when the neutral axis passes through the center of gravity and in (29) when the axis is the lower edge of the rectangle.

$$(28) \quad I = \frac{bd^3}{12}.$$

$$(29) \quad I = \frac{bd^3}{3}.$$

Interpret (28) and (29).

The moment of inertia of a hollow rectangle ( $I_h$ ), may be obtained by subtracting the moment of inertia ( $I_1$ ) of the interior rectangle, from the moment of inertia ( $I$ ) of the outside rectangle.

$$(30) \quad I_h = I - I_1 = \frac{1}{12}(bd^3 - b_1d_1^3).$$

**Ex. 13.** Fig. 104 illustrates a hollow rectangular section turned over on its side. Compare the moment of inertia of the section in this position with the moment of inertia of the section in its normal position.  $b = 4$  ins.,  $b_1 = 3$  ins.,  $d = 8$  ins.,  $d_1 = 7$  ins.

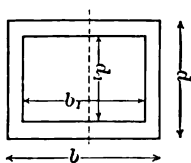


FIG. 104.

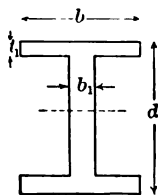


FIG. 105.—I-Beam.

**13.** The moment of inertia of an **I-beam** section, Fig. 105, may be computed from (28) by subtracting the moment of inertia of the two vacant rectangular spaces from the moment of inertia of the circumscribing rectangle as expressed in (31).

$$(31) \quad I = \frac{1}{12}\{bd^3 - (b-t)(d-2t)^3\}.$$

**Ex. 14.** Fig. 106 illustrates a **T-beam** section. By the use of (29) and the principle of Ex. 13, we obtain (32), where  $c_1$  and  $c$  are the respective distances of the outside fibers from the neutral axis.

$$(32) \quad I = \frac{1}{3} \{ t c^3 + b c_1^3 - (b - t)(c_1 - t_1)^3 \}.$$

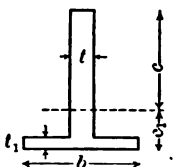


FIG. 106.—T Beam.

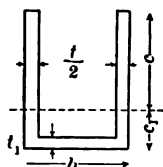


FIG. 107.—Channel.

**Ex. 15.** Fig. 107 illustrates a **channel** section. Show that (32) is also the formula for the moment of inertia of a channel.

**Ex. 16.** Determine the area and the moment of inertia of a standard I-beam with the following dimensions:  $d = 18$  ins.,  $b = 6$  ins.,  $t = .46$  in. The dimensions of the flange are taken from a manufacturer's table.

**Ex. 17.** Determine the area and the moment of inertia of a standard T-bar with the following dimensions:  $b = 4$  ins., thickness of flange  $t_1 = \frac{1}{2}$  to  $\frac{5}{8}$  ins., thickness of stem  $t = \frac{1}{2}$  to  $\frac{5}{8}$  ins., depth  $c = 2.82$  ins., and  $c_1 = 1.18$  ins. Check with a manufacturer's table.

**Ex. 18.** Determine the area and the moment of inertia of a standard channel with the following dimensions:  $b = 15$  ins.,  $c_1 = .79$  in.,  $c = 2.61$  ins.,  $t = .4$  in., and the average thickness of flange  $= .4$  in. Check with a manufacturer's table.

**Ex. 19.** Make a drawing from the detail measurement of the section of a standard 100-lb. T-rail. Determine its area and moment of inertia and check with a manufacturer's table.

**14. Comparative Strengths of Beams of Equal Length and of Equal Cross-section.** From (27) we obtain (33).

$$(33) \quad S = \frac{c}{I} M. \quad \therefore S \propto M.$$

The ratio  $\frac{I}{c}$  is called the section modulus, and contains all the dimensions of the cross-section. Therefore for a

constant section modulus the stress in the outside fiber will vary directly as the bending moment. If a number of beams are compared for strength, we shall assume that they have the same section modulus and the same unit stress, and therefore the bending moments will be equal by (27). Consider four beams of equal cross-section: (1) A cantilever loaded at the end with  $W_1$ ; (2) a cantilever loaded uniformly with  $W_2$ ; (3) a simple beam loaded at the middle with  $W_3$ ; and (4) a simple beam loaded uniformly with  $W_4$ . The respective bending moments are  $M_1, M_2, M_3, M_4$ . We then have (34).

$$(34) \quad M_1 = M_2 = M_3 = M_4.$$

For beams of equal length  $l$ , we obtain

$$(35) \quad M_1 = W_1 l, \quad M_2 = \frac{W_2 l^2}{2}, \quad M_3 = \frac{W_3 l}{4}, \quad M_4 = \frac{W_4 l}{8}.$$

$$(36) \quad \therefore W_1 l = \frac{W_2 l^2}{2} = \frac{W_3 l}{4} = \frac{W_4 l}{8}.$$

$$(37) \quad \therefore W_1 = \frac{W_2}{2} = \frac{W_3}{4} = \frac{W_4}{8}.$$

*The interpretation of (37) states that the uniformly loaded simple beam will sustain twice the load of a simple beam loaded at the middle, and the latter will sustain twice the load of a cantilever loaded uniformly and again the last will sustain twice the load of a cantilever loaded at the end. Therefore, the relative strength of the four beams in the order mentioned is 8 : 4 : 2 : 1.*

**Ex. 20.** A uniformly loaded standard I-beam has its supports 20 ft. distant. The beam is a 6-in. B 17 Cambria beam. Determine the safe load allowing a fiber stress of 12,500 lbs. Determine the safe load if the beam is used as a cantilever. Determine the equivalent Carnegie beam.

15. The **stiffness** of a beam is the ratio of its deflection to its span. Its **deflection** is the lateral displacement from its normal position.

16. Columns and struts are subjected to a direct compression stress due to a load  $P$ , which causes unit stress  $S_d$ , as expressed in (1) and (2), and also to a stress  $S_1$  due to bending. The total stress  $S$  is the sum of the contributing stresses as stated in (38).

$$(38) \quad S = S_d + S_1.$$

$$(39) \quad S = \frac{P}{A} + \frac{cM}{I} \quad \text{Sub. in (38) from (2) and (27).}$$

The only force causing the bending moment  $M$  is the load  $P$ , and its lever arm ( $f$ ) is the lateral deflection of the center line of the column from its normal position.

$$(40) \quad \therefore M = Pf.$$

$$(41) \quad I = Ar^2.$$

(41) expresses the moment of inertia of the cross-section whose area is  $A$  and whose radius of gyration is  $r$ . Substituting (40) and (41) in (39) we obtain (42).

$$(42) \quad S = \frac{P}{A} \left( 1 + \frac{cf}{r^2} \right).$$

Interpret (42).

The deflection of a beam varies directly as the square of its length, and therefore if the proportionality factor be taken as  $\frac{q}{c}$  we obtain (43).

$$f \propto l^2; \quad \therefore (43) \quad f = \frac{q}{c} l^2 \quad \text{and} \quad \therefore (44) \quad cf = ql^2.$$

$$(45) \quad \therefore \frac{P}{A} = \frac{S}{1 + \frac{ql^2}{r^2}}.$$

(45) is the result of substituting (44) in (42) and solving for  $\frac{P}{A}$ . The factor  $q$  is a numeric constant, which depends upon the kind of material, and the arrangement of the ends of the columns. Interpret (45) and solve for  $P$ ,  $A$ ,  $q$ ,  $l$ , and  $r$ , and interpret each of the resulting equations.

**Ex. 21.** Determine the safe load on a column for which  $S = 8000$  lbs. per square inch,  $A = 3 \times 4$ ,  $\frac{l^2}{r^2} = 4800$ , and  $q = \frac{1}{3000}$ .

**17.** The moment of inertia ( $M$ ) of a circle about its diameter ( $d$ ) is expressed in (46).

$$(46) \quad M = \frac{\pi d^4}{64}.$$

**Ex. 22.** Divide (46) by the area of the circle and determine the radius of gyration of the circle referred to the diameter.

**Ex. 23.** Construct a circle with two perpendicular diameters. The radius of gyration is  $r_1$ . Lay off the two lines parallel respectively to the perpendicular diameter and at distances therefrom equal to the radius of gyration. The intersection of these lines determines a point, whose distance from the center is  $r$ . A radius equal to  $r$  is called the polar radius of gyration. Show that  $r^2 = 2r_1^2$ .

**18.** A polar moment of inertia ( $J$ ) is a second moment of an area about a point called a pole. In the case of the circle the pole is the center of the circle. The polar moment is the product of the area of the figure times the square of its polar radius of gyration as expressed in (47).

$$(47) \quad J = Ar^2.$$

But

$$A = \frac{\pi d^2}{4} \quad \text{and} \quad r_1 = \frac{d}{4} \quad \text{and} \quad r^2 = 2r_1^2 = \frac{d^2}{8}.$$

$$(48) \quad \therefore J = \frac{\pi d^2}{4} \times \frac{d^2}{8} = \frac{\pi d^4}{32}.$$

Therefore the polar moment of inertia is twice its axial moment of inertia.

**Ex. 24.** The polar moment  $J_a$  of an annulus whose external and internal diameters are  $d_1$  and  $d_2$  respectively is given in (49).

$$(49) \quad J_a = \frac{\pi}{32}(d_1^4 - d_2^4).$$

**Ex. 25.** Show that (49) reduces to (50) where  $A$  expresses the area of the annulus.

$$(50) \quad J_a = \frac{(d_1^2 + d_2^2)A}{8}.$$

**19.** The polar moment of inertia of any section is the sum of the moments of inertia taken with respect to any two perpendicular axes.

**Ex. 26. Round Shaft to Transmit Power.**

$$(51) \quad S_s d^3 = 321000 \frac{H}{n}.$$

Notation:

$H$  = horse-power transmitted;

$n$  = number revolutions per minute;

$d$  = diameter in inches;

$S_s$  = unit stress for shearing per square inch.

Solve for  $S_s$ ,  $d$ ,  $H$ , and  $n$ , and interpret the resulting equation. Determine the unit stress in a shaft from the following data:

$$H = 25; \quad n = 100; \quad d = 2.5 \text{ ins.}$$

What is the factor of safety for this shaft?

**20. Combined Bending and Twisting.** When a shaft is subject to both bending and twisting then (52) expresses the unit stress  $S_1$  due to the greatest bending stress  $S$  and the torsional shearing unit stress  $S_s$ .

$$(52) \quad S_1 = \frac{S + \sqrt{4S_s^2 + S^2}}{2}$$

$$(53) \quad S = \frac{4Pl}{\pi d^3}.$$

Substitute in (52) the value of  $S$  from (53), and the value of  $S_s$  from (51), and simplify.  $P$  represents the load on the shaft and  $l$  the length of the shaft.

The last equation has been modified by A. E. Wiener, who suggests (54) for the bearing portion of a shaft of diameter  $d_b$ , and (55) for the core portion of a shaft of diameter  $d_c$ , in which  $W$  is the capacity of the generator in watts,  $k_b$  and  $k_c$  are constants.  $k_b$  ranges in value from .0025 for a high-speed drum armature to .005 for a low-speed ring armature.  $k_c$  ranges in value from 1 for a 1-KW. machine to 1.8 for a 2000-KW. machine.

$$(54) \quad d_b = k_b \sqrt{P \sqrt{n}}.$$

$$(55) \quad d_c = k_c \sqrt[4]{\frac{P}{n}}.$$

**Ex. 27.** Determine  $d_b$  and  $d_c$  for a 10 KW. machine for which  $n = 1700$ ,  $k_b = .0025$ , and  $k_c = 1.2$ .

**21. Pulley Diameter.** The diameter  $d_p$  of a pulley is expressed in (56) in which  $v$  is the velocity of the belt in feet per minute.

$$(56) \quad d_p = \frac{12v}{\pi n}.$$

Simplify (56).

**22. Armature Bearings.** The length ( $l$ ) of an armature bearing is expressed in (57) in which  $k_l$  is a constant ranging from .1 to .225 for high-speed and from .15 to .3 for low-speed armatures.

$$(57) \quad l = k_l d_b \sqrt{n}.$$

**Ex. 28.** Determine  $l$  for Ex. 27 using  $k_l = .2$ .

**23. Riveted Joints.** Notation:

$p$  = pitch of rivets;

$d$  = diameter of rivets;

$t$  = thickness of plates;

$S_t$  = tensile strength or resistance of plates to tension;  
 $S_s$  = shearing strength or resistance of plates to shearing;  
 $S_c$  = crushing strength or resistance of plates to crushing.

Considering a width of the joint equal to the pitch of the rivets:

58) The resistance of this portion to tearing  $= (p-d)tS_t$ .

59) The resistance of this portion to shearing  $= .7854d^2S_s$ .

60) The resistance of this portion to crushing  $= dtS_c$ .

61) The resistance of the solid plate to tearing  $= p t S_t$ .

**Ex. 29.** When tearing resistance equals shearing resistance, then (58) = (59). What will be the value of  $p$ ? What will be the value of  $d$ ?

**Ex. 30.** If in a width equal to  $p$ , there are  $n$  rivets in single shear then

$$(62) \quad (p-d)tS_t = .7854d^2nS_s.$$

Solve for  $p$ ,  $d$ ,  $t$ ,  $S_t$ ,  $S_s$ , and  $n$ . Interpret (62) and the resulting equations.

**Ex. 31.** If the  $n$  rivets are in double shear then

$$(63) \quad (p-d)tS_t = .7854d^2 2nS_s.$$

Solve for  $p$ ,  $d$ ,  $t$ ,  $n$ ,  $S_t$ , and  $S_s$ . Interpret (63) and the resulting equations.

**Ex. 32.** If the resistance to shearing equals the crushing resistance, then (59) = (60). What will be the value of  $d$ ?

Allow  $S_c = 2S_s$  and substitute in the last equation.

**24. Efficiency of a Joint.** The efficiency of a joint is the ratio  $\frac{(58)}{(61)}$  or  $\frac{(59)}{(61)}$  whichever is least. Interpret this ratio and express the ratio as efficiency per cent.

TABLE XI. TEST UPON A SERIES RAILWAY MOTOR

Point in Fig. 108. .	A	B	C	D	E	F	G	H	J	K
Speed (R.P.M.) . . .	2050	1230	950	805	705	630	575	530	495	470
Current (amperes)	15	20	25	30	35	40	45	50	55	60

By noting in Table XI the change in current with the corresponding successive values in speeds we can form an idea of the influence of the current variations. We can appreciate these facts if the eye can see them recorded with smaller intervals of change, or preferably by a **continuous record**. This may be accomplished with the aid of cross-section paper, as shown in Fig. 108.

**2. Cross-section Paper.** Squared cross-section paper is a sheet of paper which has been ruled carefully with horizontal and vertical lines. These are spaced equally, and usually there are **10 or 20 divisions to the inch**. Every fifth or tenth line is **accentuated** to facilitate the counting of spaces and the avoidance of errors. The location of any point on the paper is determined by its proximity to the nearest horizontal and the nearest vertical line. The bounding lines serve as beginning lines or **lines of reference** from which all other lines are numbered in order. The two lines of reference or zero lines are called the **axes**. The position of a point is determined by its **distance** from the **two axes of reference**, and accordingly every point is designated by a **pair of values**. An 8.5"×5.5" perforated paper has a ruled space 7.5"×5" subdivided into twentieths of an inch. Every tenth line is accentuated.

**3. Plotting Tabulated Data.** Upon the cross-section paper we **plot**, i.e., locate a series of points corresponding to each pair of values in the data of the table. A **smooth curve** is drawn through the plotted points with the aid of a **flexible rule** or a **French curve**.

The resulting curve is designated in terms of its variables, and is described as a **locus** (position of points) and is called a **graph** (a writing or pictured fact).

A pair of values is required to locate each point. It is necessary to choose suitable scales to be applied to the axes so that the respective values of the variables may fall within the range of the paper.

Beginning at the lower left-hand edge of the cross-section paper, ink over the accentuated horizontal and vertical lines bounding the paper. (See Fig. 108.)

These lines are called the **axes of reference**, and usually extend beyond their intersection. The latter is called the **origin** and is the zero for measurement on the respective axes.

On the two axes of reference lay off distances to suitable scales so as to comprehend the magnitudes of the table.

The first pair of values to be plotted is

A
2050
15

and the last pair is

K
470
60

The greatest value of the current is 60. There are 100 divisions in the horizontal direction of the paper. If we lay off a scale for amperes on the horizontal axis, then one division on the paper will correspond conveniently to one ampere. It will require 60 divisions for 60 amperes and therefore 3 ins. of length in the horizontal direction (assuming 20 divisions per inch).

Examining the table again we notice that the highest magnitude for speed is 2050. There are 150 divisions in the vertical direction of the paper. If we select one division

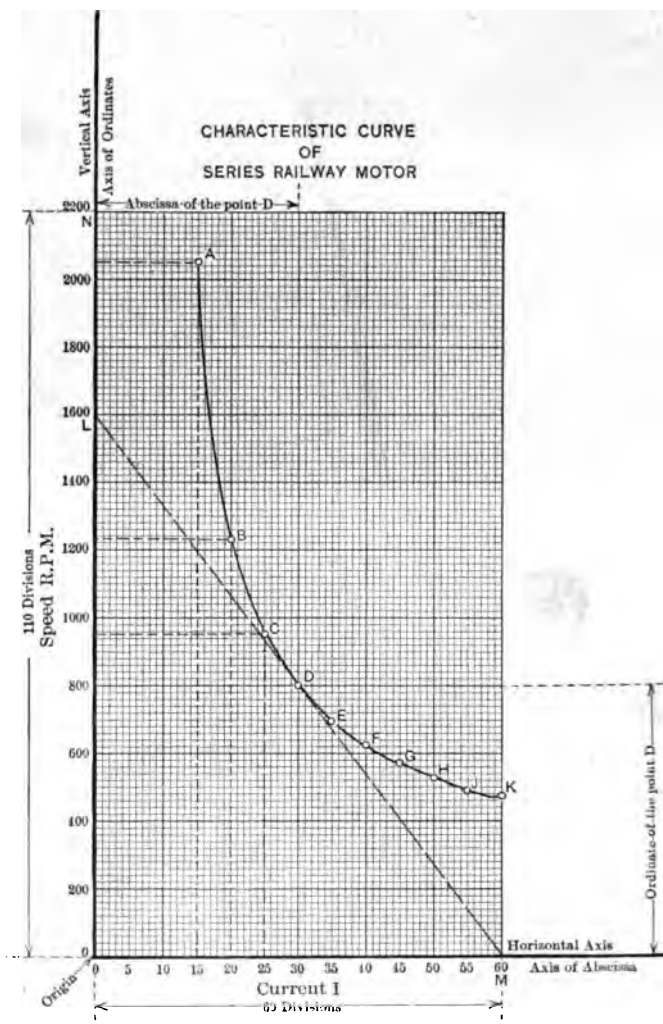


FIG. 108—The Plotting of Tabulated Data.

to correspond to 20 R.P.M. the paper will not be long enough. Suppose we choose one division to correspond to 25 R.P.M. Then 82 divisions will suffice or slightly more than 4 ins. of length.

Label the two axes Current *I* and Speed R.P.M. respectively. At convenient intervals on the axes mark and designate units of division. Every 5 amperes and every 200 revolutions should suffice to aid the eye in reading the scale.

The first point to be located is *A*, which corresponds to the first pair of associated values in Table XI.

On the horizontal (*I*) axis locate the point of division corresponding to 15 amperes. Move the finger upward over the vertical line passing through this point. Pause where it is intersected by the horizontal line passing through 2050 located on the vertical (R.P.M.) axis. Mark this point of intersection by making it the center of a small circle of  $\frac{1}{2}$  in. radius and label it *A*. The distance measured from the vertical axis is called the **abscissa** of the point, whereas the distance measured from the horizontal axis is called the **ordinate** of the point.

The second point to be plotted is *B* which corresponds to the second pair of associated values in the table.

Locate 20 on the *I* axis and 1230 on the R.P.M. axis. The intersection of the vertical and horizontal lines passing through these respective axial divisions is the second plot. Indicate it with a circle of  $\frac{1}{2}$  in. radius and label it *B*.

Proceed in like manner for the remaining pairs of values. In this way we shall have plotted ten points: *A*, *B*, *C*, *D*, *E*, *F*, *G*, *H*, *J*, *K*.

A smooth curve is drawn through the consecutive plots. Every point on the curve has a definite pair of values for its *I* and R.P.M. These are determined by the respective horizontal and vertical lines passing through the specified point on the curve. The intersection of the men-

tioned lines with the axes gives the magnitude in amperes and revolutions per minute respectively.

The curve is a succession of points. It is for this reason that the curve makes a continuous record and therefore the curve will supplement the values given in Table XI. The amplification of the table by means of the curve is called **interpolation**.

It is not difficult, although not always advisable, to project the curve beyond the limiting values of the table. Such an extension of the table is called **extrapolation**. This is identical with the process by which the human mind projects the reason and predicts with prophetic inspiration future happenings which at present are outside of the realm of recorded knowledge.

From the shape of the curve we observe that the rate of variation of the speed is not uniform and that for small currents the speed changes more rapidly than for larger currents. The **slope** of the curve is measured by the slope of its tangent. The slope of the curve at *D* is measured by the slope of the tangent *LD*.

*The tracing of the curves exemplifies the reversibility of all mathematical processes. It is just as easy to prepare the data for a table from a curve as it is to trace a curve from the data of the table. This law of duality has been illustrated in every pair of operations in the preceding chapters and will recur in all the chapters to follow.*

**Ex. 1.** Construct the magnetization curve for a machine from the data in Table XII. The curve is shown in Fig. 109. The extrapolation of the curve is shown by the dotted line and is

TABLE XII. MAGNETIZATION CURVE FOR A MACHINE

Flux per pole $\Phi$ . . . . .	$1 \times 10^6$	$2 \times 10^6$	$3 \times 10^6$	$3.5 \times 10^6$	$4 \times 10^6$
Field current <i>I</i> . . . . .	13.18	24.77	38.63	47.72	59.09

constructed upon the assumption that when there is no field current there will be no flux. This is not strictly true unless there is no residual magnetism.

What current will be necessary in order to produce a flux  $= 2.5 \times 10^6$ ?

What flux will be produced by a current of 30 amperes?

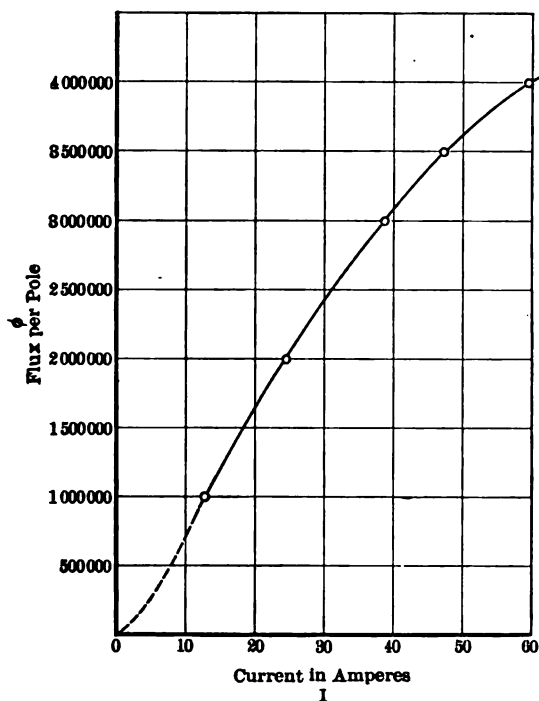


FIG. 109.—Magnetization Curve for a Machine.

What flux will be produced by a current of 35 amperes? In this figure as well as for subsequent figures, the one-tenth lines have been omitted for clearness.

**Ex. 2.** Construct a curve from the data in Table XIII. This curve is shown in Fig. 110 in which the three points *A*, *B*, and *C* appear to be on a straight line. Construct the figure so that the flux is read on the vertical axis.

TABLE XIII

Point.	Flux $\Phi$ .	Ampere Turns, $IN$ .
A	1,600,000 lines	12,070
B	1,700,000 "	13,207
C	1,800,000 "	14,105

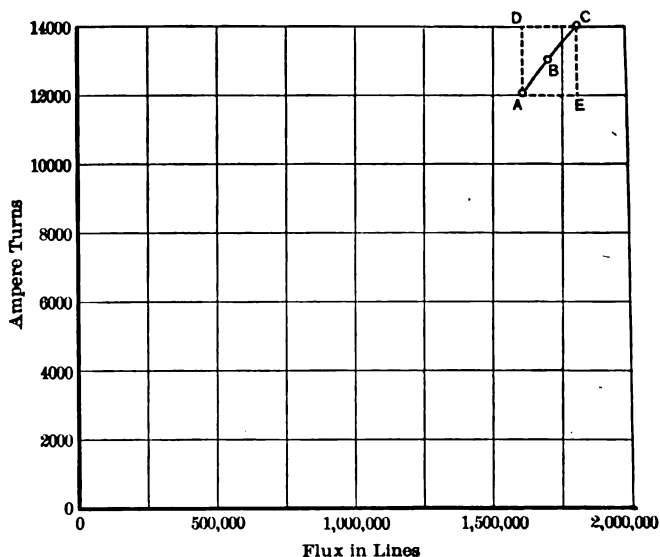


FIG. 110.—The Normal Position of Axes.

**4. Transformation of the Axes.** After the plotting of Table XIII it will be observed that the curve occupies a very small part of the paper. The value of the curve would be enhanced if larger scales were chosen. To increase the scale would necessitate a larger sheet of paper if we wished to preserve the axes and origin with the curve. We can dispense with the axes since we are concerned with *that area* of the paper only which is occupied by the curve.

sider the quadrangle  $ADCE$  enclosing and bounding the curve which extends horizontally from  $1.6 \times 10^6$  to  $1.8 \times 10^6$  and vertically from 12070 to 14105.

shall **magnify** this area by considering the point  $(1.6 \times 10^6, 12070)$  moved down to the usual position of the lower left-hand marginal intersection as shown in Fig. 1.

111. Locate the point  $1.8 \times 10^6$  at the right-hand limit of the horizontal axis. The two points  $1.6 \times 10^6$  and  $1.8 \times 10^6$  will be separated by 100 divisions. In like manner locate 14105 near the upper limit of the vertical axis.

Plot the curve to the new scale. The gain in size is illustrated by the magnification of errors incidental to plotting of abundant data. The slope of the curve becomes steeper with the magnification.

This operation is called the **transformation of the coordinates**. It is virtually equivalent to **shifting the origin** of the scales to the limits of the paper. The scales may be entered in any convenient horizontal or vertical bounding line. The work is simplified by plotting megalines instead of lines.

1. Use the principle of paragraph 4 in plotting Table XIV.

TABLE XIV

Flux $\Phi$ .	Ampere Turns, $IN$ .
5000	2500
6500	3175
8250	5500

1. Construct curves between  $B$  and  $H$  expressed in terms of  $B/H$  for sheet steel, wrought iron, soft cast steel, and cast iron. Plot these upon the same sheet of paper, using the data in Table XV.

These curves are called magnetization curves because they show the relation between the flux and the magnetizing force.

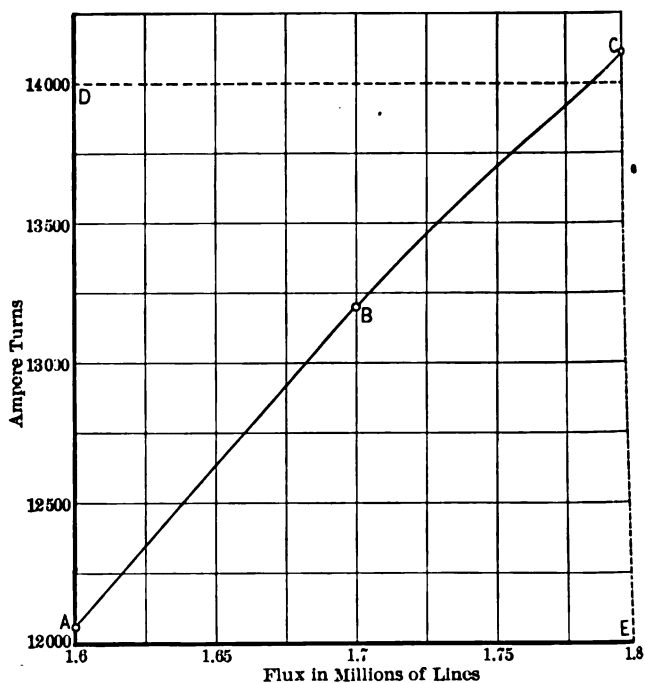


FIG. 111.—The Transformation of the Coordinate Axes.

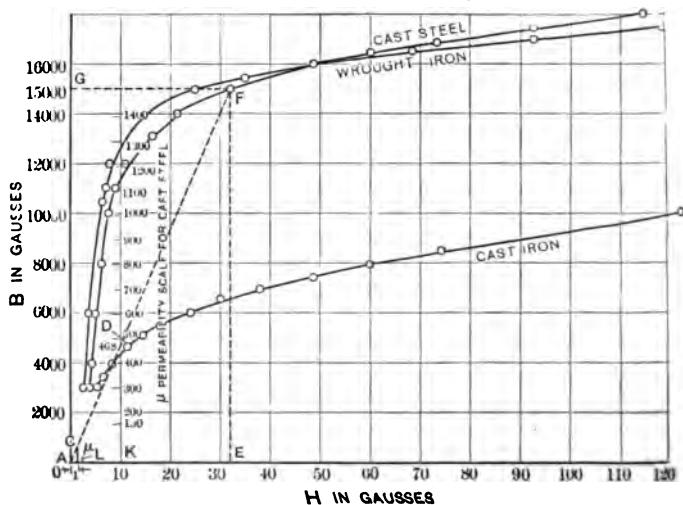


FIG. 112.—Magnetization Curves.

The same curves may be used to express the relation between the flux density per square inch read vertically and the ampere-turns per inch of length read horizontally or the ampere-turns per centimeter of length read horizontally. Supplement Fig. 12 by adding the three scales referred to above. Add the curve for sheet steel.

TABLE XV. VALUES OF  $B$  AND  $H$  IN GAUßES

$B$	$H$			
	Sheet Steel.	Cast Steel.	Wrought Iron.	Cast Iron.
3,000	1.3	2.8	2.0	5.0
3,500	.....	.....	.....	6.5
4,000	.....	3.4	.....	8.5
4,500	.....	.....	.....	11.0
5,000	.....	.....	.....	14.5
5,500	.....	.....	.....	18.5
6,000	2.3	4.5	3.5	24.0
6,500	2.4	.....	.....	30.0
7,000	.....	.....	.....	38.5
7,500	.....	.....	.....	49.0
8,000	.....	5.8	.....	60.0
8,500	.....	.....	.....	74.1
10,000	3.9	7.5	.....	124
10,500	4.1	.....	6.0	.....
11,000	.....	9.0	6.5	.....
12,000	5.0	11.5	7.9	.....
13,000	6.0	16	10.0	.....
14,000	9.0	21.5	15.0	.....
15,000	15.5	32.0	25.0	.....
15,500	.....	.....	35.0	.....
16,000	27.0	49.0	49.0	.....
16,500	37.5	60.0	69.0	.....
17,000	52.5	74.0	93.0	.....
17,500	70.0	93.0	120	.....
18,000	92.0	115	.....	.....

**Ex. 5.** Prepare table XVI as described below, using the values determined from the four curves of Ex. 4. Divide a page horizontally into seven columns and allow space under each column heading for 13 entries. The first column will be headed  $H$  and under it make 12 entries varying in steps of 10 from 10 to 120

inclusive. Make the corresponding entries under the following column headings: Ampere-turns per centimeter length, Ampere-turns per inch length; under double columns headed respectively cast iron, cast steel, wrought iron and sheet steel, make an entry for  $B$  in kilo-gausses and also an entry in kilo-maxwells per inch.

**5. Plotting of a Third Variable.** In the discussion of permeability in Chapter IX, page 185, it was shown that

$$\mu = \frac{B}{H},$$

which may be written

$$(1) \quad \frac{\mu}{1} = \frac{B}{H}.$$

Locate any point such as  $F$  on one of the curves. Drop the ordinate  $FE$ . Then  $FE$  = the  $B$  of the point ( $F$ ).  $AE$  = the  $H$  of the point ( $F$ ). Join  $A$  with  $F$ . Let  $AL = 1$  unit, draw  $CL$ , then  $AF$  intersects  $CL$  at  $C$ .

$$(2) \quad \frac{CL}{AL} = \frac{FE}{AE} \text{ ————— similar right } \triangle s$$

$$(3) \quad \therefore \frac{CL}{1} = \frac{B}{H} \text{ ————— subs. for } FE \text{ and } AE$$

$$(4) \quad \therefore CL = \mu \text{ ————— from (1) and (3) and =}$$

For every point on the curve there is a definite point on  $CL$ . The latter point is the  $\mu$  value of the former because it expresses the ratio of the  $B$  to the  $H$  of the former.

The values of  $\mu$  may be too close together on  $CL$  for close estimation but this is avoided by reading  $\mu$  on another line such as  $KD$  which is parallel to  $CL$ .

Triangles  $DKA$  and  $CLA$  are similar right triangles and therefore  $DK = 10\mu$ . In other words  $DK$  is proportional to  $\mu$ . For the point  $F$ ,  $B = 15000$  and  $H = 32$

therefore  $\mu = 468$ . Therefore either  $C$  or  $D$  may be marked 468. By proceeding in like manner to join other points on the curve with  $A$  the intersection on  $DK$  will give the corresponding  $\mu$  values of the respective points. If the points are carefully selected the values of  $\mu$  may be expressed in multiples of 100.

**Ex. 6.** Calibrate the line  $DK$  in gradations of 100 by using other points on the cast steel curve, show that the calibrations of  $DK$  apply to all magnetization curves.

**Ex. 7. Logarithmic Curve.** Consult a table of logarithms (three places will suffice). On the horizontal axis plot numbers from 1 to 15. On the vertical axis plot the corresponding logarithms.

On the same sheet plot a second curve for Napierian logs.

Show how additional or supplementary scales may enable the curves to be used for numbers ranging from 10 to 150, or for any multiple of 1 to 15.

In the case of common logarithms the vertical scale should read mantissas and the corresponding characteristic should be added mentally.

All logarithm curves pass through the point 1 on the horizontal axis. Why?

In Fig. 113  $1LR$  is the curve of common logarithms and  $1JP$  is the curve of Napierian logarithms. The vertical scale for these curves is the one closest to the axis and will be called the **axial** or **first** scale. A **second** scale appears further to the left of the vertical axis and applies to the curve  $1KQ$ . The latter is a magnified curve of common logarithms and has been plotted on the outside scale for closer reading.  $1KQ$  represents the common logarithms of squares of numbers when the axial scale is used. Why?

The Napierian curve may be obtained graphically from the curves  $1LR$  and  $1KQ$ .

The difference between the corresponding ordinates of  $1JP$  and  $1KQ$  equals .303 times the corresponding ordinate of  $1LR$ . Two of these differences are shown as  $c_1$  and  $c$ . Construct a right triangle with a base line equal to 2.303 units and an altitude  $y_c$ . Draw  $y_c$  parallel to  $y_c$  within the triangle at one unit's distance from the vertex.

We then obtain

$$\frac{y_c}{2.303} = \frac{y_c}{1},$$

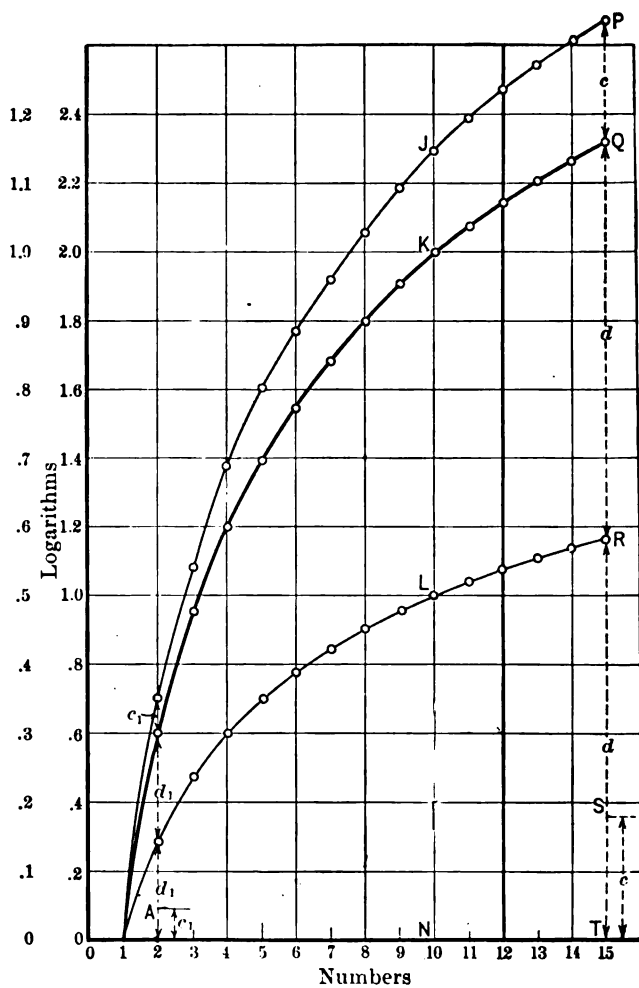


FIG. 113.—Logarithmic Curves.

and therefore  $y_c = 2.303y_e$ . This **graphic multiplication** method may be applied to obtain the Napierian curve  $IJP$  from the common curve  $ILR$ . This is illustrated as follows. Extend a line joining  $L$  with 9 until it intersects the ordinate passing through 11.303. Carry the intersection horizontally across to intersect  $LN$  and thereby locate  $J$  on  $IJP$ . In the right triangle  $LN$  and  $JN$  will be the measure of  $y_e$  and  $y_c$  respectively.

**Ex. 8. Periodic Curves.** Consult a natural sine and cosine table (three places will suffice).

Construct a table XVII, showing the range of numeric variation and the algebraic signs for the eight trigonometric functions with regard to four quadrants.

What angles correspond to the **greatest positive** and **greatest negative** values of their sines? What are the angles whose sines are zero?

What angles correspond to the **greatest positive** and **greatest negative** values of their cosines? What is  $\cos^{-1}(0)$ ?

Turn a sheet of cross-section paper so that its greatest length is horizontal and draw the horizontal axis through the center of the paper. Beginning at the left end of the axis locate points on it corresponding to every  $15^\circ$  of angle, ranging from  $0^\circ$  to  $360^\circ$ . Through these points of division draw light vertical lines. Since sines and cosines of angles lie between  $(+1)$  and  $(-1)$ , then 20 divisions in the vertical direction will correspond to 1.000 in the tables. Above or below the  $15^\circ$  points of division depending upon the algebraic sign, lay off the corresponding values of the sines of the respective angles.

For values beyond  $90^\circ$  apply the definitions of sine and cosine with due regard to algebraic signs.

The functions of angles of the second quadrant have numeric values corresponding to the like named functions of their supplements.

To obtain the values of functions for angles of the third quadrant subtract  $180^\circ$  from the angle.

To obtain the values of functions for angles of the fourth quadrant subtract the angle from  $360^\circ$ .

Through all the plotted points pass a smooth curve which takes the name **sine curve** according to the name of the plotted function.

In a like manner construct a **cosine curve**. Use the same axis and the same scale which was used for the sine curve. See Fig. 114.

What are the features of similarity and dissimilarity in the sine and cosine curves?

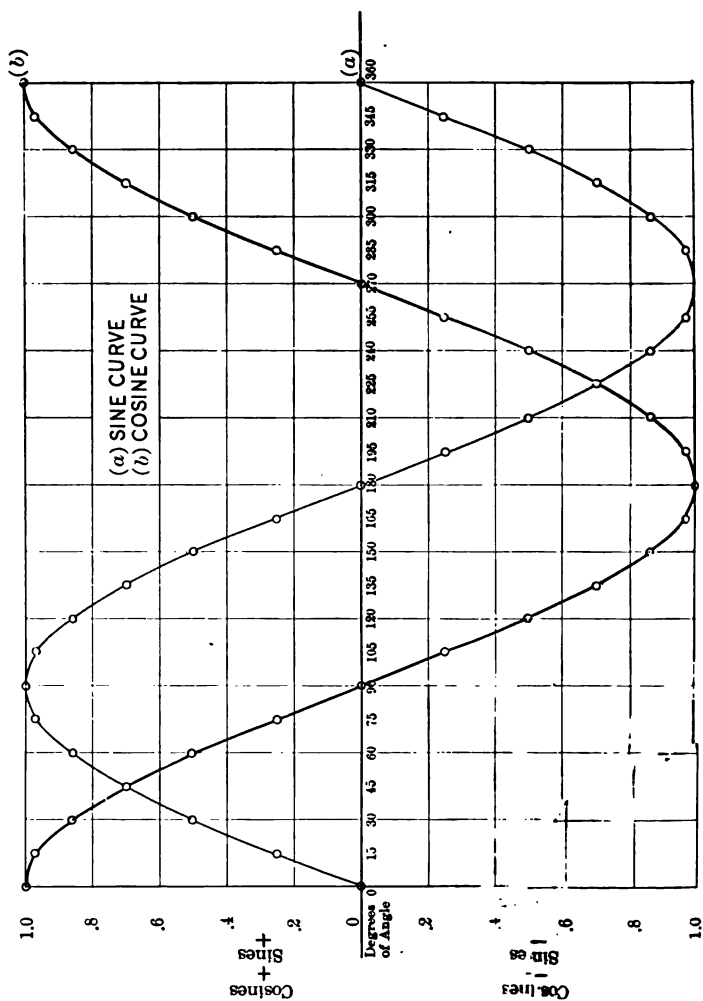


FIG. 114.—Periodic Curves.

Show from the curves that the sine of an angle of the first quadrant equals the cosine of its complement.

That portion of the curves lying between the zero ordinates of the curves is called an arc or arch or loop. Two consecutive arches constitute a complete cycle.

Where two curves appear on one sheet of paper they should be marked in some manner so as to be identified easily. Thus in Fig. 114 the sine curve is marked (a) and the cosine curve is marked (b).

The curves may be extended indefinitely to the right and to the left of the zero division. It will be seen that the curves repeat in both directions with a definite rhythm or in precisely equal intervals on the axes and hence are said to be periodic.

**6. Plotting of Formulas.** We have seen that a series of experimental results or a calculated logarithmic or trigonometric table may be plotted and a curve drawn so as to pass as evenly as possible through the points. It is not possible to draw a smooth curve through all the plotted points of experimental data. About an even number of points should be distributed on both sides of the curve. Misplaced points are due to the errors of observation and experimental defects and therefore the resulting curve should approximate to the average of these deviations. Data which when plotted show large deviations should be discarded.

Any formula may have two of its letters regarded as variables. The remaining letters will be considered fixed or constant while the variables will change through a range of coordinated or associated values. If we substitute assumed values for one variable it is called an **independent**, then the other variable is called a **dependent** and the latter may be solved for definite values according to the relation of the variables in the formula. These pairs of values may be tabulated. The tables may be used to plot a curve.

In this way the curve becomes the pictorial representation of the formula.

*A definite curve corresponds to every formula and conversely a definite formula corresponds to every curve.*

In the paragraphs which follow methods are illustrated for deriving the equation, i.e., formula corresponding to a curve.

### 7. The Graphic Representation of Ohm's Law. $I = \frac{E}{R}$ .

The formula of Ohm's Law contains three quantities  $I$ ,  $E$ , and  $R$ . The values of  $I$  will depend upon the value of the ratio  $\frac{E}{R}$ . If we consider a circuit having a constant resistance,  $R=5$  ohms, then as the impressed voltage  $E$  is made to vary, there will be a corresponding variation in  $I$ . For every value assigned to  $E$  there will be a corresponding dependent value of  $I$ . If  $E$  is made to vary from 0 to 10 volts in gradations of 1 volt then by substituting these values and also the value of  $R=5$  we obtain a corresponding range of values for  $I$ .

The associated pairs of values of  $E$  and  $I$  are arranged and entered in order in Table XVIII as shown. It is advisable to fill in the entire  $E$  column first by beginning with the least value, i.e., 0, and ending with the highest assigned value 10. As each value of  $I$  is calculated make the entry in the  $I$  column opposite the corresponding value of  $E$ .

Suppose  $E=0$  then  $I = \frac{E}{R} = \frac{0}{5} = 0$ . Under the  $I$  column enter 0 opposite 0 in the  $E$  column. When  $E=1$  then  $I = \frac{1}{5} = .2$ . Under the  $I$  column enter .2 opposite 1 in the  $E$  column. Finally when  $E=10$  then  $I = \frac{10}{5} = 2$ . Under the  $I$  column enter 2 opposite 10 in the  $E$  column. From the data so obtained we proceed to plot the corresponding curve in the usual manner.

Plot these values labeling the longer axis  $E$  and the shorter axis  $I$ . The resulting graph or curve is a straight line but a straight line is also a curve in the graphic sense.

TABLE XVIII.  $R=5\Omega$ .

$E$	$I$	$E$	$I$
0	0	6	1.2
1	.2	7	1.4
2	.4	8	1.6
3	.6	9	1.8
4	.8	10	2.0
5	1.0		

The curve passes through the origin. This condition will occur whenever the values of both variables are zero simultaneously.

The curve is said to be the **locus** of the formula, i.e., the position collectively of every point whose pairs of values validate the formula upon substitution.

The pair of values corresponding to any point are the respective distances measured parallel to the axes.

They are spoken of collectively as the **coordinates** of a point.

For purposes of distinction the first mentioned distance "the abscissa" is measured always from the vertical axis.

The second mentioned distance "the ordinate" is measured always from the horizontal axis.

Points are designated by enclosing their abscissa and ordinate values in a parenthesis.

$P \equiv (1, 0.2)$  means  $P$  is the point corresponding to an abscissa 1 and an ordinate 0.2. A comma separates the two enclosed values and a correspondence symbol ( $\equiv$ ) is placed between a letter and the corresponding parenthesis.

$P \equiv (8, 1.6)$  means locate a point 8 units distance along the horizontal axial direction and 1.6 unit in the vertical axial direction. This is a convention that must be observed for the above notation, although the point might be located equally well, by proceeding first in the vertical direction and then in the horizontal direction.

The carrying capacity of insulated copper wires for interior wiring according to the National Electric Code is given in Table XIX.

TABLE XIX. CARRYING CAPACITY OF INSULATED COPPER WIRES FOR INTERIOR WIRING, NATIONAL ELECTRICAL CODE

<i>A</i> B. & S. Co.	<i>B</i> Circular Mils.	<i>C</i> Rubber Covered Wires, Amperes.	<i>D</i> Weather Proof Wires, Amperes.	<i>E</i> Circular Mils.	<i>F</i> Rubber Covered Wires, Amperes.	<i>G</i> Weather Proof Wires, Amperes.
18	1,624	3	5	200,000	200	300
16	2,583	6	8	300,000	270	400
14	4,107	12	16	400,000	330	500
12	6,530	17	23	500,000	390	590
10	10,380	24	32	600,000	450	680
8	16,510	33	46	700,000	500	760
6	26,250	46	65	800,000	550	840
5	33,100	54	77	900,000	600	920
4	41,740	65	92	1,000,000	650	1000
3	52,630	76	110	1,100,000	690	1080
2	66,370	90	131	1,200,000	730	1150
1	83,690	107	156	1,300,000	770	1220
0	105,500	127	185	1,400,000	810	1290
00	133,100	150	220	1,500,000	850	1360
000	167,800	177	262	1,600,000	890	1430
0000	211,600	210	312	1,700,000	930	1490
				1,800,000	970	1550
				1,900,000	1010	1610
				2,000,000	1050	1670

**Ex. 9.** Label the horizontal axis for the *A* and *B* scales and the vertical axis for the *C* and *D* scales. Plot one curve for values of *A* and *C* and another curve for values of *A* and *D*.

**Ex. 10.** Label the horizontal axis *E* and the vertical axis *F* and *G*. Plot one curve for values of *E* and *F* and another curve for values of *E* and *G*.

**Ex. 11.** Plot the following table XX for temperature coefficient of copper at different initial temperatures in Centigrade. Supplement the Centigrade scale with a scale for reading Fahrenheit.

TABLE XX. TEMPERATURE COEFFICIENTS

Initial Temperature Centigrade.	Temperature Coefficient in Per Cent per Degree Centigrade.	Initial Temperature, Centigrade.	Temperature Coefficient in Per Cent per Degree Centigrade.
0	.420	26	.379
1	.418	27	.377
2	.417	28	.376
3	.415	29	.374
4	.413	30	.373
5	.411	31	.372
6	.410	32	.370
7	.408	33	.369
8	.406	34	.368
9	.405	35	.366
10	.403	36	.365
11	.402	37	.364
12	.400	38	.362
13	.398	39	.361
14	.397	40	.360
15	.395	41	.358
16	.394	42	.357
17	.392	43	.356
18	.391	44	.355
19	.389	45	.353
20	.388	46	.352
21	.386	47	.351
22	.385	48	.350
23	.383	49	.348
24	.382	50	.347
25	.380		

## CHAPTER XV

### LINEAR GRAPHS

1. In the graphic work which follows prepare the tables completely before proceeding to plot. Begin by substituting a zero value or the least assigned value for one of the variables. Label curves to correspond to the tables. Designate the axes in full and with symbolic abbreviations. Where the subject matter bears a title enter it upon the cross-section paper. Tables may be entered on the cross-section paper but it is preferable to enter them with the numeric calculations upon a sheet of plain paper which is inserted in the work book so as to face the graph.

The cross-section paper upon which the graph has been drawn is called a **plate** and is numbered in Roman notation and bears a cross reference to the page of tables, calculations and other descriptive matter.

**Ex. 1.** Prepare a table of values and plot  $I = \frac{E}{R}$ , wherein  $R$  is constantly equal to 1 ohm and  $E$  varies from 0 to 12 volts. This curve is shown as *ACE* in Fig. 115. The (a) scales apply to it.

**Ex. 2. Variation of Coulombs with Time.** Prepare a table of values and plot  $Q = It$ , where  $I$  is constantly equal to 10 amp. and  $t$  varies from 0 to 60.

This curve is shown as *LMN* in Fig. 115. The (b) scales apply to it.

**Ex. 3. Variation Metallic Deposition with Current.** Prepare a table of values and plot

$$I = \frac{W}{K_2 t}; K_2 \text{ is constantly } = 0.0003386,$$

$$t \text{ is constantly } = 1 \text{ hr. } = 60 \text{ min. } = 3600 \text{ sec.}$$

$$I \text{ varies from 2.5 to 10.}$$

This curve is shown as *QS* in Fig. 115. The (c) scales apply to it.

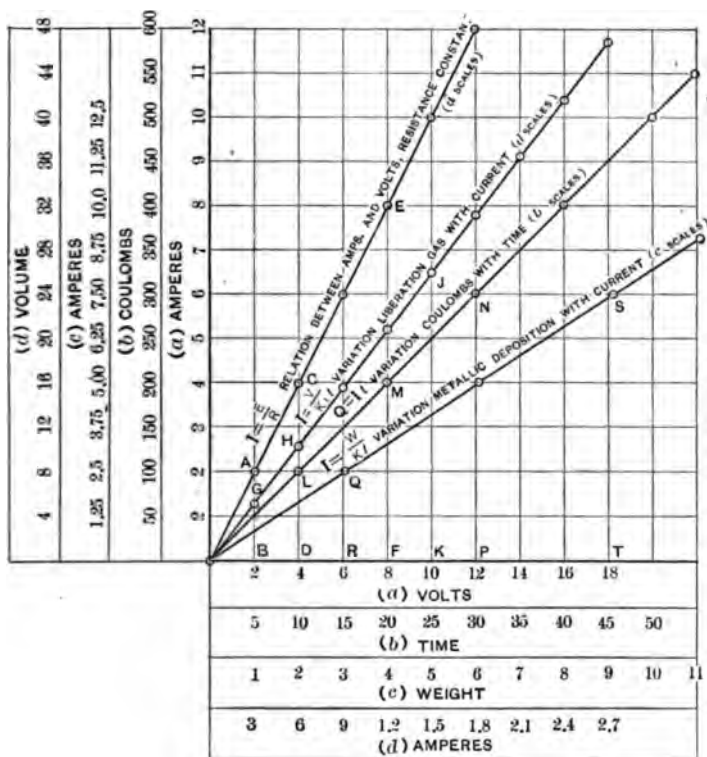


FIG. 115.—The Graphs of Simple Equations.

**Ex. 4. Variation Liberation Gas with Current.** Prepare a table of values and plot

$$I = \frac{V}{K_1 t}; K_1 \text{ is constantly } = 0.1733,$$

$t$  is constantly = 100 sec.

$I$  varies from 0.3 to 2.7.

This curve is shown as *GHJ* in Fig. 115. The (d) scales apply to it.

**Ex. 5.** On one sheet of cross-section paper plot three graphs of  $I = \frac{E}{R}$ . These may be obtained by preparing three distinct tables regarding  $E$  and  $I$  as the variables but by assigning a definite value to  $R$  for each table as follows: (a) when  $R = .1\Omega$ , (b) when  $R = 10\Omega$ , (c) when  $R = 100\Omega$ . Label the graphs (a), (b) and (c) to correspond to the tables. Plot  $I$  vertically and  $E$  horizontally. From the preceding plots of Ohm's Law it was observed that in each case the graph is a straight line. The three graphs of Ex. 5 will be straight lines. A straight line is completely determined by two of its points and therefore the above tables should contain only two pairs of entries. Since one of the pairs of values indicates that the line passes through the origin, it is advisable for accuracy that the other pair of values should locate a point at considerable distance from the origin.

2. The **slope** of a straight line is the numeric value of the tangent of its angle of inclination to the horizontal when the scales of both axes are equal. Whenever the scales of the two axes are unequal the slope is the ratio of the number of units in the perpendicular to the number of units in the corresponding projection of the line.

In Fig. 115 the slope of

$$AE = \frac{EF}{OF} = \frac{8}{8} = 1, \quad \text{the slope of } LN = \frac{NP}{OP} = \frac{300}{30} = 10.$$

**Ex. 6.** Determine the slope of  $QS$  and  $GJ$  in Fig. 115.

**Ex. 7.** Determine the slopes of graphs  $a$ ,  $b$  and  $c$  in Ex. 5. How does the value of  $R$  affect the slope? How does the value of the reciprocal of  $R$  affect the slope? What is the relation of reciprocal of  $R$  to  $E$ ?

3. It is customary to plot the **dependent variable** vertically and the **independent variable** horizontally. The slope will be numerically the same as the coefficient of the independent variable. In any equation either variable may be made the dependent variable. The formula or

equation must be transformed so that the dependent variable has a coefficient and exponent equal to unity.

Following the convention of signs established for trigonometry distances measured to the right and above the origin are positive, and on the contrary distances measured to the left and below the origin are negative.

**Ex. 8. Centigrade-Fahrenheit Conversion.** Plot  $C = \frac{5}{9}(F - 32)$ . Transform and plot  $F$  as the dependent variable. Draw the axes through the center of the paper. The range of  $C$  is to be from  $-300^\circ$  to  $+300^\circ$ . This graph is illustrated as (a) in Fig. 116. The line is not extended beyond  $-273^\circ$  C. Why? Measure the slope of the graph and compare it with the coefficient of the independent variable. What is the value of the intercepted distance on the vertical axis between the origin and the graph?

**Ex. 9. Copper Temperature Coefficient.** Plot  $R_T = R_0(1 + .0042T)$ .  $R_0 = 9.59$  ohms which is the mil foot resistance of annealed copper wire at  $0^\circ$  C. The values of  $T$  are in centigrade and range from  $-20^\circ$  to  $+50^\circ$ . This graph is shown as (b) in Fig. 116. Measure the slope of the graph and show that it is numerically the same as  $.0042R_0$ . By means of the Centigrade-Fahrenheit conversion scales show that  $.0024$  is the coefficient of  $T$  when expressed in Fahrenheit degrees. The value of  $R_0$  is read off at the point where the graph cuts the vertical axis. Plot Ex. 9 on the same plate as Ex. 8, using  $R_T$  as the dependent variable.

**Ex. 10. Variation of E.M.F. and Resistance.**

$$\text{Plot } I = \frac{E}{R + r}.$$

$I$  constantly = 10 amps.,

$r$  constantly = 2 ohms,

$R$  varies from 0 to 10 ohms.

4. The plots of examples (1) . . . (10) are characterized by graphs which are straight lines. The slopes of the straight graphs as well as their intersections with the axes of reference is very easily predetermined before the formula is plotted.

The formulas that we have dealt with are of the first degree and are represented by straight line graphs. A

formula is said to be linear when it is an equation of the first degree. Why are the formulas of Ex. (1) . . . (10) of the first degree?

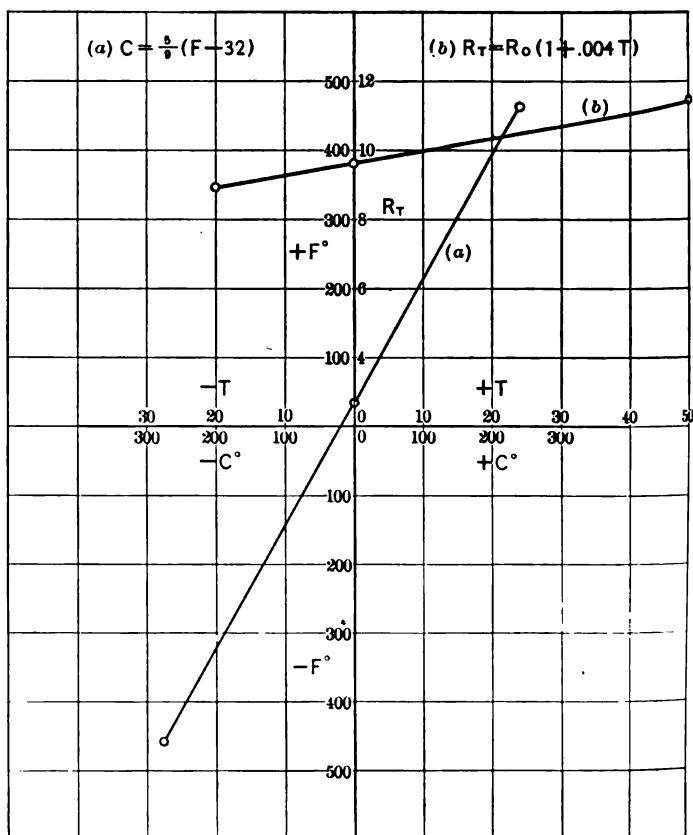


FIG. 116.—(a) Centigrade-Fahrenheit Conversion.  
(b) Copper Temperature Coefficient.

The formulas (1) . . . (10) may be regarded as forms of one simple equation, viz.:

$$(1) \quad y = a + bx.$$

This is shown by assigning to  $x$ ,  $y$ ,  $a$ , and  $b$  those values which will identify (1) with each equation in the group 1) . . . (10).

To show that

2)  $I = \frac{E}{R}$  is a form of  $y = a + bx$ ,

make  $y$  in (1)  $\equiv I$  in (2),

"  $x$  " "  $E$  "

"  $b$  " "  $\frac{1}{R}$  "

"  $a$  " " zero in (2) (missing term)

then  $y = a + bx$  becomes  $I = \frac{E}{R}$  and the latter is said to be a special form of the linear equation  $y = a + bx$ .

To show that

3)  $Q = It$  is a form of  $y = a + bx$ ,

make  $y$  in (1)  $\equiv Q$  in (3),

"  $x$  " "  $t$  "

"  $b$  " "  $I$  "

"  $a$  " " zero in (3) (missing term).

To show  $R_T = R_0(1 + 0.004T)$  is a form of  $y = a + bx$ , first transform it, then  $R_T = R_0 + 0.004R_0T$ .

Then show that

4)  $R_T = R_0 + 0.004R_0T$

is a form of  $y = a + bx$

make  $y$  in (1)  $\equiv R_T$  in (4).

"  $x$  " "  $T$  "

"  $b$  " "  $0.004R_0$  "

"  $a$  " "  $R_0$  "

then  $y = a + bx$  becomes  $R_T = R_0 + 0.004R_0T$  and therefore  $R_T = R_0(1 + 0.004T)$  is said to be a special form of (1).

Show how the formulas in Ex. (3), (4), (8), and (10) are interpreted as special forms of the linear equation  $y = a + bx$ .

**5. Degree of an Equation.** We have noted that equations contain two kinds of quantities, variable elements (variables) and fixed quantities (constants). The latter are generally numeric or literal factors and constitute the coefficients of the variables in their respective terms. There may be one isolated constant called the absolute term.

Variables enter equations with exponents which may be integral or fractional, numeric or literal, positive or negative.

The **degree** of a **term** is the numeric value of the exponent of the variable in that term. When several variables occur as factors in a term then the degree of the term equals the sum of the exponent of the variables.

The **degree** of an **equation** is equal to the degree of its highest term.

The degree of an equation is determined only after it is free from fractional and negative exponents, radicals and denominators with variable factors.

Why does  $y = a + bx$  represent an equation of the first degree? What is the degree of the group of equations which are special forms of (1)? Why?

**Ex. 11.** State the degree of the following equations:

$$\left. \begin{array}{ll} (5) & x = y^2 \\ (6) & x = ay^3 \\ (7) & x = c + dy \\ (8) & x = \frac{a}{y} \end{array} \right\} x \text{ and } y \text{ are variables}$$

$$(9) \quad pv^n = R \dots \dots p \text{ and } v \text{ are variables}$$

$$(10) \quad W = IE \dots \dots I \text{ and } E \text{ are variables.}$$

$$\begin{array}{ll}
 (11) & I = \frac{E}{R} \\
 (12) & I = \frac{E}{R+r} \\
 (13) & 5y = 2 + 0.3x \\
 (14) & y = 7 - 5x \\
 (15) & y = 3 - 2.5x
 \end{array}
 \left. \vphantom{\begin{array}{l} (11) \\ (12) \\ (13) \\ (14) \\ (15) \end{array}} \right\} \begin{array}{l} I \text{ and } R \text{ are variables.} \\ \\ x \text{ and } y \text{ are variables.} \end{array}$$

6. In the preceding examples we have seen that every first degree equation is plotted as a linear graph, i.e., a straight line. Two questions arise: first, is every equation of the first degree represented by a linear graph, and second, shall we interpret every linear graph as an equation of a straight line? We shall now proceed to the proof of these statements.  $x$  is the independent (variable),  $y$  the dependent (variable) and  $a$  and  $b$  are fixed numeric values (constants) as expressed in the general equation of the first degree:

$$(1) \qquad y = a + bx.$$

Whatever is true of (1) is true in general for all first degree equations. Assign to  $x$ , any six values designated by  $x_1, x_0, x_2, x_3, x_4, x_5$ , and then substitute these in succession in (1), and obtain the corresponding values  $y_1, y_0, y_2, y_3, y_4, y_5$ , from (1) as shown in (a), (b), (c), (d), (e), (f). Letters are preferable to numbers in this discussion.

$$\begin{array}{ll}
 (a) & y_1 = a + bx_1, \\
 (b) & y_0 = a + bx_0, \\
 (c) & y_1 = a + bx_2, \\
 (d) & y_3 = a + bx_3, \\
 (e) & y_4 = a + bx_4, \\
 (f) & y_5 = a + bx_5,
 \end{array}$$

These pairs of values are tabulated in Table XXI and are then plotted as points  $P_1, P_0, P_2, P_3, P_4$ , and  $P_5$  in Fig. 117.

TABLE XXI. COORDINATES OF POINTS

Plotted point . . . . .	$P_1$	$P_0$	$P_2$	$P_3$	$P_4$	$P_5$
Values of $x$ . . . . .	$x_1$	$x_0$	$x_2$	$x_3$	$x_4$	$x_5$
Values of $y$ . . . . .	$y_1$	$y_0$	$y_2$	$y_3$	$y_4$	$y_5$

Construct the ordinates at each point, and from each point and between the adjacent ordinates draw parallels

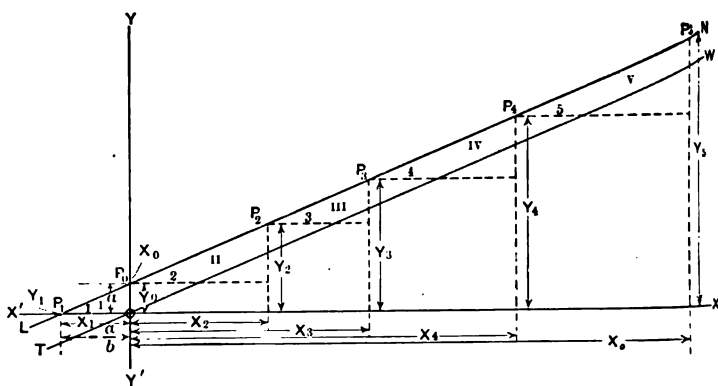


FIG. 117.—A Linear Graph Corresponds to Simple Equation.

to the  $XX'$  axis. These construction lines are represented by dash lines. Join each pair of consecutive points by straight lines:  $P_5P_4, P_4P_3, P_3P_2, P_2P_0, P_0P_1$ . In this manner we have formed five right triangles V, IV, III, II, and I. The pairs of sides are  $y_5 - y_4$  and  $x_5 - x_4, y_4 - y_3$  and  $x_4 - x_3, y_3 - y_2$  and  $x_3 - x_2, y_2 - y_0$  and  $x_2 - x_0$ , and  $y_0 - y_1$  and  $x_0 - x_1$ . These pairs of sides have equal ratios

which may be obtained by subtracting the six consecutive equations in the group (a), (b), (c), (d), (e), (f) above.

$$(g) \quad \therefore \frac{y_5 - y_4}{x_5 - x_4} = \frac{y_4 - y_3}{x_4 - x_3} = \frac{y_3 - y_2}{x_3 - x_2} = \frac{y_2 - y_0}{x_2 - x_0} = \frac{y_0 - y_1}{x_0 - x_1} = b.$$

These equal ratios express the value of the slopes of  $P_5P_4$ ,  $P_4P_3$ ,  $P_3P_2$ ,  $P_2P_0$ , and  $P_0P_1$ , i.e., the tangents of 5, 4, 3, 2, and 1 are equal and therefore angles  $5=4=3=2=1$ . Each of the five triangles contain a right angle. Therefore the sum of the component angles at each of the points,  $P_4$ ,  $P_3$ ,  $P_2$ , and  $P_0$  equals  $180^\circ$ . In other words  $P_5P_4P_3$  is a straight angle and therefore  $P_5P_3$  is a straight line. By a like authority the following pairs of lines form straight angles,  $P_4P_3$  and  $P_3P_2$ ,  $P_3P_2$  and  $P_2P_0$ , and  $P_2P_0$  and  $P_0P_1$ . Therefore  $P_5$ ,  $P_4$ ,  $P_3$ ,  $P_2$ ,  $P_0$ ,  $P_1$  all lie on one straight line  $LN$  and since these points were plotted from **any** assumed values which would satisfy (1) as shown in (a), (b), (c), (d), (e), (f), then **all points** which satisfy (1) will lie on  $LN$  or  $LN$  produced. Conversely, the slope measured at every point on  $LN$  will satisfy equation (1). Therefore:

*Every equation of the first degree is represented by a straight line and every straight line is interpreted as an equation of the first degree.*

**7. Intercepts.** When we substitute in (1) the value for  $x=0$  then (1) reduces to  $y=a+b \times 0=a$ .

This means that the line cuts the  $Y$  axis at a distance  $a$  from the origin. This distance is called the **Y intercept**.

On the other hand, if the equation is solved for  $x$  and the value for  $y=0$  is substituted the equation reduces to

$$x = -\frac{a}{b}.$$

This means that the line cuts the  $X$  axis at a distance  $-\frac{a}{b}$  from the origin. This is a negative distance, and

therefore it is laid off to the left of the origin. It is designated as the **X** intercept.

**8. The Slope of a Linear Equation.** The line  $LN$  ( $y=a+bx$ ) makes an angle 1 with the  $X$  axis. The slope of

$$LN = \frac{y_2 - y_0}{x_2 - x_0},$$

but

$$\frac{y_2 - y_0}{x_2 - x_0} = b,$$

by equation (g).

Therefore the slope of  $LN=b$ , which is the coefficient of the independent variable  $x$  in  $y=a+bx$ .

*In equation  $y=a+bx$  we have by recapitulation:*

*$a$  is the intercepts on the  $Y$  axis,*

*$-\frac{a}{b}$  is the intercepts on the  $X$  axis,*

*$b$  is the slope of the line.*

In Fig. 117 a line  $TW$  is drawn parallel to  $LN$  so as to pass through  $O$ . What is its intercepts on the  $Y$  axis? Therefore  $a=0$ .

The slope of  $TW$  equals the slope of  $LN$ .

Therefore the equation of a line passing through the origin is  $y=bx$ .

Every linear equation from which the absolute term ( $a$ ), i.e., constant term, is missing represents a line through what point?

For such a line show that its slope =

$$\frac{\text{the ordinate of a point}}{\text{the abscissa of the same point}}.$$

Equation (1) may be transformed into (h) in which the denominators of  $x$  and  $y$  represent the intercepts measured in the  $X$  and  $Y$  directions respectively.

$$(h) \quad \frac{y}{a} + \frac{x}{-a} = 1.$$

*Observation. Plotting a First Degree Equation. Two points are sufficient to determine a straight line. Therefore before attempting the plotting of formulas of the first degree determine the intercepts of the graph. Then through the pair of points so determined draw a straight line which corresponds to the formula.*

*This is quickly done by substituting  $y=0$  and solving for  $x$ , then substituting  $x=0$  and solving for  $y$ .*

*Should  $x$  and  $y$  both equal zero simultaneously, then the line passes through the origin. In such cases an additional pair of values will have to be used for a second point. There is the alternative, however, of using the slope at the origin.*

### 9. Writing the Equation for a Given Straight Line.

I. The equation of a straight line may be obtained by substituting in (1) or (h), the value of  $b$ , which is the slope of the graph, and the value of  $a$ , which is the intercepts on the  $Y$  axis.

II. The equation of a straight line may be obtained by substituting in (h) the values of  $a$  and  $\frac{-a}{b}$ , which are the respective intercepts of the graph on the  $Y$  and  $X$  axes.

III. It is always possible to measure the slope of a line but not always convenient to determine the intercepts. In such cases make use of (g). Determine  $b$  the slope of the line and the coordinates  $(x_1, y_1)$  for any point on the line. Substitute these three values in (g) and simplify.

$$(g) \quad \frac{y-y_1}{x-x_1} = b.$$

IV. A fourth method used in obtaining the equation of a straight line is of more general application to all classes of curves.

We wish to express the line in the form (1) and therefore it is necessary to determine the values of  $a$  and  $b$ . Two points on the curve give us the coordinates  $P_1 \equiv (x_1 y_1)$  and  $P_2 \equiv (x_2 y_2)$ . Substitute the coordinates in (1) and obtain (a) and (c).

$$(a) \qquad y_1 = a + bx_1,$$

$$(c) \qquad y_2 = a + bx_2.$$

From the equations (a) and (c) the constants  $a$  and  $b$  are determined by one of the methods of elimination for the solution of simultaneous equations.

**10. Graphic Representation of Simultaneous Equations.** Two linear equations when plotted give two corresponding straight lines which by geometry intersect in a single point. This common point has a pair of coordinate values which when substituted in both given equations satisfies or validates them. No other point on either line validates both equations. The coordinates of the intersection should agree with the numeric solution obtained by elimination, using algebraic methods.

**Ex. 12.** Determine the solution of the following simultaneous equations:

$$(a) \qquad 3x + y = 3,$$

$$(b) \qquad 5x + 2y = 4.$$

Determine the intercepts for (a) and enter these in a table labeled (a) and then determine the intercepts for (b) and enter these in a table labeled (b). Plot directly on the axes from the

3. Extend the linear graphs until they intersect. Determine abscissa and ordinate of the point of intersection and write

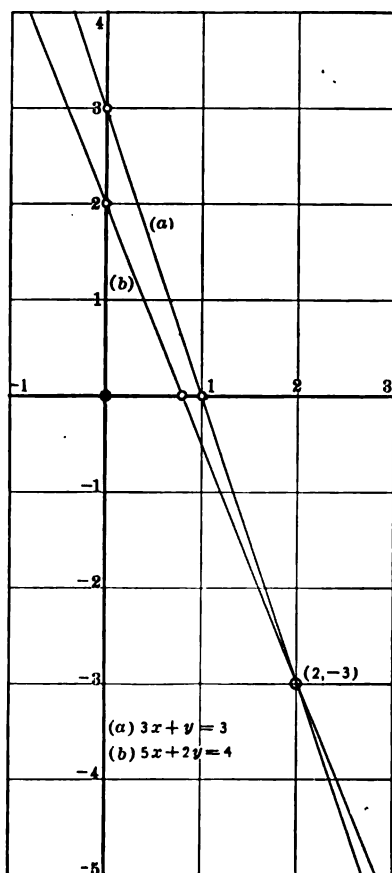
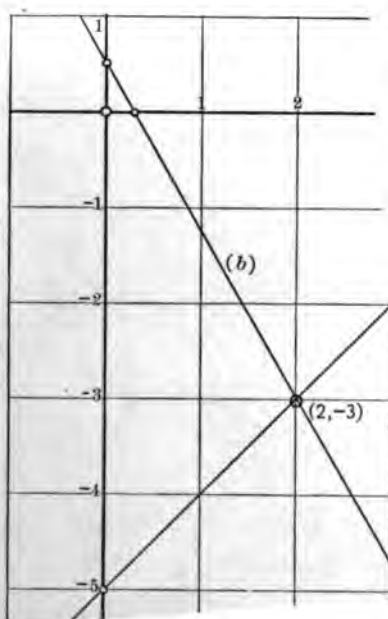


FIG. 118.—Graphic Solution of Simultaneous Linear Equations.

respective values for  $x$  and  $y$ . These graphs are illustrated in Fig. 118. The intersection gives  $x = 2$  and  $y = -3$ . An enlargement of the horizontal scale would define the intersection more distinctly.

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it is seen that the intercepts for (b) are



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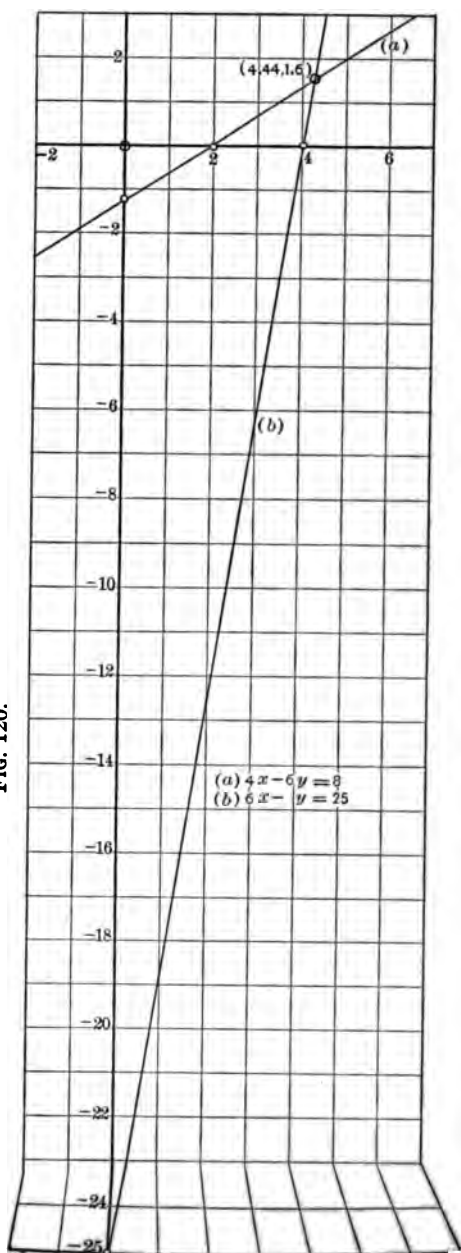
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Fig. 120.



## CHAPTER XVI

### NON-LINEAR ALGEBRAIC EQUATIONS

1. **THE** first degree equation (1) is called a linear formula from the fact that it corresponds to a straight line graph when plotted to two rectangular axes.

$$(1) \qquad y = a + bx.$$

$$(2) \qquad y_1 = bx.$$

We have observed that (2) is a special form of (1) when the  $a$  term becomes zero. (1) and (2) have the same slope and are therefore parallel lines. If the distance  $a$  is added to (2) then (1) may be written (3),

$$(3) \qquad y = a + y_1.$$

The interpretation of (3) states that the ordinates of curve (1) are  $a$  greater than the ordinates of (2). The two elements which change the position of a line are its slope  $b$  and its vertical axial intercepts  $a$ .

2. A **non-linear equation** is one which differs from (1) in some degree, i.e., one whose graph is not a straight line as illustrated in the algebraic and non-algebraic equations (4) . . . (17).

ALGEBRAIC.

(4)  $y = a + bx^{\frac{1}{2}}$ .

(5)  $y = a + bx^2$ .

(6)  $y = bx^3$ .

(7)  $y = a + bx^n$ .

(8)  $y = a + bx^n + cx^m$ .

(9)  $y = bx^{\pm m}$ .

(10)  $y = a + bx + cx^2 + dx^3 + ex^4 + \dots$

NON-ALGEBRAIC.

(11)  $y = \sin x$ .

(12)  $y = \log x$ .

(13)  $y = a + b \sin x$ .

(14)  $y = \sin^{-1}(x)$ .

(15)  $y = e^x$ .

(16)  $y = e^{ax} \sin (bx + c)$ .

(17)  $R = \theta^2$ .

Algebraic equations involve only the operations of addition, subtraction, multiplication, division, and commensurable powers of the variable.

**3. Parabolic Curves.** The equation  $y = x^m$  represents a standard parabola. There are infinite number of parabolic curves obeying the law that the ordinate is equal to a definite power of the abscissa. For every positive integral or positive fractional power of  $x$  there is a definite standard parabola. Plot the graphs for  $y = x^m$  for the following values of the exponent  $m$ .

Curve of Cube Roots.

Ex. 1.  $m = \frac{1}{3} = \frac{2}{6} \frac{\text{odd}}{\text{odd}}$ .

Curve of Solenoid Winding.

Ex. 6.  $m = \frac{4}{3} = \frac{4}{6} \frac{\text{even}}{\text{odd}}$ .

Curve of Square Roots.

Ex. 2.  $m = \frac{1}{2} = \frac{2}{4} \frac{\text{odd}}{\text{even}}$ .

Curve of Carrying Capacity.

Ex. 7.  $m = \frac{3}{2} = 1.5 \frac{\text{odd}}{\text{even}}$ .

Curve of Fusing Effects of Currents. Curve of Squares of Copper Losses.

Ex. 3.  $m = \frac{2}{3} = \frac{4}{6} \frac{\text{even}}{\text{odd}}$ .

Ex. 8.  $m = 2 = \frac{2}{1} \frac{\text{even}}{\text{odd}}$ .

Curve of Cubes or Luminous Intensity.

Ex. 4.  $m = \frac{3}{4} = .75 \frac{\text{odd}}{\text{even}}$ .

Ex. 9.  $m = 3 = \frac{3}{1} \frac{\text{odd}}{\text{odd}}$ .

Curve of Energy Radiation of Black Body.

Ex. 5.  $m = 1 = \frac{1}{1} \frac{\text{odd}}{\text{odd}}$ .

Ex. 10.  $m = 4 = \frac{\text{even}}{\text{odd}}$ .

Prepare and label a table of values for  $x$  and  $y$  in each example. Assume ten values for  $x$  and determine the corresponding values for  $y$ . The values must be so chosen that they may be located on the axes of the cross-section paper. The origin of the axes should be located at the center of the paper and equal scales chosen for both axes. Plot the ten curves on the same sheet of paper to the same scale. Arrange a blank sheet of paper to contain the ten tables and place this sheet so as to face the graphs. Letter and number each graph to correspond to a like letter or number which appears in the label of the table so as to facilitate cross-reference.

4. Fig. 121 represents the graphs of  $y = x^m$  for the following values of  $m$ :  $\frac{1}{4}$ ,  $\frac{1}{3}$ ,  $\frac{1}{2}$ , 1, 2, 3, 4, which correspond to graphs (f), (e), (d), (g), (a), (b), and (c) respectively.

Each parabola is divided at the origin into two equal parts. Curve (c), which has the greatest exponent shows the least divergence from the Y axis, whereas curve (f) which has the least exponent, shows the greatest divergence from the Y axis. The numeric order of the exponents is also the numeric order of the graphs. (g) is recognized as a straight line and separates the graphs which have integral exponents from the graphs which have decimal exponents.

Every parabola passes through the common point (1, 1) which is obtained for each equation by substituting  $x=1$  in  $y=x^m$ . Why? The area near the origin is enlarged, shown in the upper group of graphs in Fig. 121, which may be readily identified by their letters. In each case half the parabola is located in the first quadrant. The second half of the parabola may be located in the second, third, fourth quadrant, depending upon whether the exponent reduces to a ratio, between an even and an odd number, between two odd numbers, or between an odd and an even number. A quantity with a fractional exponent may be written in the radical form. The denominator of the exponent becomes the index of the root and the numerator of the exponent becomes the power index of the quantity. Thus

$$(18) \quad y = x^{\frac{3}{5}} = \sqrt[5]{x^3} \quad \text{and} \quad y = x^{\frac{2}{3}} = \sqrt[3]{x^2}.$$

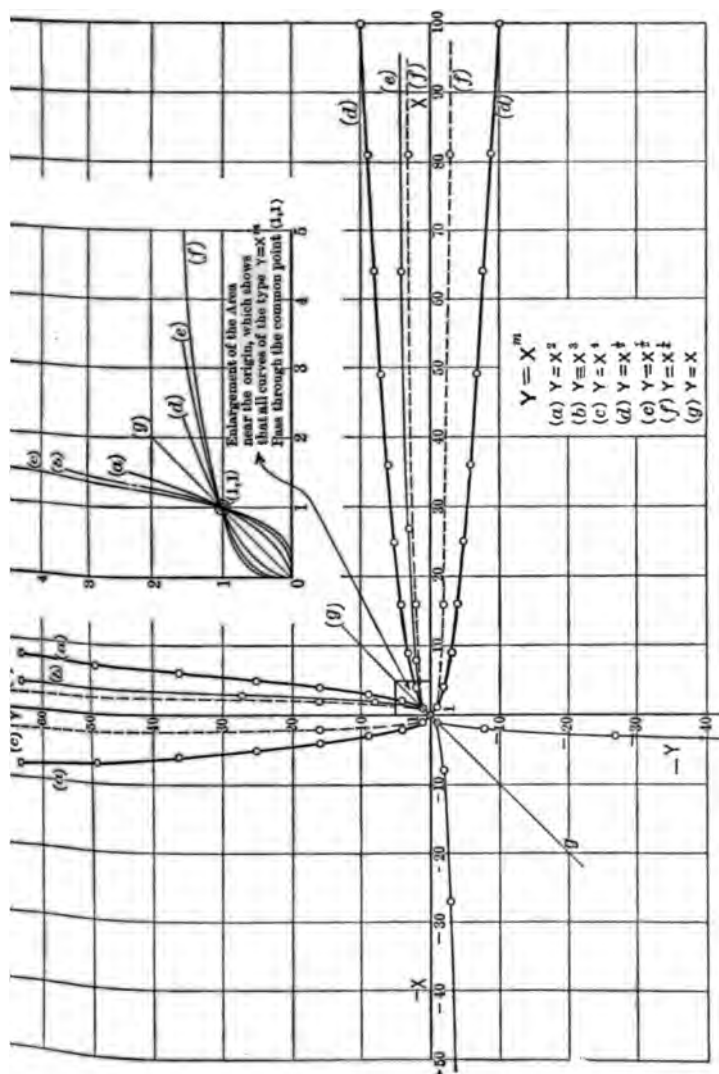
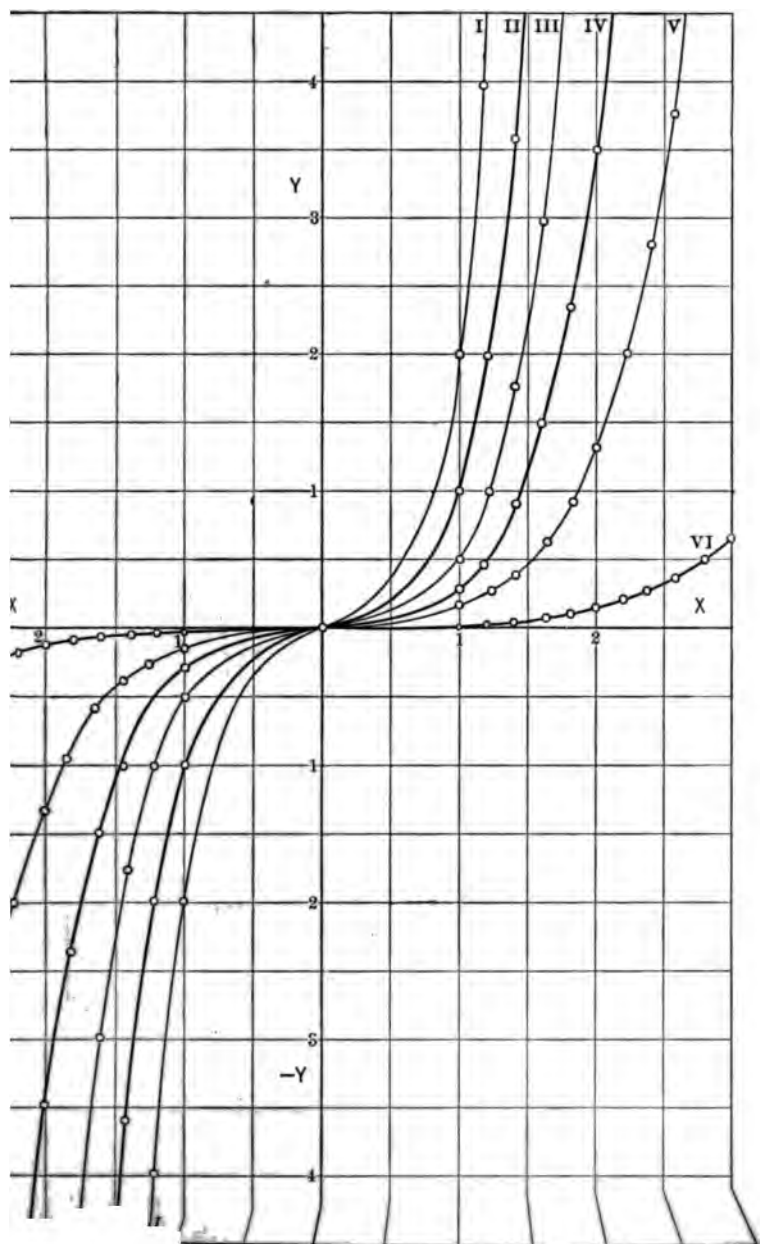


FIG. 121.—The Family of Standard Parabolas.

Exponent of $x$ .	Equivalent Root and Power.	Assigned vs	
		$x$	
$\frac{\text{even}}{\text{odd}}$	$\frac{\text{odd}}{\text{odd}} \sqrt{x^{\text{even}}}$	+	
		-	
$\frac{\text{odd}}{\text{odd}}$	$\frac{\text{odd}}{\text{odd}} \sqrt{x^{\text{odd}}}$	+	
		-	
$\frac{\text{odd}}{\text{even}}$	$\frac{\text{even}}{\text{even}} \sqrt{x^{\text{odd}}}$	+	
		+	

**5. Families of Curves.** A family of graphs which is plotted by varying one of the arbitrary constants in an equation of  $m$  in  $y = bx^m$  gives a family represented partially in Fig. 121. The family of parabolas which are represented in Fig. 122.

**Ex. 11.** Plot  $y = bx^2$ . Prepare and corresponding graphs, when  $b = .1, .5, 1$  effect of changing the constant  $b$ .



—Odd-odd Parabolas  $y = bx^{3.5}$ .

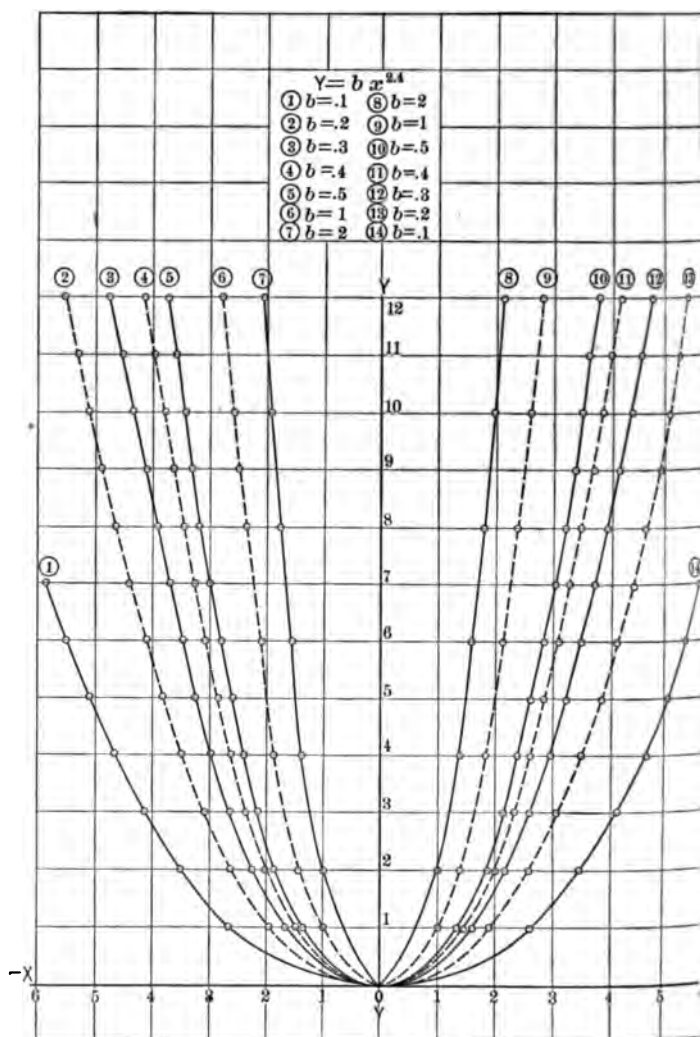


FIG. 123.—Even-odd Parabolas.

parabolas are identified by the number index which gives the value of the parameter  $b$ . The exponent  $2.4 = \frac{12}{5} = 1\frac{2}{5}$ , which comes under the even-odd class. Curves of this class are symmetrical to the  $Y$  axis, i.e., on every perpendicular to the  $Y$  axis there is a pair of alternate points equally distant from the axis. The axis cuts a parabola at its vertex.

**Ex. 12.** Plot the data from Ex. 18, Chapter IX, which corresponds to equation,

$$P_h = .0035B^{1.6}.$$

**Ex. 13.** Plot the following equations on one sheet of paper to the same scale and axes.

$$19) \quad y_1 = cx^2, \qquad (20) \quad y_2 = a + cx^2,$$

and

$$21) \quad y_3 = a + bx + cx^2,$$

**8. The Composition of Curves by Addition and Subtraction.** The interpretation of (20) states that the ordinates of  $y_2$  exceed the ordinates of  $y_1$  by the constant amount  $a$ . The graph of (20) will be a duplication of (19), but the curve will be moved vertically upward a distance  $a$  without alteration in shape. Substituting (20) and (22) in (21) we obtain (23).

$$(22) \quad y_4 = bx.$$

$$(21) \quad y_3 = a + bx + cx^2. \qquad \therefore (23) \quad y_3 = y_2 + bx = y_2 + y_4.$$

(22) is the equation of a straight line. The interpretation of (23) states that the graph for  $y_3$  may be obtained by adding the corresponding ordinates of graphs (20) and (22). Check the result from calculation by adding the ordinates graphically with a bow dividers. What change has taken place in the position of the axis of the graph (21) compared with (19) and (20).

**9. Hyperbolic Curves.** The equation  $y = x^{-m}$  represents a **standard hyperbola**. There are an infinite number of hyperbolic curves obeying the law that the ordinate equal to the reciprocal of a definite power of the absciss. For every negative integral or fractional power of  $x$  there is a definite standard hyperbola.

$$(24) \quad y = x^{-m} = \frac{1}{x^m}.$$

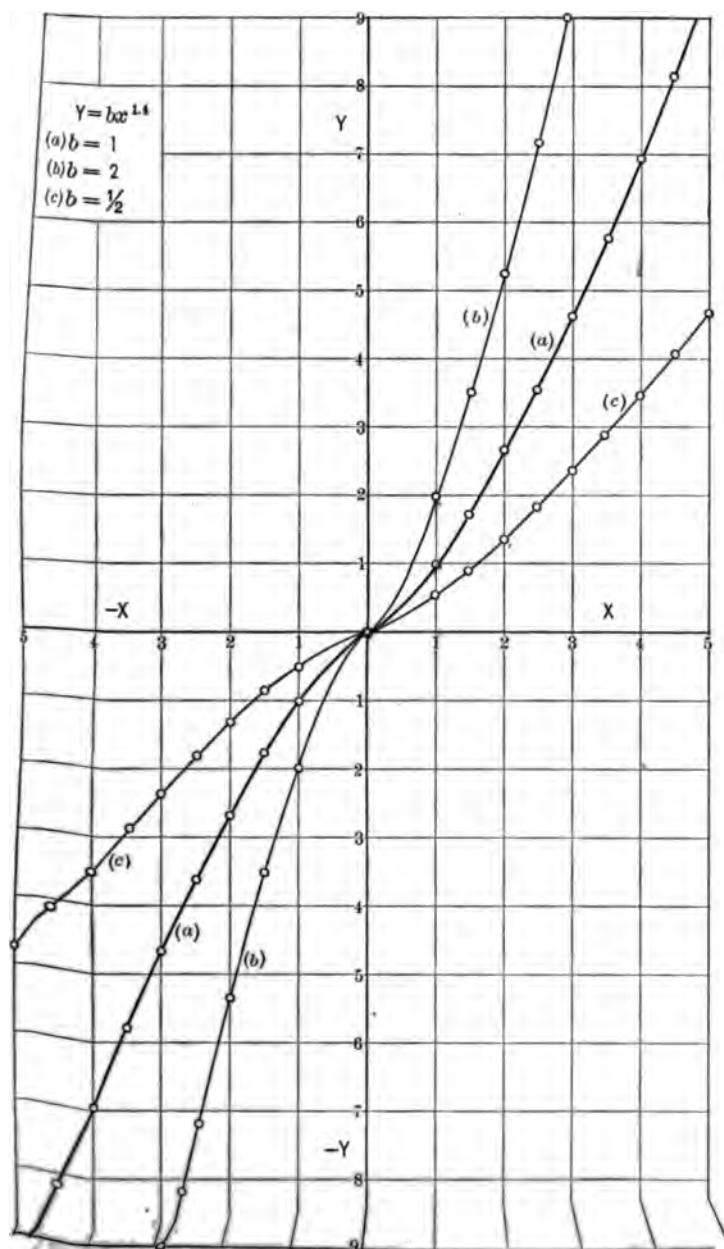
The interpretation of (24) states that for every definite assigned value of  $m$  the value of the ordinate of the standard hyperbola is numerically the reciprocal of the ordinate of the corresponding standard parabola. The tables for plotting hyperbolas may be very readily obtained from the tables for plotting parabolas. In both tables the decimal and integral values of  $x$  are the same but the values of  $y$  in the table for hyperbolas are obtained by reciprocating the values of  $y$  in the corresponding table for parabolas. The standard hyperbolas pass through the common point (1, 1). The hyperbolas have their two halves called **branch** distinctly separated. The quadrant positions of hyperbolas are obtained from Table XXII.

**10.** Fig. 124 shows three odd-odd parabolas obtained by plotting  $y = bx^{1.4}$ . Graphs (a), (b), and (c) correspond to the respective values 1, 2, .5 for the parameter  $b$ .

**11.** Fig. 125 shows three odd-odd hyperbolas obtained by plotting  $y = bx^{-1.4}$ . The numeric values of the ordinates of (a), (b), (c), in Fig. 125, are the reciprocals of the numeric values of the ordinates of (a), (b), (c), respectively in Fig. 124. A point indefinitely near the origin in Fig. 124 is indefinitely removed from the origin in Fig. 125.

Integral values in the one family of curves are decimal values in the other family of curves.

**12. Powers, Roots, and Reciprocals.** Parabolas may be used to obtain powers and roots of numbers and hyperbolas



—Odd-odd Parabolas.

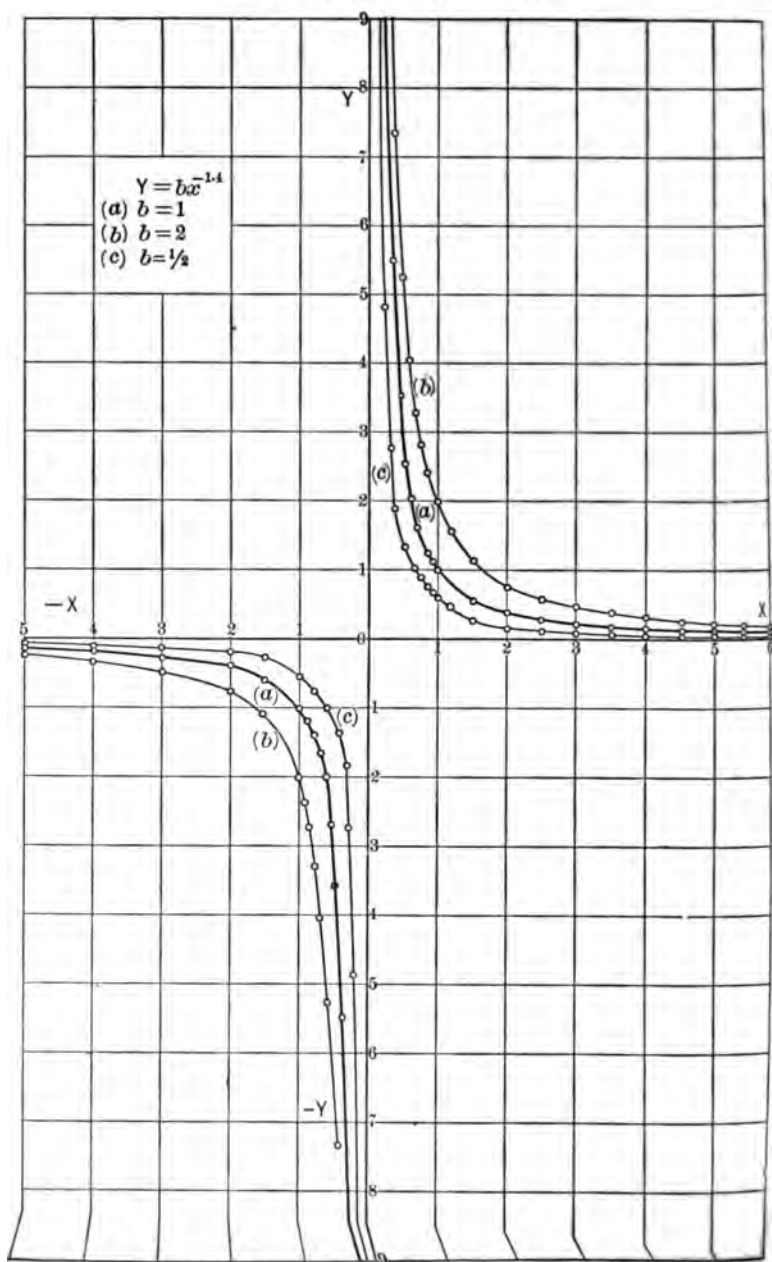


FIG. 125.—Odd-odd Hyperbolas.

may be used to obtain reciprocals of numbers, reciprocals of powers and reciprocals of roots of numbers.

13. Fig. 126 shows a family of odd-even hyperbolas plotted from  $y = bx^{-1.3}$  in which the values of  $b$  are .1, .5, 1, 2, 10. The curves are identified by the reference index of lines showing light, heavy, solid, dash, and dot-dash lines. Compare the family of parabolas of Fig. 124 with the family of hyperbolas of Fig. 125 by noting their relative position toward the  $Y$  axis. What is the change in position of hyperbolic graphs when the parameter  $b$  is increased or decreased.

14. Other non-linear algebraic equations are represented in (26), (27), (28), and (29), which are known as **power series**.

$$(25) \quad y = a + bx + cx^2 + dx^3 + ex^4 + \dots + kx^m.$$

All composite parabolic equations are modified forms of (25) and (26) from which they may be obtained by assigning definite values to the coefficients  $a, b, c, h, e$ , etc.

$$(26) \quad y^n = a + bx + cx^2 + dx^3 + ex^4 + \dots + kx^m.$$

The degree of  $x$  in (25) and (26) is numerically the same as the highest exponent  $m$  in any of its terms. (25) is a single valued function of  $y$  whereas (26) is a multivalued function of  $y$ . For every single value of  $y$  in (25) there will be  $m$  values of  $x$ . Such an equation when plotted will represent a graph which rises and falls with no regularity. The number of **crests** and **troughs** in the graph will be one less than the degree of the equation. At the crest of any graph there will be a **maximum point** whose ordinate is numerically greater than the ordinates of the contiguous points. At the trough of any graph there will be a **minimum point** whose ordinate is numerically less than the ordinates of the contiguous points.

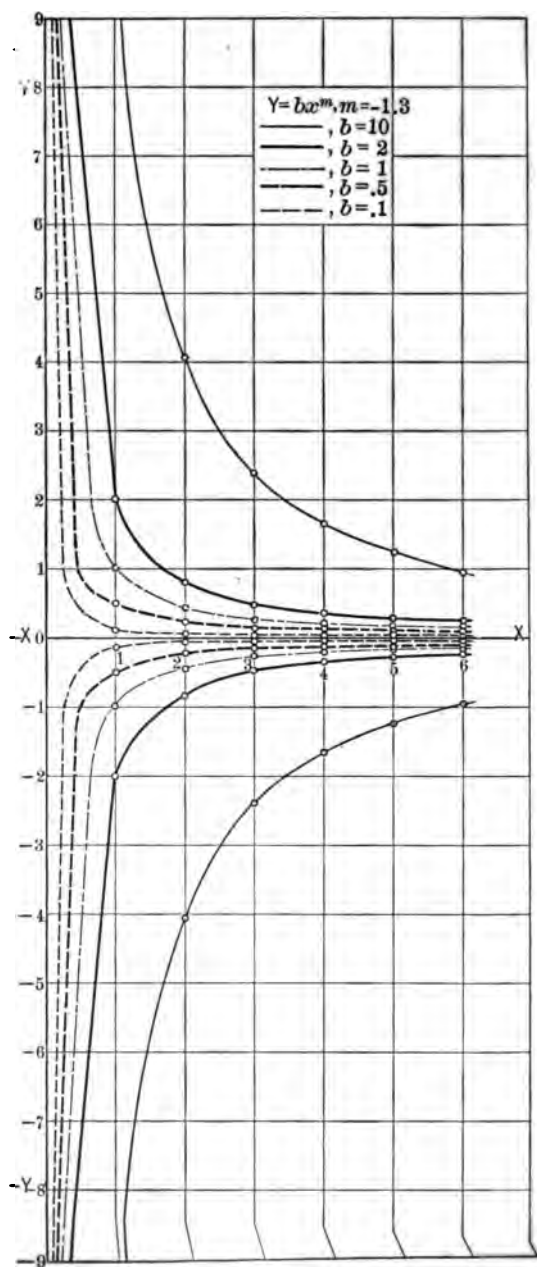


FIG. 12c

Hyperbolas.

In Fig. 170  $P$  and  $U$  are maximum points and  $A$  is a minimum point. The interpretation of (26) states that there are  $n$  different values of  $y$  which will produce the same set of values of  $x$ . As an illustration let  $n=2$ ,  $a=6.25$ ,  $b=-.25$ , and all other coefficients equal zero. After substitution and transposition (26) reduces to (1) of Ex. 17, which is the ellipse shown in Fig. 129. For every value of  $y$  (+ or -) there are two values of  $x$  and likewise for every value of  $x$  (+ or -) there are two values of  $y$ . Parabolas with integral exponents are forms of (26), whereas parabolas with fractional exponents are forms of (26).

Other power series are given by (27) and (28) of which hyperbolas are corresponding forms.

$$(1) \quad y = a + \frac{b}{x} + \frac{c}{x^2} + \frac{d}{x^3} + \frac{e}{x^4} + \dots$$

$$(2) \quad y^n = a + \frac{b}{x} + \frac{c}{x^2} + \frac{d}{x^3} + \frac{e}{x^4} + \dots$$

In order to abbreviate the right hand-members of (1) . . . (28), we use the symbol  $f(x)$ , which reads **function of  $x$** . A function is a quantity which depends on another quantity for its value. Since  $y$  in (25) . . . (28) depends upon the value of  $x$  then the left-hand members of the equations equal functions of  $x$ . Other distinguishing symbols follow.

Therefore (25) . . . (28) may be abbreviated respectively as (29) . . . (32).

$$(1) \quad y = f_1(x). \quad (30) \quad y = f_3(x) \quad \text{or} \quad y = \phi(x).$$

$$(2) \quad y = f_2(x) \quad \text{or} \quad y = F(x). \quad (32) \quad y = f_4(x) \quad \text{or} \quad y = \psi(x).$$

The symbol for function may be indicated by  $f$ , or  $f$  with subscript, or by  $\phi$  or  $\psi$ , etc. The dependent variable is

always a function of the independent variable. What functions of  $x$  are represented in  $y = \sin x$  and  $y = \log x$ .

**15. Conic Sections.** A quadratic equation in  $y$  and  $x$  may contain the first and second powers of both  $y$  and  $x$ , a product term  $xy$ , and a constant or absolute term. Such a complete quadratic equation, i.e., one which contains all the mentioned terms is represented in (33).

$$(33) \quad ax^2 + bxy + cy^2 + dx + ey + f = 0.$$

There are four types, i.e., four distinct families of curves corresponding to (33). They include **circles, ellipses, conic parabolas, and conic hyperbolas**, and are known as conic sections. They were recognized by the Greeks as the sections of cylinders, spheres, and cones.

**16.** Fig. 15, page 61, shows how the conic sections may be obtained from a right circular cone, i.e., a cone having a circular base with its apex located vertically above the center of the base. When a cutting plane is parallel to the base the section will be a circle, such as  $MNPQ$ . If the cutting plane is rotated slightly so as to be non-parallel to the base, the section becomes an ellipse, such as  $DBEC$ . If the angle of the cutting plane be further increased the ratio of the major and minor diameters of the ellipse is increased. When the cutting plane is rotated further so as to be parallel to the element  $O 10$  the section becomes a parabola, such as  $KFCGL$ . If the angle of the cutting plane be further increased the cutting plane will cut both nappes of the cone and the resulting sections become hyperbolas.

**17. Conic Test.** A study of the effect of changing the values of the coefficients of (33) leads to a precise way of determining the conditions which result in a curve of each type. The numeric values of  $a$ ,  $b$ , and  $c$  are sufficient to determine the kind of conic according to the following relations. All terms of a quadratic equation must be trans-

osed to one member of the equation, as shown in (33),  
nd then  $A$  is the coefficient of the  $x^2$  term,  $C$  is the coeffi-  
ient of the  $y^2$  term, and  $B$  is the coefficient of the  $xy$  term.  
ultiply  $4A$  by  $C$  and compare it with  $B^2$  as follows:

When  $B^2 < 4AC$  the equation (33) represents an ellipse.

When  $B^2 = 4AC$  the equation (33) represents a parabola.

When  $B^2 > 4AC$  the equation (33) represents an hyper-  
bola.

When  $B^2 = 0$  and  $A = C$  the equation (33) represents a  
circle.

When  $B = 0$  then  $B^2$  is greater than any negative value of  
the product  $4AC$ .

**Ex. 14.** Apply the conic test and substantiate the designation  
of the following simplified forms of conic formulas.  $r$  is the  
radius of the circle.  $a$  and  $b$  are the respective semimajor and  
semiminor axes.  $H$  and  $K$  are the respective coordinates of the  
center of an eccentric conic, i.e., one whose center is not at the  
origin.  $c, d, k, m, n$ , are arbitrary constants.

(34)  $x^2 + y^2 = r^2$ . The equation of a circle

whose center is at the origin.

(34A)  $x^2 + y^2 = 25$ . What is the value of  $r$ ?

(34B)  $(x-H)^2 + (y-K)^2 = r^2$ . The equation of an eccentric circle.

See Figs. 127 and 130.

(35)  $(x-5)^2 + (y-6) = 49$ . What are the values of  $r, H$ , and  $K$ .

(36)  $(x-3)^2 + (y+3) = 16$ . What are the values of  $r, H$ , and  $K$ .

There will be one definite circle corresponding to a definite  
value assigned to the radius  $r$ .

A circle may be moved about in a plane by changing the  
numeric values of the coordinates of its center. The equation (36)

holds true for any position in the four quadrants, as may be shown from the relations of the sides of the right angle triangles.

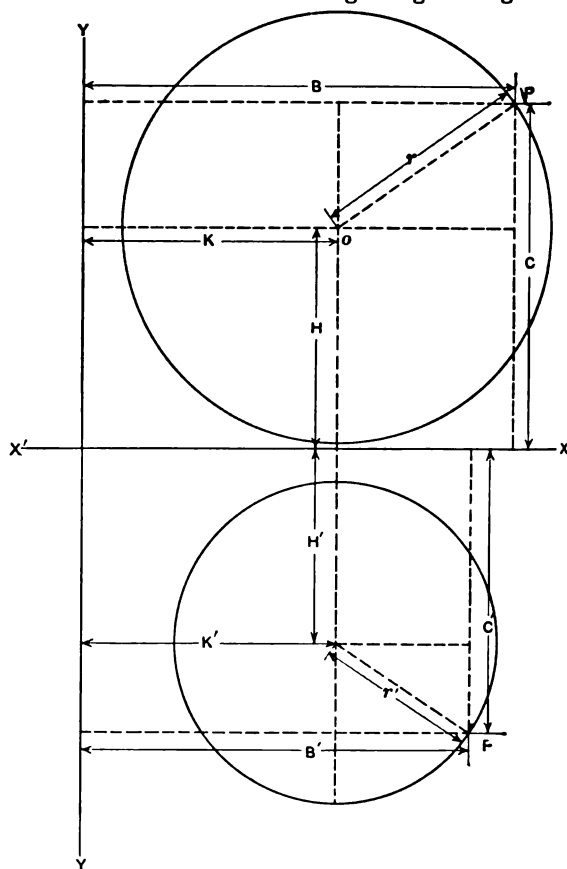


FIG. 127.—The Circle  $(x-H)^2 + (y-K)^2 = r^2$ .

$$(37) \quad \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

The equation of an ellipse.

See Figs. 129 and 133. The center is at the origin.

$$(38) \quad \frac{(x-H)^2}{a^2} + \frac{(y-K)^2}{b^2} = 1. \quad \text{The equation of an eccentric ellipse}$$

For every definite pair of values of  $a$  and  $b$  there is a definite ellipse. An ellipse may be moved vertically and horizontally in the plane by changing the respective coordinates of its center. A family of ellipses which have common foci are **confocal**.

)  $\frac{x^2}{49} + \frac{y^2}{36} = 1$ . What are the values of  $a$  and  $b$ ?

)  $\frac{x^2}{36} + \frac{y^2}{49} = 1$ . What are the values of  $a$  and  $b$ ?

What comparison is there between the ellipses (39) and (40)?

)  $\frac{(x-3)^2}{36} + \frac{(y-4)^2}{25} = 1$ . What are the values of  $a$ ,  $b$ ,  $H$ , and  $K$ ?

)  $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ . The equation of an hyperbola.

See Fig. 135. The center is at the origin.

)  $\frac{(x-H)^2}{a^2} - \frac{(y-K)^2}{b^2} = 1$ . The equation of an eccentric hyperbola.

For every definite pair of values of  $a$  and  $b$  there is a definite hyperbola. An hyperbola may be moved vertically and horizontally in the plane by changing the respective coordinates of its center. Confocal hyperbolas have common foci.

)  $\frac{x^2}{49} - \frac{y^2}{36} = 1$ . What are the values of  $a$  and  $b$ ?

)  $\frac{x^2}{36} - \frac{y^2}{49} = 1$ . What are the values of  $a$  and  $b$ ?

)  $\frac{(x-2)^2}{25} - \frac{(y^2-3.5)^2}{16} = 1$ . What are the values of  $a$ ,  $b$ ,  $H$ , and  $K$ ?

)  $(x-2)^2 - (y+1)^2 = 1$ . What are the values of  $a$ ,  $b$ ,  $H$ , and  $K$ ?

)  $y = kx^2$ . The equation of a parabola.

The vertex is at the origin and the curve is symmetrical to the  $Y$  axis. Its focus is  $\frac{k}{4}$  units from the vertex.

(49)  $x = my^2$ .

The equation of a parabola

The vertex is at the origin and the curve is symmetrical to the  $X$  axis. Its focus is  $\frac{m}{4}$  units from the vertex.

Describe the parabolas

(50)  $y = 8x^2$ ,

(51)  $x = 6$

(52)  $y = -4x^2$

and

(53)  $x = -$

The diameter of a parabola is that portion of the curve between the vertex and last ordinate of the curve. A parabola and its entire diameter cannot be represented by a finite line extending indefinitely from the vertex.

(54)

$y = (c + kx)^2$ .

The equation of a parabola whose vertex is located vertically above the origin but otherwise like (48).

(55)

$x = (d + my)^2$ .

The equation of a parabola whose vertex is located horizontally to the right of the origin but otherwise like (48). Describe the parabolas (56A)  $y = (1 + 8x)^2$ , (56B)  $y =$

(56C)  $x = (1 + 4y)^2$ ,

(56D)  $x = (-2 -$

Write equations for a circle, ellipse, parabola, and hyperbola using the data:  $r = 4$ ,  $a = 3$ ,  $b = 4$ ,  $H = 1$ ,  $K = 2$ ,  $k = m = 10$ ,  $d = 2.5$ . Describe each curve.

Plot

(57)

$x^2 + y^2 = a^2$ .

# NON-LINEAR ALGEBRAIC EQUATIONS 335

$a$  to have a special value according to the following assignment:

$a = 1, 1.5, 2, 2.5, 3, 3.5, 4, 4.5, 5.$

Plot

$$(58) \quad \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

$a$  and  $b$  are to have specially assigned values in pairs as follows:

$a$	1	1	2	2	3	4	3	3	2	1
$b$	2	3	3	4	4	3	2	1	1	1

Plot

$$(59) \quad \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1.$$

$a$  and  $b$  are to have specially assigned values the same as above.

Plot

$$(60) \quad y = kx^2 \quad \text{and} \quad (61) \quad x = my^2,$$

Don the same sheet of paper. Use the same pair of axes for both curves and assume

$$k = 4 = m.$$

What significant relation do these equations bear to each other? What mechanical operation is equivalent to interchanging  $x$  and  $y$  in the equation?

The meaning of the  $xy$  term, equations (34) to (61), present standard curves which contain no  $xy$  term and for which corresponding graphs have their diameters parallel to the axes. The introduction of the  $xy$  term has the effect of rotating the diameter of a conic through an angle  $\theta^\circ$  from its normal position. The angle of rotation is measured

by the tangent as expressed in (62) in which  $A$ ,  $B$ , and  $C$  are the coefficients of (33).

$$(62) \quad \tan 2\theta = \frac{B}{A-C}.$$

**Ex. 15.** Describe the curve and its position and then plot following equations:

$$(63) \quad y = \frac{ax}{c+dx}, \quad \text{when } a=2.5, c=3, d=.5.$$

$$(64) \quad y = (a+bx)^{-1}, \quad \text{when } a=1, b=8.$$

$$(65) \quad xy = ax + by, \quad \text{when } a=1, b=5.$$

$$(66) \quad ch = T^2, \quad \text{when } T=1.5.$$

$$(67) \quad A = P \left( 1 + \frac{r}{100} \right), \quad \text{when } A=100.$$

$$(68) \quad \frac{1}{u} = a + bp, \quad \text{when } a=1, b=.5.$$

$$(69) \quad q = a + \frac{b}{p}, \quad \text{when } a=1, b=5.$$

$$(70) \quad h = \frac{2kfv^2}{dg}, \quad \text{when } k=.2, f=25, v=10, d=$$

$$(71) \quad (y-x)(x+2y-3) = 7.$$

$$(72) \quad 5x^2 + 2xy + 5y^2 - 12x - 12y = 0.$$

$$(73) \quad x^2 - 2xy + y^2 - 8x + 16 = 0.$$

The solution of non-linear simultaneous equations is determination of the pairs of values of the variables which will satisfy the given equations. The solution may be determined graphically by observing the coordinates of the points of intersection of the graphs.

In general, two or more simultaneous equations have many solutions as the product obtained by multiplying degree of each equation. This is substantiated in the graphic work which follows.

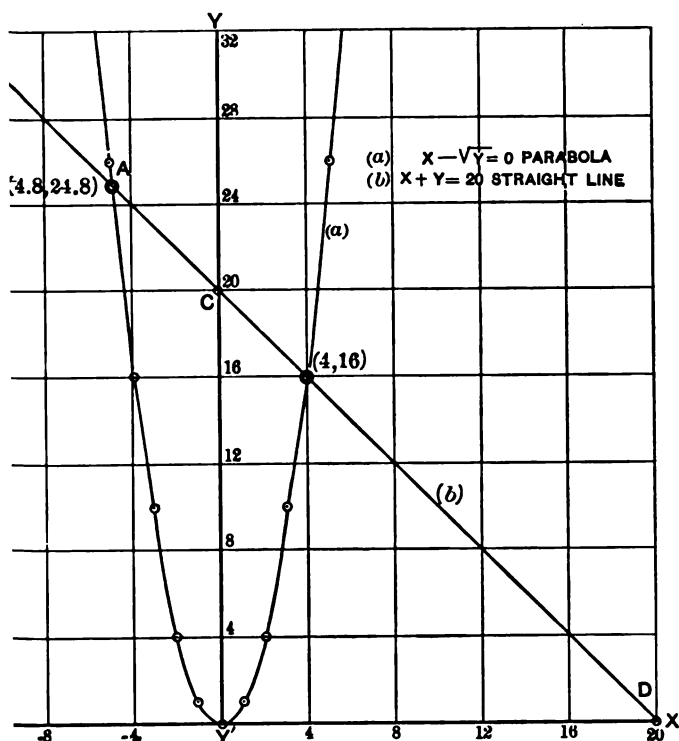


FIG. 128.—Graphic Solution of Simultaneous Equations.

olve graphically the following simultaneous pairs of tions:

1. 16. (a)  $x - \sqrt{y} = 0$ .

(b)  $x + y = 20$ . See Fig. 128.

he intersections give  $(-4.8, 24.8)$  and  $(4, 16)$ .

Equation (a) represents a parabola and being devoid of a constant term the vertex is at the origin. Equation (b) represents a line whose slope is  $(-1)$  and whose  $X$  intercept is  $(+20)$ .

**Ex. 17.** (1)  $x^2 + 4y^2 = 25$ .

(2)  $2xy = 12$ .

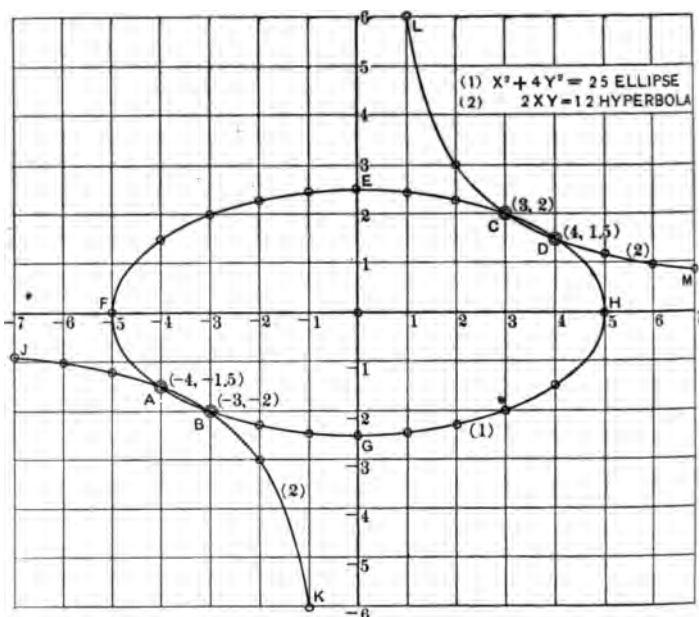


FIG. 129.—Non-linear Simultaneous Equations.

The four solutions are  $(3, 2)$ ,  $(4, 1.5)$ ,  $(-3, -2)$ , and  $(-4, -1.5)$ . Why are there four solutions? Are the graphs normal? See Fig. 129.

**Ex. 18.** (a)  $4(x + y) = 3xy$ .

(b)  $x + y + x^2 + y^2 = 26$ .

These equations are plotted in Fig. 130. Describe the graphs and state the solutions of the equations.

**Ex. 19.** (1)  $x + y = 5.$

(2)  $x^2 + y^2 = 65.$

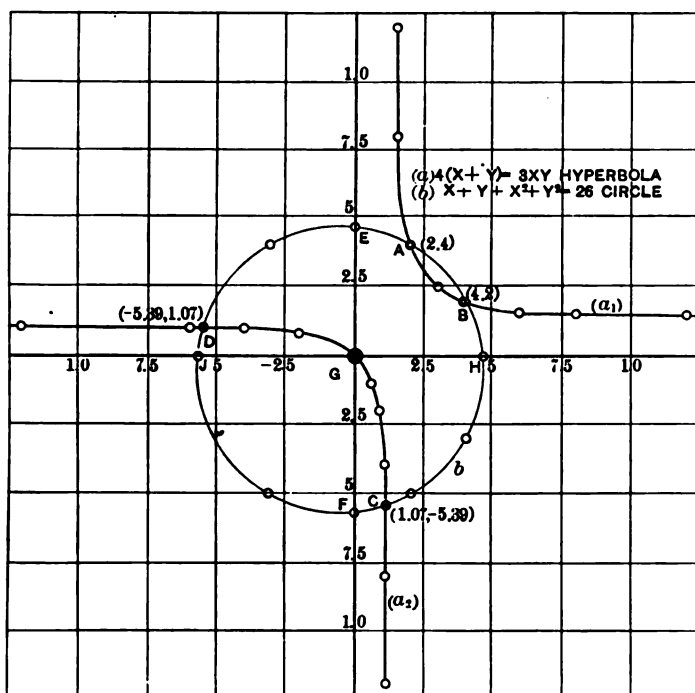


FIG. 130.—Two Simultaneous Quadratics Showing Four Solutions.

How many solutions should there be for these equations? State their values. See Fig. 131.

**Ex. 20.** Plot

(74)  $4x^2 + 4xy + y^2 - 8x - 41y + 400 = 0$

on the same cross-section paper with the graphs of Ex. 16. It will be observed that the above equation may be reduced to (75) and (76).

$$(75) \quad (2x + y - 20)^2 = y.$$

$$(76) \quad 2x + y - 20 = \sqrt{y}.$$

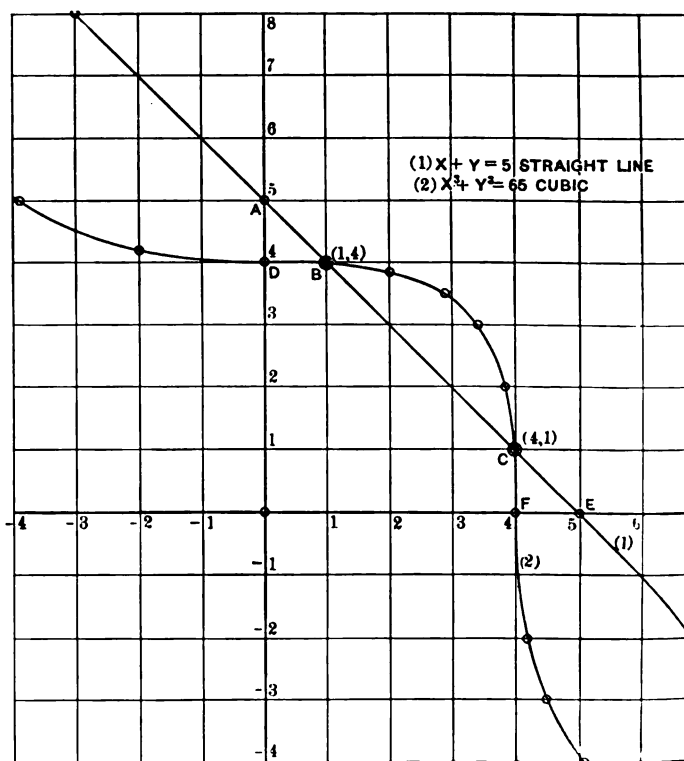


FIG. 131.—Line, Linear and Non-linear Simultaneous Equation.

Equation (76) is the sum of (a) and (b) in Ex. 16.

**Ex. 21.** Plot

$$(77) \quad x^2 - 3xy + y^2 + 5x + 5y - 26 = 0,$$

on the same cross-section paper with the graphs of Ex. 18. Add the equations of Ex. 18 and compare with Ex. 21.

**Ex. 22.** Plot

(78) 
$$x^2 - xy + y^2 = 13,$$

on the same cross-section paper with the graphs of Ex. 19. Divide (2) by (1) in Ex. 19 and compare with Ex. 22.

## CHAPTER XVII

### ECCENTRICITY OF CONICS

1. THE ellipse, parabola, and hyperbola may be constructed with the aid of the straightedge and compass without the necessity of plotting points from calculations.

The conics mentioned above have the common property that every point on each curve preserves a **constant ratio** between its distance from a fixed point (**focus** or **pole**) to the distance from a fixed line (**directrix**). These distances are designated as the **focal** and **directral distances** respectively. In Fig. 132,  $P$  is a point on the curve.  $F$  is the focus.  $DD'$  is the directrix.

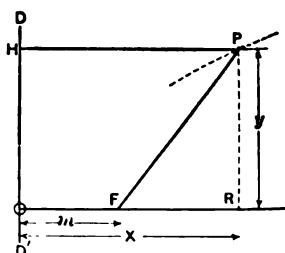


FIG. 132.—The Analytic Definition of a Conic.

$$(1) \quad \frac{\text{Focal distance}}{\text{Directral distance}} = \frac{PF}{PH} = \frac{\sqrt{y^2 + (x-m)^2}}{x} = e = \text{a numeric constant.}$$

2. The fixed ratio  $e$  is called the eccentricity of a conic. For any one definite curve  $e$  is fixed in value, i.e., a numeric constant, but  $e$  may have any positive value. For all values less than one ( $e < 1$ ),  $e$  determines an **ellipse**, and for all values of  $e$  in excess of one ( $e > 1$ ),  $e$  determines an **hyperbola**. When  $e = 1$ , then the conic is a **parabola**.

The parabola lies at the boundary between ellipses and hyperbolas.

The conic test observed in the last chapter is consistent with the following order of magnitudes:

Ellipses are constructed for values of  $e < 1$ ;

Parabolas are constructed for values of  $e = 1$ ;

Hyperbolas are constructed for values of  $e > 1$ ;

Circles are not constructed for values of  $e = 0$ ;

since the directral distance is infinite.

The directral distance  $PH$  is measured perpendicular to the directrix  $DD'$ .

The focal distance is measured radially from the focus  $F$ .

The focus is located at a distance  $m$  from the directrix, and the vertex lies between the directrix and the focus. When the eccentricity of a conic is known the equation for the graph may be obtained by substituting the values of  $e$  and  $m$  in (1) or in its simplified form given in (2).

$$(2) \quad (1 - e^2)x^2 - 2mx + y^2 + m^2 = 0.$$

3. The ellipse, Fig. 133, the parabola, Fig. 134, and the hyperbola, Fig. 135, may be constructed with the aid of the straightedge and compass without the necessity of plotting points from calculations.

In each case the directrix is shown as a vertical line. The horizontal axis is drawn through a point  $O$  on the directrix and through the focus  $F$ . The diagonal lines  $OS$  and  $OS'$  are drawn so that their slope equals  $e$ , which is the numeric value of the eccentricity of the desired conic. Then any perpendicular drawn between the axis and the diagonal  $OS$  may be used for the focal distance of a point  $P$  on the conic. The corresponding directral distance of the perpendicular equals the directral distance of the point  $P$  on the conic. The points on the curve are located by intersecting the measured perpendicular with an arc centered at the focus and with a radius equal to the measured perpendicular.

The point of intersection of any curve and the axis called the **vertex**.

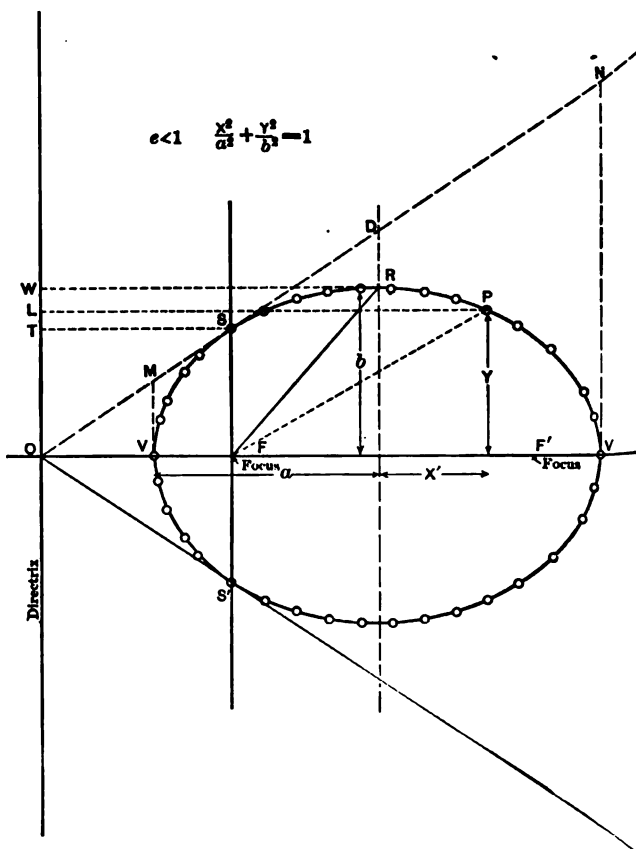


FIG. 133.—The Construction of an Ellipse from its Analytic Definition

The parabola has a single vertex and a single focus. The vertex lies midway between the focus and directrix.

The ellipse has two vertices  $V$  and  $V'$  and two foci  $F$  and  $F'$  symmetrically located on the horizontal axis.

[illegible]

Fig. 134.—The Analytic Construction of the Parabola  $y = 2mx$ .

mel, an instrument used for constructing the ellipse.  
e are two equal perpendiculars which can be intersected

by the arc centered at  $F$ . Therefore an hyperbola has two branches which lie on opposite sides of the directrix. It has two vertices  $V$  and  $V'$  and two foci  $F$  and  $F'$  symmetrically located about the center on the horizontal axis.

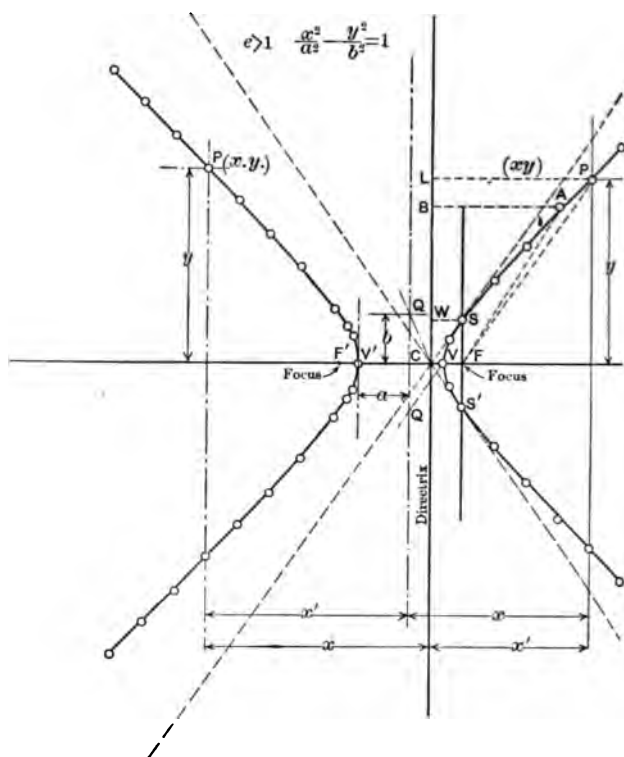


FIG. 135.—The Analytic Construction of the Branches of the Hyperbola from its Right Focus.

The major diameter  $VV'$  of an ellipse or hyperbola is the segment of the horizontal axis lying between the vertices. The center  $C$  is the mid point of the major diameter. The minor diameter is a segment of the perpendicular bisector

of the major diameter. The semimajor and semiminor diameters are designated by  $a$  and  $b$  respectively. Table XXIII gives the properties of conics.

TABLE XXIII. PROPERTIES OF ELLIPSES AND HYPERBOLAS

Properties.	Ellipse.	Hyperbola.
1. Value of $e$ in terms of $b$ and $a$ . . . . .	$\pm\sqrt{1-\frac{b^2}{a^2}}$	$\pm\sqrt{1+\frac{b^2}{a^2}}$
2. Value of $b$ in terms of $a$ and $e$ . . . . .	$\pm a\sqrt{1-e^2}$	$\pm a\sqrt{e^2-1}$
3. Distance of extremity of minor axis from focus . . . . .	$\sqrt{b^2+a^2e^2}$	$\sqrt{b^2+a^2e^2}$
4. Distance of focus from center . . . . .	$\pm ae$	$\pm ae$
5. Distance from focus to near vertex . . . . .	$a(1-e)$	$a(e-1)$
6. Distance from focus to far vertex . . . . .	$a(1+e)$	$a(1+e)$
7. Distance from directrix to center . . . . .	$\frac{a}{e}$	$\frac{a}{e}$
8. Distance from directrix to near vertex . . . . .	$\frac{a}{e}(1-e)$	$\frac{a}{e}(e-1)$
9. Distance from directrix to far vertex . . . . .	$\frac{a}{e}(1+e)$	$\frac{a}{e}(1+e)$
10. Distance from directrix to near focus . . . . .	$\frac{a}{e}(1-e^2)$	$\frac{a}{e}(e^2-1)$
11. Distance from directrix to far focus . . . . .	$\frac{a}{e}(1+e^2)$	$\frac{a}{e}(1+e^2)$
12. The semi-parameter, i.e., ordinate through focus . . . . .	$a(1-e^2)$	$a(e^2-1)$
13. The focal distances $r_1$ and $r_2$ for any point on the curve . . . . .	$r_1+r_2=2a$	$r_1-r_2=2a$

## CHAPTER XVIII

### THE FORMULATION OF GRAPHS

**1. Writing the Equation of a Curve.** When the variation of the elements of a graph are expressed as an equation the graph is said to be formulated. We shall assume that our familiarity with the families of curves plotted in the preceding chapters will enable us to recognize a parabola or an hyperbola.

Many experimental curves will approximate to the graph of (1) or (2).

$$(1) \quad y = bx^m, \quad (2) \quad y = bx^{-m}.$$

This assumption implies that for the specific curve under consideration, there will be a definite value for  $b$  and a definite value for  $m$ , and further that the curve (1) passes through the origin whereas (2) does not.

Select any two points on the curve and measure the pairs of coordinates for two points, abbreviate these by  $(x_1, y_1)$  and  $(x_2, y_2)$ . If the curve is parabolic each pair of coordinate values will validate (1), i.e., the equation (1) will hold true when  $x_1$  and  $y_1$ , and also when  $x_2$  and  $y_2$  are substituted therein. We obtain equations (a) and (b) by substituting the coordinates in (1).

$$(a) \quad y_1 = bx_1^m, \quad (b) \quad y_2 = bx_2^m.$$

(a) and (b) are simultaneous equations because the same constant values of  $b$  and  $m$  are used in both (a) and (b).

We can eliminate  $b$  from (a) and (b) by division obtaining (c). The latter can be transformed for  $m$  by logarization.

$$c) \quad \frac{y_1}{y_2} = \frac{x_1^m}{x_2^m} = \left(\frac{x_1}{x_2}\right)^m \text{ ————— Div (a) by (b)}$$

$$d) \quad \log \frac{y_1}{y_2} = m \log \frac{x_1}{x_2} \text{ ————— Log of a power}$$

$$e) \quad m = \frac{\log \frac{y_1}{y_2}}{\log \frac{x_1}{x_2}} \text{ ————— Div (d) by coef } m$$

$$f) \quad \therefore m = \frac{\log y_1 - \log y_2}{\log x_1 - \log x_2} \text{ ————— Log of quotient.}$$

The interpretation of (f) states that the exponent of the parabola is the ratio between the difference of the logarithms of the ordinates of two points, to the difference of the logarithms of the abscissas of the same two points. Logarizing (f) we obtain (g),

$$g) \quad \log y = \log b + m \log x \text{ ————— Log (f)}$$

$$h) \quad \therefore \log b = \log y - m \log x \text{ ————— Trans (g).}$$

The interpretation of (h) states that the logarithm of the factor  $b$  may be obtained by subtracting  $m$  times the logarithm of the abscissa of any point from the logarithm of the ordinate of the same point.

When the values of  $b$  and  $m$  are known they are substituted in (1).

If the graph is an hyperbola then equation (2) is applicable.

**Ex. 1.** Show that the values of  $m$  and  $b$  for an hyperbola may be computed from (i) and (j) respectively.

$$(i) \quad m = -\frac{\log y_1 - \log y_2}{\log x_1 - \log x_2}$$

$$(j) \quad \log b = \log y + m \log x.$$

Interpret (i) and (j).

**Ex. 2.** Determine the formula for the current-effort curve *EFGH* in Fig. 136, assuming the graph is a parabola passing through the origin. Use the coordinates of the points *F* and *H* in (e) and (h) and check the resulting formula for the point *G*.

**Ex. 3.** Determine the formula for the speed-current curve *ABCD* in Fig. 136, assuming the graph is an hyperbola. Use the coordinates of the points *A* and *D* in (i) and (j) and check the resulting formula for the point *C*.

2. Equations (1) and (2) may be written in the linear forms (3) and (4) by logarization. (3) and (4) are identical in form with (5) the equation of a straight line.

$$(3) \quad \log y = \log b + m \log x.$$

$$(4) \quad \log y = \log b - m \log x.$$

$$(5) \quad Y = A + BX.$$

(3) and (4) contain the variable  $\log y$  which is replaced by  $Y$  in (5), and the variable  $\log x$  which is replaced by  $X$  in (5).  $\log b$  is a constant and is replaced by the constant  $A$  in (5) and  $m$  or  $-m$  are constants also and are replaced by  $B$  in (5).

If we plot (5) we will obtain a linear graph.  $Y$  is the ordinate and  $X$  the corresponding abscissa for any point on the line.  $A$  is the intercept on the  $Y$  axis and  $B$  is the slope of the line.

To plot  $Y$  means to plot logarithms of  $y$  vertically and to plot  $X$  means to plot logarithms of  $x$  horizontally.

3. The necessity of determining the numeric values of  $\log y$  and  $\log x$  for plotting is obviated by the use of

logarithmic cross-section paper. Such paper is shown in Fig. 137, which represents two independent unit logarithmic squares called **log-units** arranged side by side for convenience. Figs. 139 and 140 represent two unit logarithmic squares arranged vertically one above the other. A number of log-units may be united for extended plotting.

A log-unit is numbered on its axes from 1 to 10 in sequence. The spacing of log paper is unlike the uniform spacing of squared cross-section paper but follows the spacing of a log scale of a slide rule. The intersection of the axes, i.e.,

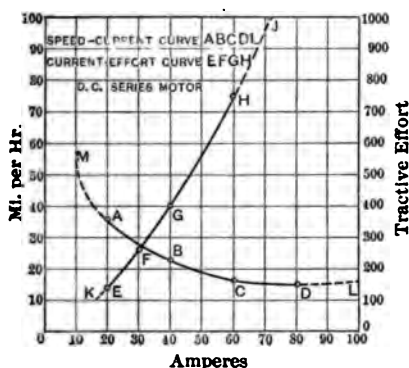


FIG. 136.—The Extrapolation of Curves from Logarithmic Plotting.

the zero point for uniform scales, is marked 1 for log scales because  $\log 1 = 0$ . The log-unit is one unit in length and therefore the extreme ends of the axes are marked 10 since  $\log 10 = 1$ . The divisions of the log paper are mantissas and therefore the scale may be multiplied by a power of ten to accommodate a range of numbers from 1-10, 10-100, 100-1000 or .1-1, .01-1, .001-.01, etc. Joining two log-units increases the range of the combined scale from 1-100 or any equivalent.

4. The points on the graphs in Fig. 136 are replotted in the two log-units of Fig. 137. For every point on the squared paper, Fig. 136, there is a corresponding point on the log

paper, Fig. 137. These corresponding points are marked with like letters for ready identification. By extending the linear graphs of Fig. 137, additional points such as *L*, *K* and *J* may be obtained and thereby extrapolate the original data. Any point such as (15, 20) of the squared paper will have the same coordinates (15, 20) on the log paper.

**Ex. 4.** Measure the slope and intercepts of *ABCD* in Fig. 137 and compare their values with the results of Ex. 2.

**Ex. 5.** Measure the slope and intercepts of *EFGH* in Fig. 137 and compare their values with the results of Ex. 3.

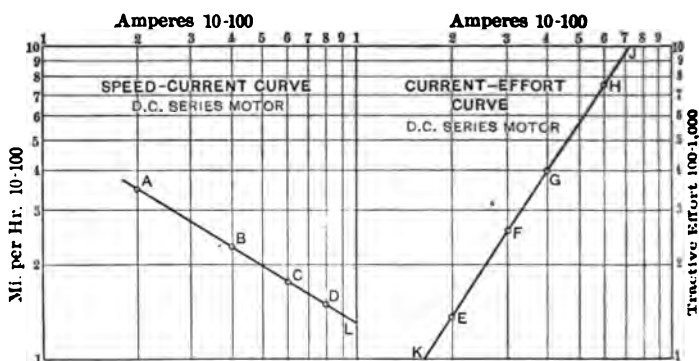


FIG. 137.—Plotting on Logarithmic Paper.

*Observation.* When parabolic and hyperbolic graphs are plotted on logarithmic paper the slope of the linear graph is the exponent of the horizontally plotted variable and the intercepts on the vertical axis is the coefficient of the same variable.

5. In Fig. 138, (a) is the standard parabola  $y = x^{1.5}$ . (c) and (d) illustrate the effect of changing the coefficient. (e) is the standard hyperbola  $y = x^{-1.5}$  which illustrates the effect of changing the sign of the exponent. Figs. 139 and 140 represent (a), (b), (c) and (d) plotted upon logarithmic paper. The log-unit *ABUV* corresponds to the small square *ABUV* of Fig. 138. The log-unit *BCWU* corresponds to the rectangle *BCWU* in Fig. 138. The log-unit *WDFU*

corresponds to the large square *WDFU* in Fig. 138. The log-unit *UFEV* corresponds to the rectangle *UFEV* in Fig. 138.

The logarithmic paper provides a means for magnifying the region close to the origin. The edges of the logarithmic paper are subdivided into tenths of the principal divisions.

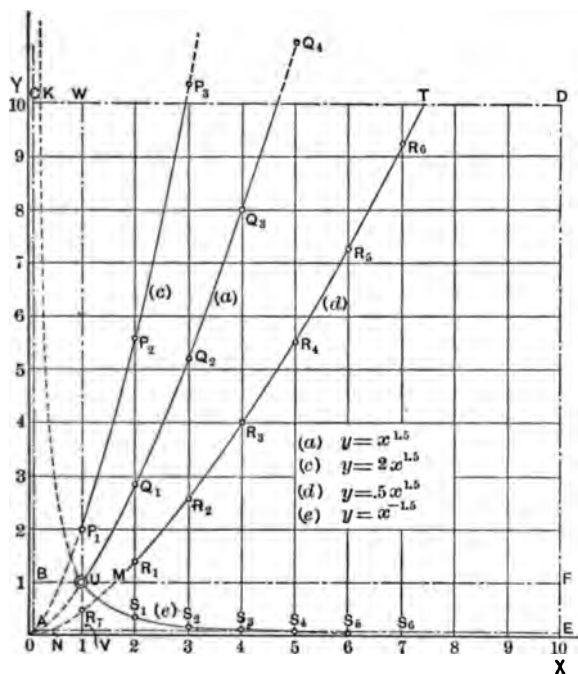
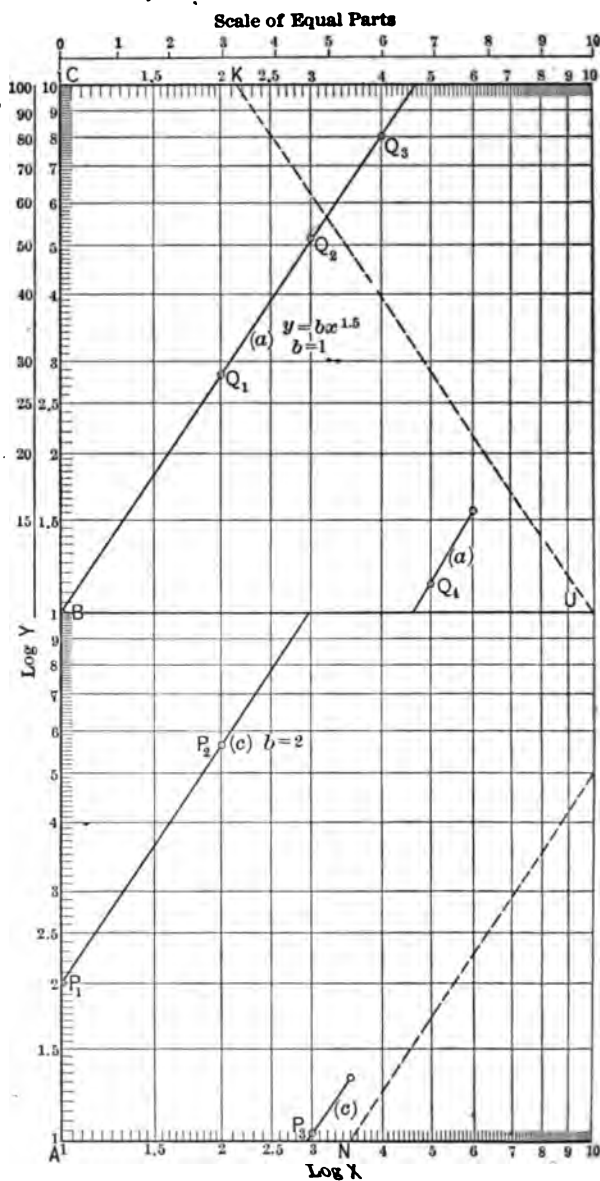
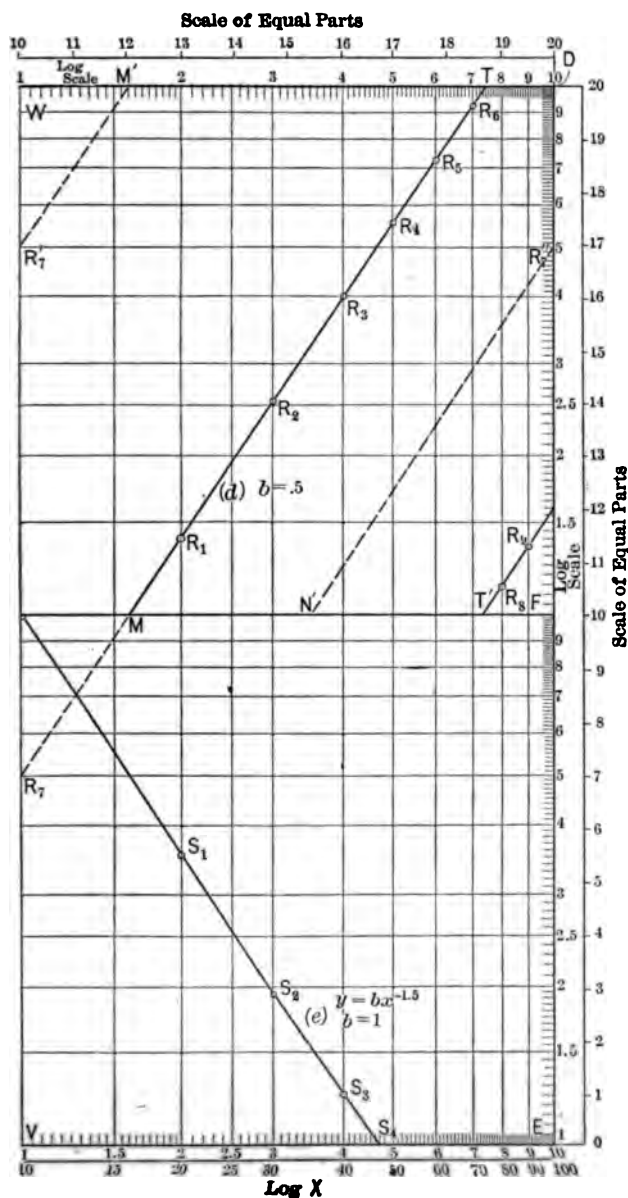


FIG. 138.—Plotting of  $y = bx^m$  on Squared Paper.

A scale of equal parts indicates the manner of graduating the principal divisions and the subdivisions of the log paper.

The graphs (a), (c) and (d) in Figs. 139 and 140 are parallel and their constant slope is measured by the tangent of the angle of inclination. The angle is  $56^\circ 20'$ . The  $\tan 56^\circ 20' = 1.5$ , which corresponds to the exponent of the variable  $x$  for the three equations (a), (b) and (d). The

FIG. 139.—Logarithmic Plotting of  $y = bx^m$ .



slope of (e) in Fig. 140 is negative. The angle is  $123^\circ 40'$ . The  $\tan 123^\circ 40' = -1.5$  which corresponds to the exponent of the variable  $x$  in (e). The vertical intercept of (c) is at  $P_1$ , indicating 2 which is the coefficient of the variable  $x$  in (c). The vertical intercept of (d) is at  $R_7$  which is obtained by extending  $MT$  through the next log-unit.  $R_7$  indicates .5 which is the coefficient of the variable  $x$  in (d). The vertical intercept of (e) is at  $U$  indicating 1 which is the coefficient of the standard hyperbola (c).

The graph (a) should have been represented in the log unit  $WUFD$  but in order to avoid over-crowding the same purpose is served by representing it in log-unit  $CBUW$  providing we retain the proper range of values of the scale divisions.

By extending the graphs into the next log-unit a greater range of values of the variables may be obtained. This extension may be provided for in a better way so as to economize in paper and space and at the same time not sacrifice any accuracy. The extension of  $MT$  gives the segments  $MR_7$  and  $R_7N$ . Locate  $M'$  vertically above  $M$  and draw  $M'R_7'$  parallel to  $MT$ . Then  $M'R_7'$  represents  $MR_7$  for which the vertical scale should be read in tenths. Locate  $R_7''$  on the same horizontal line with  $R_7'$ . Through  $R_7''$  draw  $R_7''N'$  parallel to  $MT$ . Then  $R_7''N'$  represents  $R_7N$  for which the horizontal scale should be read in tenths. Vertically under  $T$  locate  $T'$ . Draw  $T'R_9$  parallel to  $MT$ . Then  $T'R_9$  represents the extension of  $MT$  into the next higher log-unit. By continuing this process the entire graph may be represented in a single log-unit.

**Ex. 6.** Plot upon one log-unit the graphs for reading the squares and cubes of numbers and the reciprocals of numbers. Show how these graphs may be used for obtaining square roots and cube roots.

*Observation.* Logarithmic paper gives a ready means for determining the constants of a graph of the type  $y = bx^{\pm m}$ . It

serves as a quick method for plotting such a curve since it requires a minimum amount of data. It magnifies any desired region of uniformly divided cross-section paper. An entire graph may be represented in one log-unit.

6. If a given curve is not of the type  $y = bx^m$  the graph will not be linear upon logarithmic paper. The equation  $y = bx^{\pm m}$  is a close approximation if a straight line may be drawn through the average of a number of points plotted on logarithmic paper.

If very great accuracy is sought in determining the formula for a very precise determination we may use the power series (25), (26), (27) or (28), suggested in Chap. XVI. A definite number of terms usually not to exceed four, is chosen for the right-hand member. This form requires us to determine one constant for each term. Four simultaneous equations are written containing the four pairs of measured coordinates taken from the curve. The solution of four equations gives the numeric values of the constants.

We are often guided in the choice of a suitable formula by the shape of the curve. If one of the variables becomes indefinitely large while the other variable approaches a limiting value then use (6),

$$(6) \quad y = \frac{ax}{x-b}.$$

Other equations which are suggested for the formulation of graphs are given in (7), (8) and (9):

$$(7) \quad y = \frac{ax^2}{1+ex^2},$$

$$(8) \quad y = (a+bx)^n,$$

$$(9) \quad y = a+bx^n.$$

**Ex. 7.** Plot (6), (7), (8), and (9), when  $a=2$ ,  $b=1$ ,  $e=.5$ .

**Ex. 8.** Plot (6), (7), (8), and (9), when  $a=2$ ,  $b=1$ ,  $e=1$ .

**Ex. 9.** Plot (6), (7), (8), and (9), when  $a=2$ ,  $b=2$ ,  $c=1$ ,

**Ex. 10.** Plot (6), (7), (8), and (9), when  $a=1$ ,  $b=1$ ,  $c=1$ .

7. Any equation which may be restated in the linear form (5) may be plotted as a linear graph,

$$(5) \quad Y = A + BX.$$

Consider the logarithmic curves represented in Fig. 113, their equations are (10) and (11):

$$(10) \quad y = \log x;$$

$$(11) \quad y = \log_e x;$$

$$(12) \quad Y = X.$$

(10) and (11) are forms of the linear equation (12) in which  $Y$  represents  $y$ , and  $X$  represents  $\log x$ . If  $y$  be plotted vertically to a uniform scale and  $x$  horizontally to a logarithmic scale the graph of (10) will be a straight line with a slope of unity. Paper prepared in this manner is called **semi-logarithmic paper**. By means of the scale of equal parts, the logarithmic paper of Figs. 139 and 140 may be used as semi-logarithmic paper.

(6) may be transformed to read (13) which corresponds to the intercepts form (14) of (5).

$$(13) \quad \frac{\frac{y}{a}}{\frac{x}{-a}} = 1,$$

$$(14) \quad \frac{Y}{A} + \frac{X}{\frac{-A}{B}} = 1.$$

Therefore if  $y$  is plotted vertically and  $\frac{y}{x}$  is plotted horizontally both being uniformly scaled then the graph is linear. Fig. 141 shows (6) plotted in the usual way as an

hyperbola II and also (13) the equivalent of (6) plotted in the linear form I. The vertical and horizontal intercepts

I are shown as  $a$  and  $-\frac{a}{b}$  respectively, which verify (13).

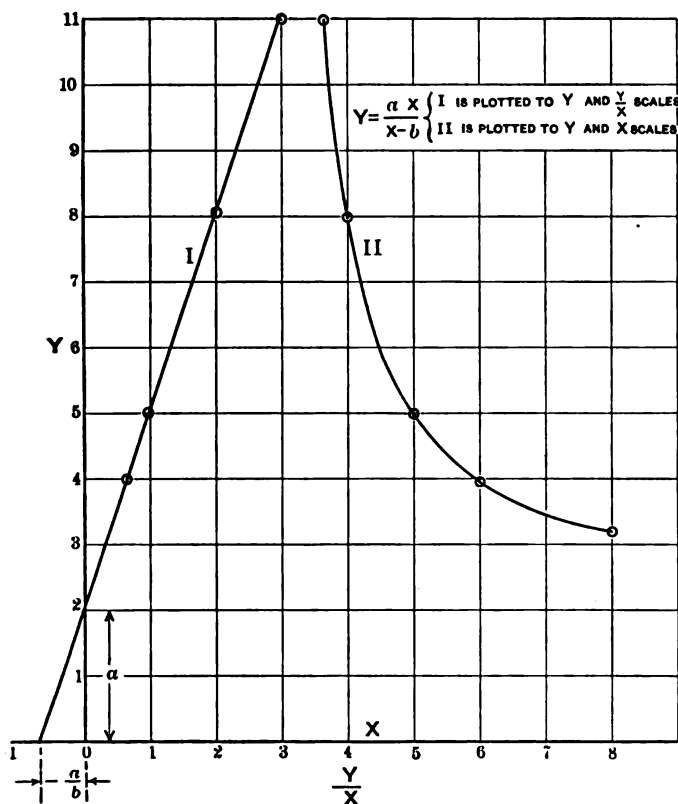


FIG. 141.—Ordinary and Linear Representation of  $y = \frac{ax}{x-b}$ .

The vertical line which is located one unit's distance to the right of the  $y$  axis is the axis of  $x$ , and is called the **index**. A straightedge which joins the origin with any

point on the linear graph I cuts the index at a point which indicates the corresponding value of  $x$ . The  $y$  axis and  $x$  index are graduated alike. The latter property serves as a method for the rapid plotting of the linear representation of (6).

(7) may be transformed to read (15) which corresponds to the intercepts form (14) of (5).

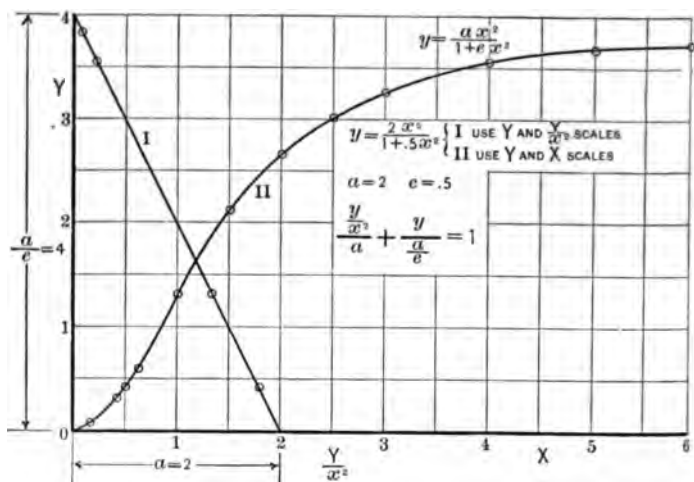


FIG. 142.—Ordinary and Linear Representation of  $y = \frac{ax^2}{1+ex^2}$ .

$$(15) \quad \frac{y}{a} + \frac{\frac{y}{x^2}}{e} = 1,$$

$$(14) \quad \frac{Y}{A} + \frac{X}{B} = 1.$$

Therefore if  $y$  is plotted vertically and  $\frac{y}{x^2}$  is plotted horizontally both being uniformly scaled then the graph

is linear. Fig. 142 shows (7) plotted in the usual way as a cubic hyperbola II and also in the linear form I.

The vertical and horizontal intercepts are shown as  $\frac{a}{e}$  and  $a$  respectively. The vertical line which is located one unit's distance to the right of the  $y$  axis is the  $x$  index. The  $x$  index should be graduated as square roots of the paralleled numbers on the  $y$  axis. A straightedge which joins the origin with any point on the linear graph I cuts the index at a point which indicates the corresponding value of  $x$ . The latter property serves as a method for the rapid plotting of the linear representation of (7).

**Ex. 11.** Show that the conic equations (37) and (42), Chapter XVI, may be plotted as linear graphs on cross-section paper in which the vertical and horizontal rulings are graduated in squares of numbers similar to the  $x$  index in the preceding paragraph.

**8.** The sine curve, Fig. 143, is expressed as equation (16) and is identified with the linear form (12).

$$(16) \quad y = \sin x.$$

$$(12) \quad Y = X.$$

In Fig. 143,  $OABCDE$  represents half of the sine loop plotted in the usual way. The vertical and horizontal scales are uniform scales of numbers and angles respectively. If the horizontal scale be made to correspond to sines of  $x$ , as shown in the sinusoidal scale of angles, then the half loop becomes a linear graph  $OE$ . Like letters in each graph represent corresponding points.

**Ex. 12.** Prepare a table of equations both algebraic and non-algebraic which are representable by linear graphs when plotted to specially scaled cross-section paper. In parallel columns indicate the nature of the scales for the two axes.

**9. Exponential Equations.** When one of the variables in an equation appears as an exponent of a constant base

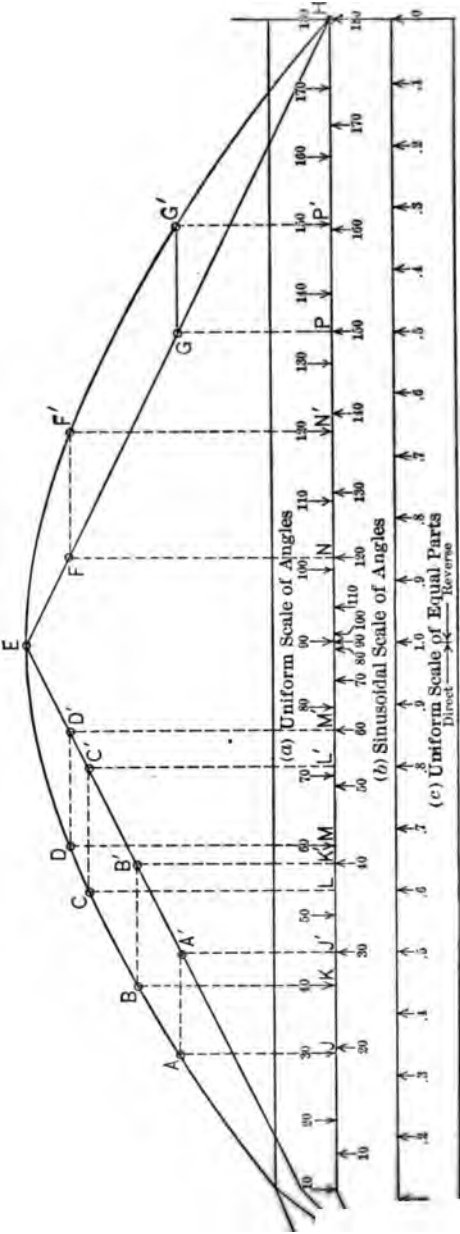


Fig. 143.—Ordinary and Linear Representation of  $y = \sin x$ .

the equation is called an **exponential** as illustrated in (17), (18), (19) and (20)..

$$(17) \quad y = a^x,$$

$$(18) \quad y = ba^{cx}.$$

$$(19) \quad y = e^{kx},$$

$$(20) \quad y - d = be^{k(x-m)}.$$

Usually the constant base is the Napierian number ( $e$ ). (17) and (18) may be transformed into (24) and (28) respectively, by the introduction of a constant multiplier for the variable exponent.

$$(21) \quad \log_e y = x \log_e a; \quad (25) \quad \log_e y = \log_e b + cx \log_e a;$$

$$(22) \quad \log_e a = \text{a constant } K; \quad (26) \quad c \log_e a = n, \log_e b = B;$$

$$(23) \quad \therefore \log_e y = Kx; \quad (27) \quad \log_e y = B + nx;$$

$$(24) \quad \therefore y = e^{Kx}; \quad (28) \quad y = e^{B+nx}.$$

There are many applications of this law. It has been styled the Compound Interest Law by Lord Kelvin.

Plot the following equations using semi-log paper and consult the **exponential table XXIII**.

**Ex. 13A.** (24)  $y = e^{kx}$ . Use the following values of  $k$ , .1, .5, 1, 2, 5.

**Ex. 13B.** (29)  $y = e^{-kx}$ . Use the following values of  $k$ , .1, .5, 1, 2, 5.

**Ex. 14.** The friction of a rope or belt on a pulley or cylinder is expressed by (30). Plot (30) in which  $T_2$  is the maximum tension and  $T_1$  the minimum tension,  $\alpha$  the angle of contact, and  $\mu$  the coefficient of friction.  $\mu$  for hemp rope on cast-iron is .2-.4, and for wire rope .5.

$$(30) \quad \frac{T_2}{T_1} = e^{\pm \mu \alpha}$$

make  $\frac{T_2}{T_1}$  the independent variable and plot  $\alpha$  from 0 to  $180^\circ$  when  $\mu = .2, .25, .3, .35, .4, .45, 5$ .

**Ex. 15.** The necessary distance ( $d$ ) for the separation of conductors to avoid corona is given in (31) in which  $D$  is the diameter in inches,  $d$  is the distance in inches,  $E$  is effective kilo-

volts between conductors and  $K$  is a constant. Assume  $E=150$  and  $K=39.9$ .

$$(31) \quad d = \frac{D}{12} \epsilon^{\frac{E}{KD \cdot 8}}.$$

**Ex. 16.** Variation of voltage with self induction.

$$(32) \quad E = RI - RI_0 \epsilon^{-\frac{Rt}{L}}.$$

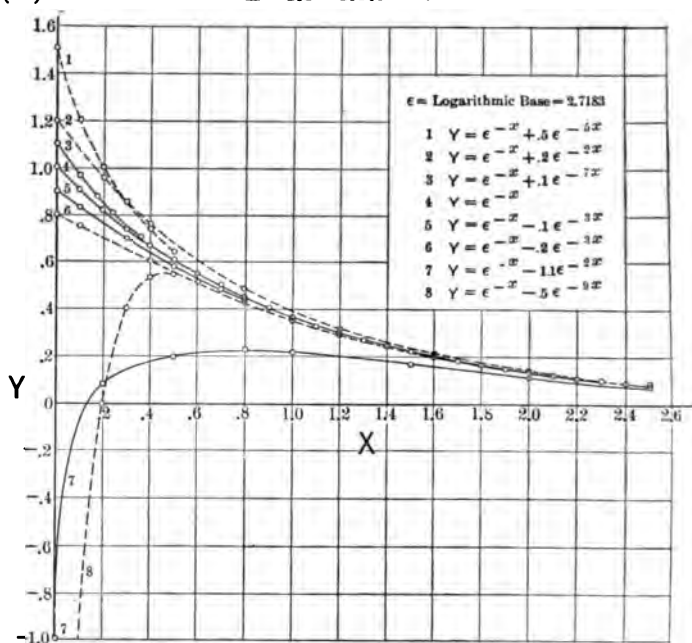


FIG. 144.—The Exponential Family.

$I_0$  is the value of the current when  $t=0$ . See Figs. 191 and 192.

10. The standard form of the exponential may be written (33) or (34), but the exponent appears negative for a majority of technical applications.

$$(33) \quad y = \epsilon^x, \quad (34) \quad y = \epsilon^{-x}.$$

(34) is illustrated as curve 4 in Fig. 144. It crosses the  $Y$  axis at the point 1. To the left of the  $Y$  axis it rises rapidly as  $x$  becomes greater negatively. On the right of the  $Y$  axis it falls less rapidly as  $X$  increases positively. The plotting of (34) on semi-logarithmic paper determines a straight line with a negative slope ( $-1$ ).

Many exponential equations are combinations of two exponential forms as represented by (35) and as illustrated

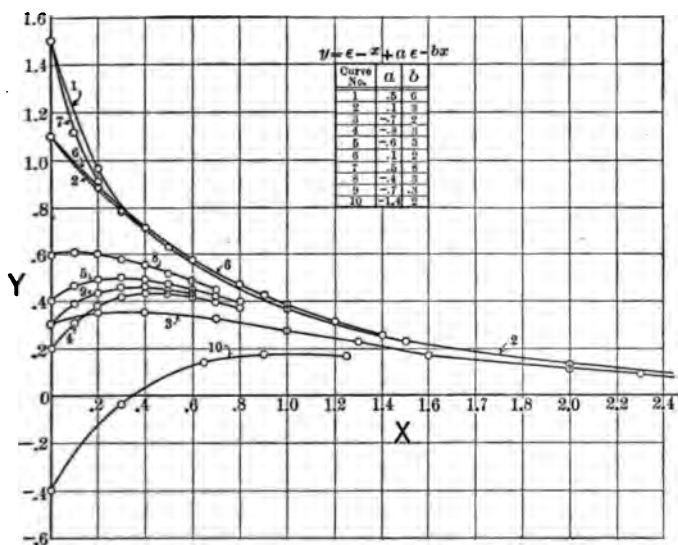


FIG. 145.—The Exponential Family.

in Fig. 144. Curves 2 and 7 may be compared to show the effect upon the standard when a second exponential is either added or subtracted. The other curves serve to show the effect of changing the constants in (35).

$$(35) \quad y = e^{-x} \pm a e^{-bx}.$$

Other exponential curves of the type (35) are shown in Fig. 145, which have been plotted separately for clarity.

**Ex. 17.** Plot  $y = \frac{1}{2}\{e^x + e^{-x}\}$ ; this particular exponential is designated an **hyperbolic cosine** and is abbreviated **cosh x**. Assume values of  $x$  between 0 and 2.

**Ex. 18.** Plot  $y = \frac{1}{2}\{e^x - e^{-x}\}$ ; this particular exponential is designated an **hyperbolic sine** and is abbreviated **sinh x**. Assume values of  $x$  between 0 and 2.

**Ex. 19.** Plot the following exponentials in Table XXIII on semi-logarithmic paper and extend the range of the tables by adding parallels to the linear graphs, as described in paragraph 5, page 356. Add the supplementary scales for each parallel line.

TABLE XXIII. EXPONENTIALS

$e^x$	Exponent $x$ .	$e^{-x}$	$e^x$	Exponent $x$ .	$e^{-x}$
1.000	.0	1.000	3.669	1.3	.273
1.105	.1	.905	4.035	1.4	.247
1.221	.2	.819	4.482	1.5	.223
1.350	.3	.741	4.953	1.6	.202
1.492	.4	.670	5.474	1.7	.183
1.649	.5	.607	6.050	1.8	.165
1.822	.6	.549	6.686	1.9	.150
2.014	.7	.497	7.389	2.0	.135
2.226	.8	.449	8.166	2.1	.122
2.460	.9	.407	9.025	2.2	.111
2.718	1.0	.368	9.974	2.3	.100
3.004	1.1	.333	11.023	2.4	.091
3.320	1.2	.301	12.182	2.5	.082

**Ex. 20.** The exponential curves shown in Figs. 144 and 145 are obtained by addition. If the component exponential curves are replotted on semi-logarithmic paper can they be added to produce the resulting curves shown in Figs. 144 and 145?

**Ex. 21.** Show how an exponential table may be constructed from a table of hyperbolic logarithms or from a table of hyperbolic sines and cosines.

## CHAPTER XIX

### THE USE OF POLAR PAPER

1. In the preceding chapters it was shown that a point was completely determined by its coordinates, i.e., its distances from two fixed lines called axes. The axes may be represented in a rectangular or oblique position and the scales thereon may be graduated uniformly or nonuniformly.

In Fig. 146, the **horizontal axial line** has a fixed left-hand extremity called a **pole** or **origin**. The point  $P_1$  may be

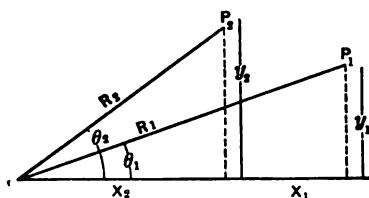


FIG. 146.—Polar Coordinates.

located by the pair of rectangular coordinates  $x_1$  and  $y_1$ , and it may be located also by a pair of **polar coordinates**  $r_1$  and  $\theta_1$ . The **radius vector** ( $R$ ) is the distance measured radially from the pole to the point. The vectorial angle ( $\theta$ ) is the angle formed by the radius vector and the horizontal axial line. Likewise the point  $P_2$  may be designated and located by its rectangular coordinates  $x_2$ ,  $y_2$  and by its polar coordinates  $R_2$ ,  $\theta_2$ .

A curve may have its formula expressed in terms of the variables  $R$  and  $\theta$  as well as in terms of the variables  $x$  and  $y$ . The relations between  $R$ ,  $\theta$ ,  $x$  and  $y$  are expressed in (1), (2), (3) and (4).

$$(1) \quad x = R \cos \theta.$$

$$(2) \quad y = R \sin \theta.$$

$$(3) \quad R = \pm \sqrt{x^2 + y^2}.$$

$$(4) \quad \tan \theta = \frac{y}{x}.$$

Therefore an equation in  $x$  and  $y$  may be rewritten in terms of  $R$  and  $\theta$  by substituting from (1) and (2). Thus (5) becomes (6) and reduces to (7).

$$(5) \quad y = bx^m.$$

$$(6) \quad R \sin \theta = bR^m \cos^m \theta.$$

$$(7) \quad R = \frac{\sec \theta}{\sqrt[m-1]{b \cot \theta}}.$$

2. In order to obviate the necessity for measuring the angles and radii vectors for polar plotting we may use **polar coordinate paper**, which is prepared for the trade. Polar paper as shown in Fig. 147, is divided by concentric circles and radial lines. When the circles are equally spaced the radii are uniformly divided. When the radii are equally spaced the circles are uniformly divided. The center of the circles is the pole. The horizontal radius extending to the right of the pole is the axial line from which positive and negative angles are read in counter-clockwise and clockwise direction respectively. The outer circumference is graduated in degrees and  $\pi$  measure. Other points of divisions may be supplemented to indicate positive and negative multiples and submultiples of radians.

When using polar coordinate paper first solve the p

formula for  $R$  and prepare a table of values for  $R$  and  $\theta$ . Designate the radian scale so as to comprehend the greatest value of  $R$  in the table and indicate the first six positive and negative radians. If  $\theta$  is expressed in degree measure then its unit value or radian measure may be obtained from Table VIII. Although the angles may be plotted in degrees

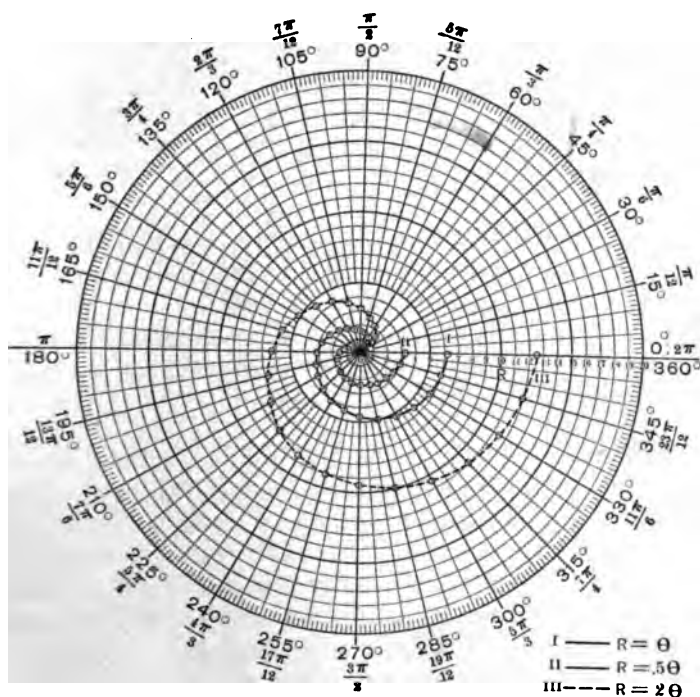


FIG. 147.—The Plotting of Polar Equations.

the solution of  $R$  can be obtained usually only after  $\theta$  is expressed in unit measure.

**Ex. 1.** Plot (8)  $R = \theta$ . The values of  $\theta$  and  $R$  are given in Table XIV. The corresponding curve is shown as I in Fig. 147.

There are two circles marking 20 unit divisions on each radius. Curve I recedes radially 6.28 units in one complete convolution.

One complete convolution corresponds to a rotation of the radius vector through  $360^\circ$ , or  $2\pi$  radians. I is known as a standard linear spiral, or the spiral of Archimedes.

TABLE XXIV. THE STANDARD LINEAR SPIRAL.

$R$	0	.262	.524	1.05	1.57	2.09	2.62	3.14	3.66	4.19	4.71	5.23	5.77	6.28
Degrees	$0^\circ$	$15^\circ$	$30^\circ$	$60^\circ$	$90^\circ$	$120^\circ$	$150^\circ$	$180^\circ$	$210^\circ$	$240^\circ$	$270^\circ$	$300^\circ$	$330^\circ$	$360^\circ$
$\theta$ Radians	0	.262	.524	1.05	1.57	2.09	2.62	3.14	3.66	4.19	4.71	5.23	5.77	6.28
$\pi$ measure	0	$\frac{\pi}{12}$	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	$\pi$	$\frac{7\pi}{6}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{11\pi}{6}$	$2\pi$

4. The graph of (9)  $R=2\theta$  is shown as curve III in Fig. 147. The radial ordinates in (9) are correspondingly twice the magnitude of the radial ordinates in (8).

The graph of (10)  $R=.5\theta$  is shown as curve II in Fig. 147. The radial ordinates of (10) are correspondingly one-half the magnitude of the radial ordinates in (8).

*Observation.* A spiral is the polar graph of a polar equation which does not contain trigonometric functions. The standard linear spiral  $R=\theta$  is the polar representation of the rectangular linear graph  $y=x$ . The radial recession or departure of the linear spiral is proportional to its angular advance. The numeric coefficient of  $\theta$  increases the departure proportionately. When the absolute or constant term is missing from the polar equation the spiral passes through the origin.

**Ex. 2.** Plot (8), (9) and (10) for negative values of  $\theta$  and  $R$ .

Negative angles are laid off clockwise in accordance with the convention of signs in trigonometry. Negative values of  $R$  are to be plotted in a contrary sense to positive values and may be laid off by extending the radial lines back through the origin. The positive and negative halves of spirals represent chine design.

Plot the following spirals for positive and negative values of  $\theta$ .

**Ex. 3.** (11)  $R = \theta^2$ , (12)  $R = 2\theta^2$ , (13)  $R = .5\theta^2$ .

**Ex. 4.** (14)  $R = \theta^3$ , (15)  $R = 2\theta^3$ , (16)  $R = .1\theta^3$ .

**Ex. 5.** (17)  $R = \theta^{-1}$ , (18)  $R = 5\theta^{-1}$ , (19)  $R = 25\theta^{-2}$ .

**Ex. 6.** (20)  $R = 1 + \theta$ , (21)  $R = 2 + .5\theta$ , (22)  $R = 2\theta + .5\theta^2$ .

Spirals (8) to (16) are parabolic forms, whereas spirals (17) to (19) are hyperbolic forms. They are so named on account of the resemblance to similar rectangular equations.

**Ex. 7.** Place a sheet of transparent polar paper upon a sheet of squared paper so that the respective pole and origin are coincident and so that the respective axial line and  $x$  axis are coincident. A straight line which passes through the pole and origin will be expressed as (23) in polar coordinates and as (24) in rectangular coordinates.

(23)  $\tan \theta = b$ , (24)  $y = bx$ .

Any linear graph which cuts the pole will have a constant slope as shown in (23).

**Ex. 8.** Show that the polar form of (25) is (26).

(25)  $y = a + bx$ . (26)  $R(\sin \theta - b \cos \theta) = a$ ,

**Ex. 9.** Show that the polar form of (27) becomes (7).

(27)  $\log y = \log b + m \log x$ .

Plot the following equations:

**Ex. 10.** (28)  $R = \sin \theta$ , (29)  $R = \cos \theta$ ,

**Ex. 11.** (30)  $R = \csc \theta$ . (31)  $R = \sec \theta$ .

5. The conics may be constructed by means of polar coordinates from (32). In such cases the focus of the conic is located at the pole.

(32)  $R = \frac{a}{1 - e \cos \theta}$ .

In (32)  $e$  is the eccentricity of the conic and  $l$  is the ordinate above the focus.

**Ex. 12.** Plot (32) when  $l=1$  and  $e=.5, 1$  and  $1.5$ .

**6.** The polar equation of a circle is given in (33).

$$(33) \quad a^2 = R^2 + r^2 - 2Rr \cos(\theta - \alpha).$$

(33) is identified with Fig. 148, in which  $a$ =radius of the circle,  $R$ =the polar ordinate of any point on the circumference and  $\theta$  the corresponding angular ordinate;  $r$ =the polar ordinate of the center and  $\alpha$  the corresponding angular ordinate. From the figure it is seen that for a given value of  $\theta$  there will be two values of  $R$  corresponding to the points  $P_1$  and  $P_2$ , respectively.

**Ex. 13.** Show that (33) reduces to (34) when the pole is coincident with the lower extremity of a vertical diameter.

$$(34) \quad R = 2r \sin \theta.$$

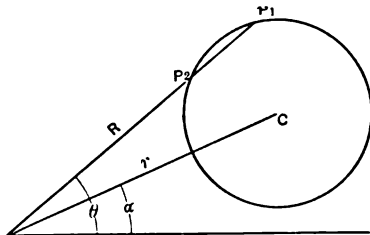


FIG. 148.—The Polar Circle.

**Ex. 14.** Show that (34) reduces to (35) when the center of the circle coincides with the pole. What is the interpretation of (35).

$$(35) \quad R = a.$$

**7. The Logarithmic Spiral.** The logarithmic spiral is also called the **equiangular spiral** and represents an exponential curve and corresponds to 4 in Fig. 144. Its equation is given in (36) and (37). It takes its name from (37). It is a curve which has considerable application in the design

of milling cutters owing to the fact that the curve cuts each radius vector with a constant angle. The angle shown in Fig. 149 is  $124^\circ 42'$ . The tangent of  $124^\circ 42' = 1.44$  which is also the modulus of the system of logarithms ( $b=2$ ) used in plotting Fig. 149.

$$(36) \quad R = b^\theta.$$

$$(37) \quad \log_b R = \theta.$$

$$(38) \quad R = e^{K\theta}.$$

Eq. (36) is a special form of (38) in which  $K$  is an arbitrary constant. When the Napierian base is used we

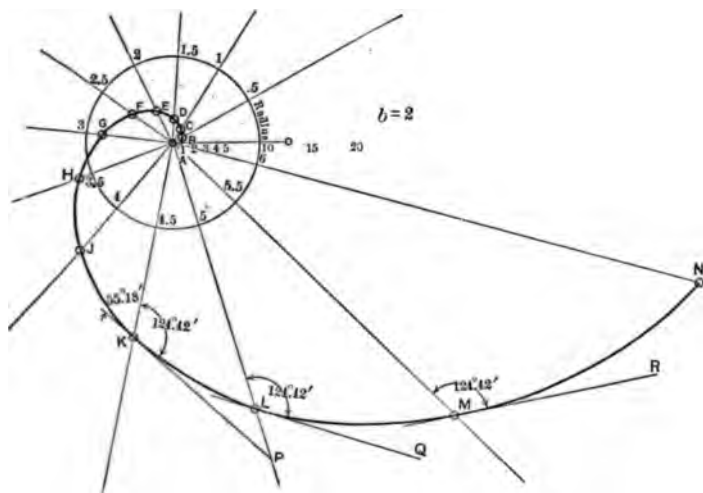


FIG. 149.—The Logarithmic or Equiangular Spiral.

have the portrayal of the exponential curves shown in Fig. 144.

A table of values between  $\theta$ ,  $K\theta$ ,  $\log R$  and  $R$  is prepared for the special value  $b=2$ . The curve is plotted between  $R$  and  $\theta$ . Any radius vector, which bisects the angle formed by its two adjacent radii vectors, is a mean proportional. In Fig. 149, the logarithmic spiral has

$$(d) \quad \therefore OK = b^{4.5} = \frac{1}{10}$$

What is the character of the function  $\theta$  if  $\theta$  is negative.

**Ex. 15.** Construct equiangular spiral; also a spiral to have its slope equal to its radius vector.

8. Polar coordinates are common measurements of the intensity of light. One form of polar paper is that represented by the recording or record cards of recording instruments. The radial lines of regular polar paper are replaced by semi-radial lines which correspond to the position of the stylus or inking point.

**9. Roulettes or Cycloidal**  
rolling curve generates a periodic curve. The special curve which results is the **generator curve**, i.e., the position of the generatrix is

in which  $r$  is the radius and  $\theta$  the angle of rotation of the generator circle.

$$(39) \quad x = r\theta - r \sin \theta,$$

$$(40) \quad y = r - r \cos \theta,$$

$$(41) \quad x = r \cos^{-1} \left\{ \frac{r-y}{r} \right\} - \sqrt{2ry-y^2},$$

$$(42) \quad x = r \operatorname{versin}^{-1} \left\{ \frac{y}{r} \right\} - \sqrt{y(2r-y)}.$$

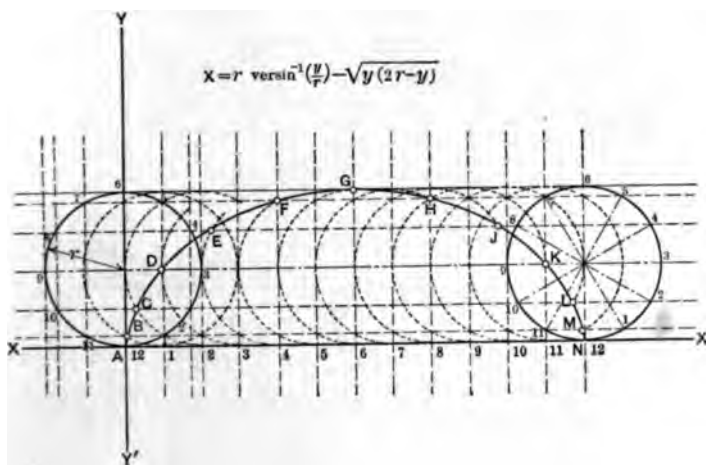


FIG. 150.—The Cycloid.

Eliminate  $\theta$  from (39) and (40) and show that (41) and (42) result. In Fig. 150,  $AN$  is laid off on the directrix  $XX'$  as the rectified circumference of the generator circle whose radius is  $r$ .  $AN$  is subdivided into 12 equal divisions corresponding to the subdivisions of the circumference. The center of the generator circle will be directly above each of these twelve numbered positions in succession corresponding to each twelfth of its completed rotation.

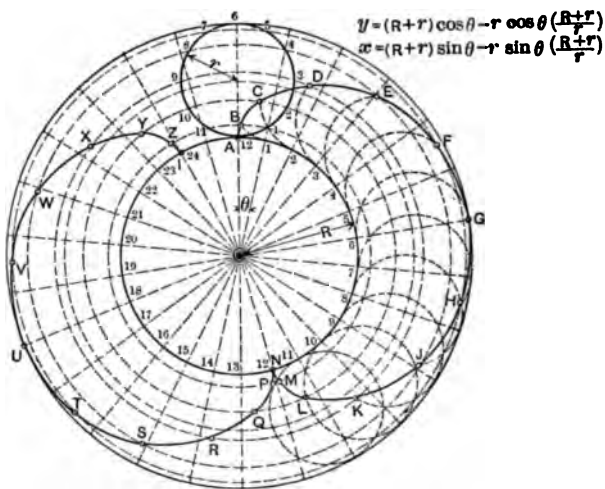


FIG. 151.—The Epicycloid.

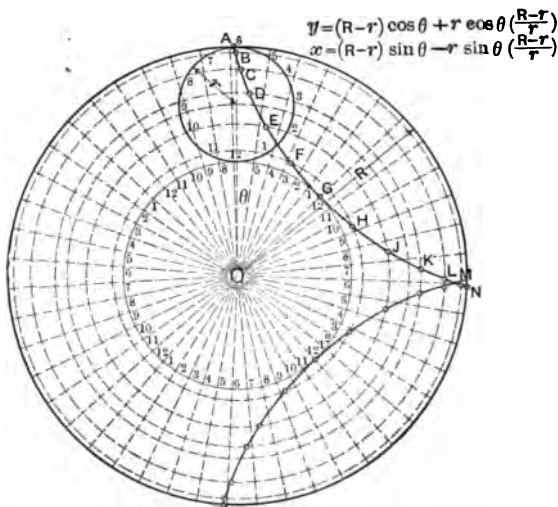


FIG. 152.—The Hypocycloid.

The respective positions of the generatrix are  $A, B, C, D, E, F, G, H, J, K, L, M, N$ . The generatrix does not rise and advance uniformly for successively equal amounts of rotation, as is indicated by the series of horizontal parallels.

10. When the directrix is a circular arc the resulting curve traced by the point  $A$  may be either an **epicycloid** or **hypocycloid**, depending upon whether the generator circle rolls outside or inside the directrix. The epicycloid is shown in Fig. 151, and its equations are given in (43) and (44). The hypocycloid is shown in Fig. 152, and its equations are given in (45) and (46).  $R$  and  $r$  are the respective radii of the director and generator circles. Not only do the generator curves rotate but they also revolve about the center  $O$ . The amount of revolution about  $O$  is expressed by  $\theta$ . The generator circumference is divided equally into twelfths. The series of concentric circles is drawn through these points to show the successive elevations of the generatrix. The arc  $AN$  of  $12 \dots 12$  equals the circumference of the generator circle and is likewise divided into twelve equal divisions.

$$(43) \quad y = (R+r) \cos \theta - r \cos \theta \left( \frac{R+r}{r} \right),$$

$$(44) \quad x = (R+r) \sin \theta - r \sin \theta \left( \frac{R+r}{r} \right),$$

$$(45) \quad y = (R-r) \cos \theta + r \cos \theta \left( \frac{R-r}{r} \right),$$

$$(46) \quad x = (R-r) \sin \theta - r \sin \theta \left( \frac{R-r}{r} \right).$$

When the generatrix lies outside or inside of the periphery of the generator circle the resulting curve will be looped and is known accordingly as a **curtate** or **prolate cycloid**.

**Roulettes** are applied industrially in machine design and also in the geometric design of engraving.

**Ex. 16.** Construct the involu

**11. The Trisection of an A**  
angle may be accomplished by

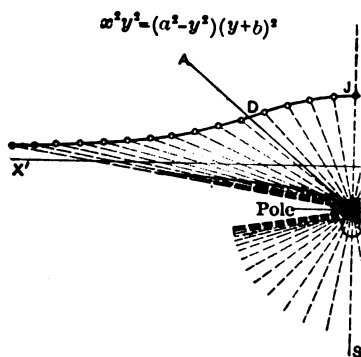


FIG. 153.—Th

in Figs. 153, and 154.  $XX'$  is  
 $O$  a pole and  $OP$ ,  $OJ$ ,  $OD$ .

s expressed in (47) and (48). The rectangular coordinates refer to the axes  $RS$  and  $XX'$ .

$$17) \quad \frac{OQ}{QP} = \frac{13}{32},$$

$$18) \quad \frac{x}{b+y} = \frac{\sqrt{a^2 - y^2}}{y},$$

$$19) \quad x^2 y^2 (a^2 - y^2) (y + b)^2.$$

It is customary to use the left end of the upper branch of the conchoid for the trisection of an angle as shown in Fig. 154. The angle  $BOA$  is given. Construct  $PR \perp OA$  and fill in the angular space with radial lines. On each radial line measured outward from  $PR$  lay off distances equal to  $2OP$ . Through the points so obtained draw the

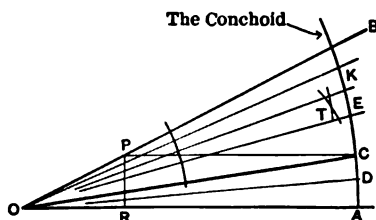


FIG. 154.—The Trisection of Angle  $BOA$

conchoid  $ADCEK$ . Construct  $PC$  parallel to  $OA$  intersecting the conchoid at  $C$ . Join  $C$  with  $O$  then  $COA = \frac{1}{3}BOA$ . The bisection of  $BOC$  with a compass completes the trisection.

It is not necessary to reconstruct the conchoid for each angle since the conchoid may be laid off on tracing paper or a matrix or a die made in celluloid or metal. If the vertex of the angle  $O$  and one of its sides  $OB$  are brought into coincidence with the pole  $O$  and axial line  $OC$  respectively, as shown in Fig. 155, then the trisection may be readily obtained by the intersection of  $VP$  a parallel to  $OB$  drawn from  $V$  the intersection of the directrix with  $OD$  the other



**Ex. 17.** By means of a conchoid trisect a radian of angle.

**13. The Multisection of an Angle.** An angle may be multisectioned, i.e., divided into any number of divisions by means of the chordel which is illustrated in Figs. 155 and 156. In Fig. 155,  $O$  is the pole and  $OB$  the directrix or axial line. The chordel is so named from the fact that the intercepted arcs are evenly divided by applying consecutively a chord or element of definite length. In Fig. 155, the element is used seven times and serves to divide an

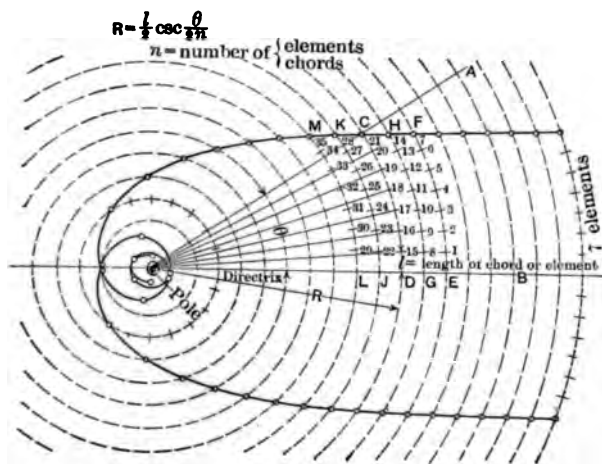


FIG. 155.—The Chordel.

angle into seven equal parts. In Fig. 156, the element is used three times and serves to divide an angle into three equal parts.

The chordel is constructed by beginning at the directrix and laying off consecutive chords on the concentric arcs so that there will be as many elements as the required numeric division of the angle. A smooth curve is then passed through the points. The chordel is symmetrical to the directrix and has one less convolution than the number of

elements. The equation of the chordel is given in (50) in which  $l$  is the length of an element,  $n$  the number of elements and  $R$  and  $\theta$  are polar coordinates.

$$(50) \quad R = \frac{l}{2} \csc \frac{\theta}{2n}.$$

In Fig. 156, the angle  $BOA$  is to be trisected by the chordal. Regard the vertex  $O$  as the pole of the chordal and the side  $OA$  as the directrix. Construct three concentric arcs and lay off a convenient element three times.

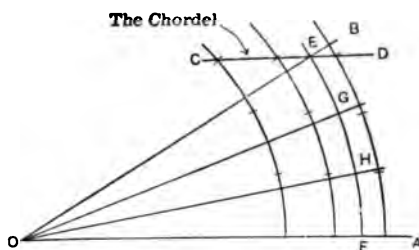


FIG. 156.—The Trisection of  $BOA$ , by Means of the Chordal.

Through the last points of division draw the chordal  $CED$  intersecting the angles' side  $OB$  at  $E$ . Draw another concentric arc  $EF$  through  $E$ , and observe that the element divides the arc  $EF$  evenly at  $G$  and  $H$ . Connect  $G$  and  $H$  with  $O$ . Then

$$\angle s \ EOG = GOH = HOF = \frac{BOA}{3}.$$

**Ex. 18.** By means of the chordal divide an angle of  $100^\circ$  into 7 parts.

**Ex. 19.** By means of the chordal divide a radian into 5 parts.

## CHAPTER XX

### SOLVING FORMULAS BY CHARTS

1. THE graphic solution of many formulas may be obtained by preparing a suitable **chart** or **diagram** of lines which may be constructed on plain paper. A straightedge or stretched thread is applied upon the chart and its intersection with a fixed graduated line called an **index** or **support** gives the desired solution.

2. The chart solution of a quadratic equation is shown in Fig. 157, in which the index is a parabolic curve. The equation of the second degree for the variable  $\omega$  is expressed in (1).

$$(1) \quad \omega^2 + u\omega + v = 0.$$

The horizontal line through the center of the chart is called the **zero line**. The left and right vertical bounding lines are designated as the **u** and **v** axes respectively. The axes are uniformly graduated for both positive and negative values. To solve a quadratic equation observe the  $u$  and  $v$  values of the given equation. Locate these values on their respective axes and join them with a straightedge. The intersection of the straightedge with the index indicates the solution for  $\omega$ .

**Ex. 1.** Verify the following solutions:

	Given	Solution
(2)	$\omega^2 + .5\omega - 1.5 = 0,$	$\omega = 1,$
(3)	$\omega^2 - .75\omega - 2.5 = 0,$	$\omega = 2,$
(4)	$\omega^2 - 2.5\omega + 1.5 = 0,$	$\omega = 1.5.$

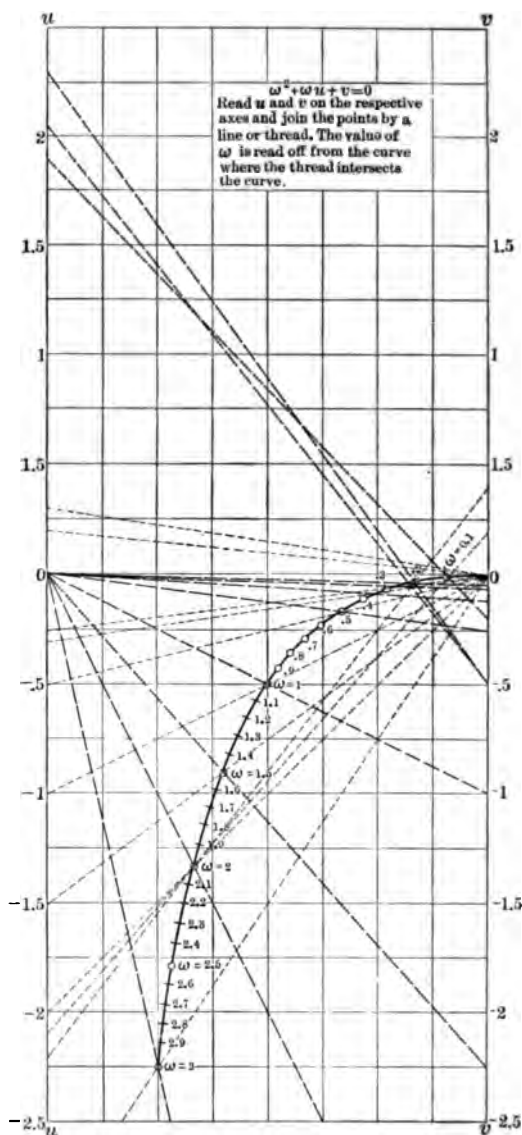


FIG. 157.—Chart Solution of a Quadratic Equation.

3. The construction of the index is obtained by locating each of its successive points of graduation by two intersecting lines. Suppose we wish to locate the point  $\omega=1$ . Substitute 1 for  $\omega$  in (1) which reduces to (5).

$$5) \quad 1+u+v=0.$$

In (5)  $u$  and  $v$  may have any pair of associated values which will validate it. Therefore, when  $u=0$ , then  $v=-1$ , and when  $v=0$ , then  $u=-1$ . Locate these pairs of points on the  $u$  and  $v$  axes respectively, and join them by straight lines. Their intersection gives the point  $\omega=1$ .

In order to locate any other point on the index such as  $\omega=2$  substitute 2 for  $\omega$  in (1) which reduces to (6).

$$6) \quad 4+2u+v=0.$$

In (6) when  $v=0$ , then  $u=-2$ , and when  $u=0$ , then  $v=-4$ . Draw straight lines through these pairs of points. Their intersection locates the point  $\omega=2$ .

Other points on the index are determined in the same manner. By extending the index to the right of the  $v$  axis negative value of  $\omega$  may be determined.

4. If the constants  $u$  and  $v$  are too large to be read on the chart then divide  $u$  by 10 and  $v$  by 100 and multiply the index reading by 10. Thus to solve (7) rewrite it as (8) and (9). Solving (9) by the chart, gives  $\omega=.6375$  and therefore  $A=10\omega=6.375$ .

$$7) \quad A^2+25A-200=0 \text{ ————— given equation.}$$

$$8) \quad 100\omega^2+250\omega-200=0 \text{ ————— subs. } A=10\omega.$$

$$9) \quad \therefore \omega^2+2.5\omega-2=0 \text{ ————— div. (8) by 100.}$$

5. The chart solution for a cubic equation is shown in Fig. 158, in which the index is also a parabolic curve. Every equation of the third degree may be reduced to the form (10) in which the second power of the unknown is missing.

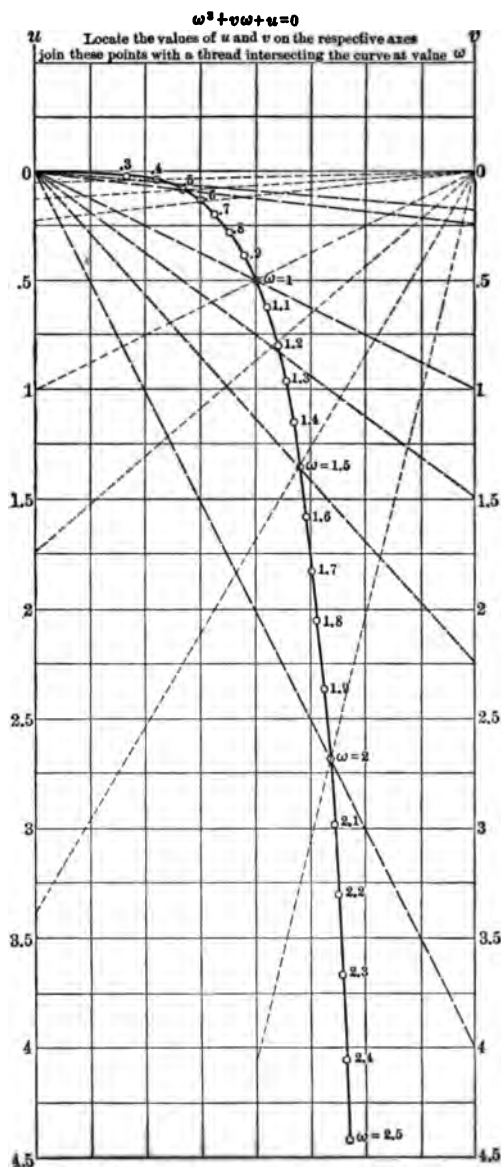


FIG. 1

1 of a Cubic Equat

Locate the constants  $u$  and  $v$  of the given equation on their respective axes in Fig. 158, and join them by a straight line. The intersection of the latter with the index indicates the solution of (10). Only negative values of  $u$  and  $v$  are shown in the chart.

$$(10) \quad \omega^3 + v\omega + u = 0.$$

**Ex. 2.** Verify the following solutions:

	Given	Solution
(11)	$\omega^3 - 3\omega - 2 = 0,$	$\omega = 2,$
(12)	$\omega^3 + .5\omega - 1.5 = 0,$	$\omega = 1,$
(13)	$\omega^3 - .75\omega + .25 = 0,$	$\omega = .5.$

6. If the numeric values of  $u$  and  $v$  are too large to be read on the chart then divide  $v$  by 100 and  $u$  by 1000 and multiply the index reading by 10.

Thus to solve (14) rewrite it as (13) and since  $\omega = .5$ , then  $y = 10\omega = 5$ .

$$(14) \quad y^3 - 75y + 250 = 0 \text{ ————— given equation.}$$

$$(15) \quad (10\omega)^3 - 75(10\omega) + 250 = 0 \text{ ————— subs. } y = 10\omega \text{ in (14).}$$

$$(16) \quad \therefore 1000\omega^3 - 750\omega + 250 = 0 \text{ ————— simplifying (15).}$$

$$(13) \quad \therefore \omega^3 - .75\omega + .25 = 0 \text{ ————— div. (16) by 1000.}$$

7. In order to reduce a complete cubic equation (17) to the required form (10) substitute  $\omega - \frac{k}{3}$  for  $z$  where  $k$  is the coefficient of  $z^2$ . Reduce the resulting equation (18) to (19) which is identical with (10).

$$(17) \quad z^3 + kz^2 + lz + m = 0,$$

$$(18) \quad \left(\omega - \frac{k}{3}\right)^3 + k\left(\omega - \frac{k}{3}\right)^2 + l\left(\omega - \frac{k}{3}\right) + m = 0,$$

$$(19) \quad \omega^3 + \left(-l - \frac{k^2}{3}\right)\omega + \left(m + \frac{4k^3}{27} - \frac{2k^2}{3} + \frac{lk}{3}\right) = 0,$$

$$(10) \quad \omega^3 + v\omega + u = 0.$$

8. The charts represented by Figs. 157 and 158, suggest a method of plotting three variables in the plane of the paper.

The charting of reciprocal formulas which are represented by (20) and (21) is shown in II of Fig. 159.

$$(20) \quad \frac{1}{R} = \frac{1}{r_1} + \frac{1}{r_2},$$

$$(21) \quad \frac{1}{C} = \frac{1}{C_1} + \frac{1}{C_2}.$$

Equation (20) is the formula for joint resistance  $R$  of two resistances  $r_1$  and  $r_2$  connected in parallel.

Equation (21) is the formula for equivalent capacity  $C$  of two condensers which are placed in series. The individual capacities are  $C_1$  and  $C_2$ .

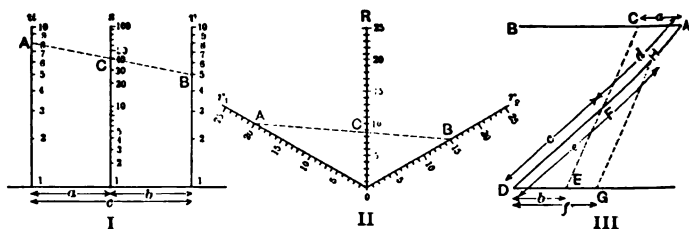


FIG. 159.—The Charting of Three or More Variables.

9. In Fig. 159, chart II represents two oblique axes on which  $r_1$  and  $r_2$  are read. The angle formed by  $r_1$  and  $r_2$  is bisected by the index which gives values of  $R$ . The three scales are uniform and equally spaced. II may be used for (21) or any similar reciprocal formula by substituting the corresponding elements for  $r_1$ ,  $r_2$  and  $R$ . It is most convenient to construct the axes at an angle of  $120^\circ$ . In the figure the straight line  $AB$  joining the point  $A$  for which  $r_1=20$  with the point  $B$  for which  $r_2=15$  intersects the index at the point  $C$  for which  $R=8.57$  when these values are substituted in (20) we obtain (22).

$$(22) \quad \frac{1}{20} + \frac{1}{15} = \frac{7}{60} = \frac{1}{8.57}.$$

The proof of the chart solution follows: join  $C$  with the 10 mark on the  $r_1$  and  $r_2$  axes and designate these respective points by  $D$  and  $E$ . Then triangles  $ADC$  and  $CEB$  are similar and triangles  $DCO$  and  $COE$  are equal equilateral triangles. Therefore,

$$(23) \quad CO = DC = CE = OD = OE.$$

$$(24) \quad \frac{AD}{DC} = \frac{CE}{EB} = \frac{AD}{DO} \text{——— homol. sds. sim } \triangle s.$$

and

$$(25) \quad \frac{AD+DO}{DO} = \frac{CE+EB}{EB} \text{——— accretion proportion.}$$

$$(26) \quad AD+DO = r_1, \quad CE+EB = OE+EB = r_2,$$

and

$$EB = r_2 - R \text{——— Mag Ax.}$$

$$(27) \quad \frac{r_1}{R} = \frac{r_2}{r_2 - R} \text{——— subs. in (25).}$$

$$(28) \quad r_1 r_2 = R(r_1 + r_2) \text{——— prod. means, trans.}$$

$$(29) \quad \frac{1}{R} = \frac{1}{r} + \frac{1}{r_2} \text{——— div Ax.}$$

Therefore any equation of the form (28) or (29) or its equivalent may be charted as shown in II, Fig. 159. How can II be supplemented to comprehend an indefinite number of elements.

10. The charting of linear formulas which are represented by (30) or (31) is shown in I, Fig. 159.

The variables  $u$  and  $v$  are scaled on the like named vertical axes which are separated by the distance  $c$ . The variable  $s$  is scaled on the index which is separated from the  $u$  and  $v$  axes by distances which are in the ratio of  $a:b$ . For (31)  $t$  may be read on the index, but the values of  $s$  are  $\frac{d}{c}$  greater than those for  $t$ .

$$(30) \quad av + bu = cs,$$

$$(31) \quad av + bu = cs + d = tc.$$

Construct a parallel to  $c$  passing through  $C$ , then by similar triangles,

$$(32) \quad \frac{u-s}{a} = \frac{s-v}{b},$$

$$(33) \quad \therefore av + bu = (a+b)s = cs.$$

11. Equations (34) and (35) reduce to the linear forms (36) and (37), which are comparable with (30) and (31).

$$(34) \quad W = I^2 R,$$

$$(35) \quad I = \frac{bd^3}{12},$$

$$(36) \quad 2 \log I + \log R = \log W,$$

$$(37) \quad 3 \log d + \log b = \log I + \log 12.$$

In preparing the chart for (36) the  $u = \log d$  and  $v = \log b$  axes and the index  $= \log I$  are graduated logarithmically as shown in I, Fig. 159. The index is located by dividing the distance  $c$  into two segments whose ratio 2:1 corresponds to the ratio of the coefficients of  $\log d$  and  $\log b$  in (36).

12. If the ranges of values of the variables are different then different scales may be used according to the relation (38) where  $l_1$ ,  $l_2$ , and  $l_3$  are the respective scales for  $\log I$ ,  $\log R$ , and  $\log W$  or their equivalents. In charting (36), the initial marks on the two axes and index read 1 to correspond to  $\log 1 = 0$ . In (37) the initial marks for the  $b$  and  $d$  scales read 1 but for the  $I$  scale the initial mark reads .0833. Therefore the 1 mark on the index will be above the corresponding 1 mark on the axes.

$$(38) \quad l_3 = \frac{l_1 l_2}{l_1 + l_2}.$$

**Ex. 3.** Prepare charts for the following:  $D = .0015ml \frac{p}{T}$ . Tension and sag in wire spans.

**Ex. 4.**  $\log e = \alpha \log t + \beta$ . Thermo-electric couple.

**Ex. 5.**  $I = \frac{E}{R}$ . Ohm's law.

**Ex. 6.**  $R_t = R_o(1 + \alpha T)$ . Temperature coefficient.

**Ex. 7.**  $I = \frac{E}{2} \sqrt{\frac{N}{R_i}}$ . To find the greatest current  $I$  from a given number of cells through an external resistance  $R$ .

**Ex. 8.**  $P = EI \cos \phi$ . Power in A.C. circuit.

**Ex. 9.**  $X = R \left( \frac{V}{V_1} - 1 \right)$ . Measurement of high resistance.

**Ex. 10.**  $B = 1317 \sqrt{\frac{P}{A}} + H$ . Traction method of magnetic test.

**Ex. 11.**  $Z = \sqrt{R^2 + \omega^2 L^2}$ . Impedance formula.

In the preceding charts it has been shown that the index may be straight or curved and the axes may be parallel or oblique. Another case arises in which the index joins the extremities of the axes forming a **Z chart** as shown in III, Fig. 159. The scales of axes  $AB$  and  $DG$  are reversed so that the respective zeros are at  $A$  and  $D$ . From the relation of the parts any of the following equations or equivalents may be solved by the  $Z$  chart.  $K = \text{constant} = c + d$ .

$$(39) \quad \frac{a}{d} = \frac{b}{c} = \frac{f}{e} = \frac{c+d}{a+b} = \frac{e-c}{f-b} = \frac{c-d}{b-a},$$

$$(40) \quad a + b = \frac{e}{f} K.$$

When the number of variables exceeds three it is possible to decompose the given equation so that the **composite chart**, i.e., **chart of charts** may be constructed which provides for a solution through successive operations. In such cases there would be a number of indexes which would in turn serve as an axis to establish a reading on the next index.

Ex. 7 may be solved by a composite chart constructed as follows if all the elements are variable. Decompose the equation as follows:

$$(41) \quad I = E\sqrt{\frac{N}{Rr}} = E\sqrt{\frac{N}{i_1}} = Ei_2.$$

Construct seven vertical lines graduated logarithmically and designated  $R, r, I, E, N, i_1, i_2$ . The scales of  $R, r, N, i_1$  will be twice those of  $I, E$  and  $i_2$ . Dispose  $R, r$  and  $i_1$  so that  $i_1$  indicates the product  $Rr$ . Dispose  $i_2$  and  $N$  with regard to  $i_1$  so that  $i_2$  indicates the ratio of  $N$  to  $i_1$ . Dispose  $I$  and  $E$  with regard to  $i_2$  so that  $I$  indicates the product  $Ei_2$ .

## CHAPTER XXI

### MEASUREMENT OF ANGLES

1. Angles are measured usually with a protractor which is graduated in degrees of arc. We read degrees of arc in measuring an angle but express the answer in degrees of angle because an angle is measured by its intercepted arc.

A degree is one-ninetieth of a right angle. The four quadrants contain each 90 degrees of angle and 90 degrees of arc.

A radius vector in making a complete rotation describes a perigon or 360 degrees of angle, as well as 360 degrees of arc.

**Ex. 1.** How many perigons are described in  $1260^\circ$ ,  $1000^\circ$ ,  $810^\circ$ ?

The ratio of the circumference to the radius of a circle is a constant of nature and is abbreviated by  $2\pi$ .

$$2\pi = 2 \times \frac{22}{7} \text{ approx.} = 2 \times 3.14159 \text{ approx.} = 6.28 \text{ approx.}$$

**Ex. 2.** What is the error in choosing  $\frac{44}{7}$ , also  $\frac{25}{4}$  for the value of  $2\pi$ ?

2. If we were to circularize the radius of a circle, i.e., bend it to the form of the circumference then the latter would contain the radius  $2\pi$  times.

The portion of the circumference exactly equaling the length of the radius is called a **radian** or a **radian of arc**. The central angle subtended by a radian of arc is also called a **radian**, i.e., a **radian of angle**.

$360^\circ$  corresponds to  $2\pi$  radians = 6.28 radians.

$$\text{One radian} = \frac{360^\circ}{2\pi} = \frac{360^\circ}{6.28} = 57.3^\circ \text{ approximately.}$$

The number of radians may be expressed in integers or in multiples of  $\pi$ . The former is designated the **unit measure** and the latter the  **$\pi$  measure** of the angle.

In any expression for the measure of an angle,  $\pi$ , 3.14 or  $180^\circ$  are equivalent and therefore may replace one another. When an angle is expressed in unit measure multiply the units by  $57.3^\circ$  and when expressed in  $\pi$  measure replace  $\pi$  by  $180^\circ$ , in order to obtain its degree measure. When the angle is expressed in degrees multiply it by  $\frac{\pi}{180^\circ}$  or  $\frac{3.14}{180^\circ}$  to express it in  $\pi$  or unit measure respectively.

**Ex. 3.** Determine the radian measure of the following angles both for unit and  $\pi$  equivalents:  $1^\circ, 2^\circ, 3^\circ, 4^\circ, 5^\circ, 10^\circ, 15^\circ, 30^\circ, 45^\circ, 60^\circ, 75^\circ, 90^\circ, 105^\circ, 120^\circ, 135^\circ, 150^\circ, 165^\circ, 180^\circ, 270^\circ, 360^\circ$ . Tabulate the three equivalents for each angle.

**Ex. 4.** Determine the degree equivalent for  $\frac{\pi}{5}, \frac{2\pi}{5}, \frac{3\pi}{4}, \frac{7\pi}{4}, \frac{9\pi}{4}, \frac{5\pi}{3}, 0.1\pi, 1.5\pi, 0.75\pi$ .

**Ex. 5.** Determine the degree equivalent for the following angles: 1.5, 2, 1.5708, 4.7124.

**Ex. 6.** Express in unit,  $\pi$  and degree measure the sums indicated with mixed notation below:

- |     |                              |     |                              |
|-----|------------------------------|-----|------------------------------|
| (a) | $\pi + 6,$                   | (h) | $30^\circ + 107^\circ,$      |
| (b) | $\frac{2\pi}{3} + 2,$        | (i) | $30^\circ + \frac{\pi}{4},$  |
| (c) | $\frac{3\pi}{4} + 2.5,$      | (j) | $180^\circ - 0.6,$           |
| (d) | $30^\circ + 2.1,$            | (k) | $90^\circ + 0.2,$            |
| (e) | $45^\circ - \frac{2\pi}{3},$ | (l) | $90^\circ + \tan^{-1}(1),$   |
| (f) | $6.2 - 15^\circ,$            | (m) | $\sin^{-1}(0.5) + 60^\circ,$ |
| (g) | $0.7\pi + 0.7,$              | (n) | $2\pi + 57.3^\circ + 2.718.$ |

Example (a) may be written,

$$\pi + 6 = 3.14 + 6 = 9.14 \text{ (radians),}$$

$$\pi + 6 = 180^\circ + 6 \times 57.3^\circ = 523.8^\circ,$$

$$523.8^\circ \times \frac{\pi}{180^\circ} = 2.91\pi \text{ (radians).}$$

unit equivalent of angles from  $1^\circ$  to  $90^\circ$  is given in Table VIII, p. 121, under the column headed radians.

3. Trigonometric formulas are verified by geometric of and also by establishing identities in the equations through the substitution of recognized elementary trigonometric relations.

Thus (1), (2), (3) may be established by constructing angle  $A$ . Its radius vector, perpendicular and projection are designated by  $RV$ ,  $\perp$  and  $\text{proj}$  respectively.

$$\perp^2 + \text{proj}^2 = \overline{RV}^2 \text{ ————— law of Rt } \triangle.$$

$$\frac{\perp^2}{\overline{RV}^2} + \frac{\text{proj}^2}{\overline{RV}^2} = 1 \text{ ————— div. (a) by } \overline{RV}^2.$$

$$\text{but } \frac{\perp^2}{\overline{RV}^2} = \sin^2 A \text{ and } \frac{\text{proj}^2}{\overline{RV}^2} = \cos^2 A \text{ —def. sin and cos.}$$

$$\therefore \sin^2 A + \cos^2 A = 1 \text{ ————— sub. (c) in (b).}$$

$$\therefore \tan^2 A + 1 = \sec^2 A \text{ ————— div. (1) by } \cos^2 A.$$

$$\therefore 1 + \cot^2 A = \csc^2 A \text{ ————— div. (1) by } \sin^2 A.$$

Other elementary forms are given in (4), (5) and (6);  $\text{vers}$  is the abbreviation for versine and  $\text{covers}$  is the abbreviation for coversine.

$$\frac{\sin A}{\cos A} = \tan A,$$

$$\text{vers } A = 1 - \cos A,$$

$$\text{covers } A = 1 - \sin A.$$

**Ex. 7.** Solve (1) . . . (6) for each function which appears in it.

**Ex. 8.** Verifying the following equation:

$$(a) \quad \frac{\tan \theta + \sec \theta - 1}{\tan \theta - \sec \theta + 1} = \tan \theta + \sec \theta.$$

$$(b) \quad \therefore \tan \theta + \sec \theta - 1 = \tan^2 \theta - \sec^2 \theta + \tan \theta + \sec \theta \text{ ————— clearing frac. in (a).}$$

$$(c) \quad \therefore -1 = \tan^2 \theta - \sec^2 \theta \text{ ————— simplifying (b)}$$

$$(d) \quad \therefore \tan^2 \theta + 1 = \sec^2 \theta \text{ ————— trans. in (c)}$$

Equation (a) is verified by reducing it to the elementary form (2).

**Ex. 9.** By means of (1) . . . (6) either singly or in combination every trigonometric function may be expressed in terms of every other trigonometric function. Express all functions of  $A$  in terms of  $\sin A$ .

**4. Functions of Algebraic Sums of Angles.** When two angles  $A_1$  and  $A_2$  are united by a plus or minus sign then the sin and cos of such a combination is expressible in terms of the sin and cos of the constituent angles as shown in (7), (8), (9) and (10). The left- and right-hand members are called the **contraction** and **expansion** respectively.

$$(7) \quad \sin (A_1 + A_2) = \sin A_1 \cos A_2 + \cos A_1 \sin A_2,$$

$$(8) \quad \sin (A_1 - A_2) = \sin A_1 \cos A_2 - \cos A_1 \sin A_2,$$

$$(9) \quad \cos (A_1 + A_2) = \cos A_1 \cos A_2 - \sin A_1 \sin A_2,$$

$$(10) \quad \cos (A_1 - A_2) = \cos A_1 \cos A_2 + \sin A_1 \sin A_2.$$

**Ex. 10.** Write the expansion for the following:

$$(a) \quad \sin (\theta + \phi), \quad (c) \quad \sin (bx + c),$$

$$(b) \quad \cos (\phi - \alpha), \quad (d) \quad \cos (\omega t + K).$$

**Ex. 11.** Write the contraction for the following:

$$(a) \quad \sin \theta \cos \phi - \cos \theta \sin \phi.$$

$$(b) \quad \cos \theta \cos \phi - \sin \theta \sin \phi.$$

**Functions of Multiple Angles.** When  $A_1 = A_2 = A$  and (9) reduce to (11) and (12) respectively. The express the relation between an angle and its second

$$\sin 2A = 2 \sin A \cos A,$$

$$\cos 2A = \cos^2 A - \sin^2 A.$$

serves to indicate that the multiplier of an angle be factored and written as a multiplier of the function,  $A \neq 2 \sin A$ .

2. Verify (11) by substituting  $A = 60^\circ, 45^\circ, 120^\circ, \pi$ .

**Functions of Half Angles.** By adding (1) and (12) in (13), by subtracting (12) from (1) we obtain (14).

$$\cos A = \sqrt{\frac{1 + \cos 2A}{2}},$$

$$\sin A = \sqrt{\frac{1 - \cos 2A}{2}}.$$

3. Substitute  $\frac{\theta}{2}$  for  $A$  in (13) and (14) and simplify.

**Tangent Functions.** The tangent functions are from the corresponding sin and cos functions by and simplification.

$$\tan (A_1 \pm A_2) = \frac{\tan A_1 \pm \tan A_2}{1 \pm \tan A_1 \tan A_2},$$

$$\tan 2A = \frac{2 \tan A}{1 + \tan^2 A},$$

$$\tan A = \frac{1 - \cos 2A}{\sin 2A} = \frac{\sec 2A - 1}{\tan 2A}.$$

4. Derive (15) from (7) and (9) for  $\tan (A_1 + A_2)$  and (10) for  $\tan (A_1 - A_2)$ .

5. Derive (16) from (11) and (12).

Derive (17) from (13) and (14), also derive (16) from

6. Substitute  $\frac{\theta}{2}$  for  $A$  in (16) and (17) and simplify.

**8. Conversion Formulas.** Another group of useful formulas is given in (18), (19), (20) and (21). This group provides a means for changing an algebraic sum into a product.

$$(18) \quad \sin A_1 + \sin A_2 = 2 \sin \frac{1}{2}(A_1 + A_2) \cos \frac{1}{2}(A_1 - A_2),$$

$$(19) \quad \sin A_1 - \sin A_2 = 2 \cos \frac{1}{2}(A_1 + A_2) \sin \frac{1}{2}(A_1 - A_2),$$

$$(20) \quad \cos A_1 + \cos A_2 = 2 \cos \frac{1}{2}(A_1 + A_2) \cos \frac{1}{2}(A_1 - A_2),$$

$$(21) \quad \cos A_1 - \cos A_2 = -2 \sin \frac{1}{2}(A_1 + A_2) \sin \frac{1}{2}(A_1 - A_2).$$

**Ex. 18.** Multiply (18) by (19) and determine the simplified value of  $\sin^2 A_1 - \sin^2 A_2$ .

**Ex. 19.** Multiply (20) by (21) and determine the simplified value of  $\cos^2 A_1 - \cos^2 A_2$ .

**9. Exponential and Trigonometric Series.** (22) and (23) express the values of the exponentials expanded into infinite series. In (22) every term is positive, whereas in (23) the terms are alternately positive and negative.

$!$  is the **factorial symbol** and means the product of the numbers from 1 to and including the indicated number.

$$(22) \quad e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \frac{z^4}{4!} + \frac{z^5}{5!} + \frac{z^6}{6!} + \dots$$

$$(23) \quad e^{-z} = 1 - z + \frac{z^2}{2!} - \frac{z^3}{3!} + \frac{z^4}{4!} - \frac{z^5}{5!} + \frac{z^6}{6!} - \dots$$

(23) is the reciprocal of (22) and may be obtained by division.  $\sin z$  and  $\cos z$  are also expanded into the series in (24) and (25) respectively.

$$(24) \quad \sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \dots$$

$$(25) \quad \cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \dots$$

It will be observed that (24) and (25) may be built from (22) and (23) by the selection of definite terms.

One-half the difference between (22) and (23) is written as (26) which is called the hyperbolic sine and is abbreviated  $\sinh z$ . One-half the sum of (22) and (23) is written as (27) which is called the hyperbolic cosine and is abbreviated  $\cosh z$ . The ratio of (26) to (27) is written (28) which is called the hyperbolic tangent and is abbreviated  $\tanh z$ . The reciprocals of  $\sinh$ ,  $\cosh$  and  $\tanh$  are  $\operatorname{csch}$ ,  $\operatorname{sech}$  and  $\operatorname{coth}$  respectively.

$$(26) \quad \sinh z = \frac{e^z - e^{-z}}{2} = z + \frac{z^3}{3!} + \frac{z^5}{5!} + \frac{z^7}{7!} + \dots$$

$$(27) \quad \cosh z = \frac{e^z + e^{-z}}{2} = 1 + \frac{z^2}{2!} + \frac{z^4}{4!} + \frac{z^6}{6!} + \dots$$

$$(28) \quad \tanh z = \frac{\sinh z}{\cosh z} = \frac{e^z - e^{-z}}{e^z + e^{-z}}.$$

**Ex. 20.** The following formulas are used for the determination of the electromotive force and current required at the transmission end of a track circuit used for signaling. Simplify the formulas by substituting hyperbolic functions to replace the exponential notation.

$$(a) \quad e = \frac{E_1}{2} \left( e^{x\sqrt{\frac{r}{r_1}}} + e^{-x\sqrt{\frac{r}{r_1}}} \right) + \frac{rr_1}{2\left(\frac{r}{r_1}\right)^{\frac{1}{2}}} \left( e^{x\left(\frac{r}{r_1}\right)^{\frac{1}{2}}} - e^{-x\left(\frac{r}{r_1}\right)^{\frac{1}{2}}} \right).$$

$$(b) \quad i = \frac{E_1}{2\sqrt{rr_1}} \left( e^{x\sqrt{\frac{r}{r_1}}} - e^{-x\sqrt{\frac{r}{r_1}}} \right) + i_1 + \frac{i_1\sqrt{\frac{r}{r_1}}}{2} \left[ \left( e^{x\left(\frac{r}{r_1}\right)^{\frac{1}{2}}} - 1 \right) + \left( e^{-x\left(\frac{r}{r_1}\right)^{\frac{1}{2}}} - 1 \right) \right].$$

Calculate  $e$  and  $i$  for the following values  $r_1 = .9$ ,  $r_2 = 2.5$ ,  $E_1 = 1$ ,  $i_1 = 1$  when  $x = 1, 2, 3, 4, 5 \dots 15$ .

10. The hyperbolic functions are related by the following formulas.

$$(29) \quad \cosh^2 z - \sinh^2 z = 1,$$

$$(30) \quad 1 - \tanh^2 z = \operatorname{sech}^2 z,$$

$$(31) \quad \coth^2 z - 1 = \operatorname{csch}^2 z,$$

$$(32) \quad \sinh (z \pm u) = \sinh z \cosh u \pm \cosh z \sinh u,$$

$$(33) \quad \cosh (z \pm u) = \cosh z \cosh u \pm \sinh z \sinh u,$$

$$(34) \quad \tanh (z \pm u) = \frac{\tanh z \pm \tanh u}{1 \pm \tanh z \tanh u}.$$

The numeric values of hyperbolic functions are obtained from Table XXIII, page 366.

**Ex. 21.** Derive the formulas for  $\sinh 2z$ ,  $\cosh 2z$ ,  $\tanh 2z$ ,  $\sinh \frac{z}{2}$ ,  $\cosh \frac{z}{2}$ .

**Ex. 22.** In (22) substitute  $z=1$  and calculate  $e$  by adding the numeric values of the first twelve terms carried to 8 decimal places.

## CHAPTER XXII

### HARMONIC MOTION

1. A point which moves **uniformly** in a circle projects orthogonally into a point which moves **non-uniformly** along any diameter of the circle. The latter motion is called **simple harmonic motion** and is definite because it obeys a precise law.

In Fig. 160 the arm  $CP$  rotates about the center  $O$ . When the arm rotates with **uniform angular velocity**, then every point, such as  $P$ , in the arm  $CP$  rotates with like uniform angular velocity. Therefore,  $N$  and  $M$  the respective orthogonal projections of  $P$  on the vertical and horizontal diameters move with simple harmonic motion. The displacement of  $N$  from the center  $C$  is  $CN$  and is numerically equal to the length of the arm  $CP$  multiplied by the sine of  $\theta$ , the angle of its rotation measured from the initial position  $CO$ , as expressed in (1).

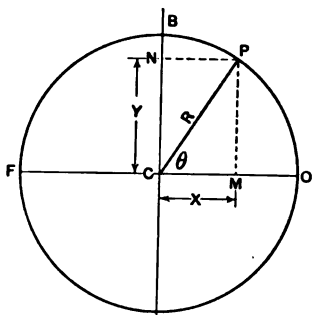


FIG. 160.—The Projection of Uniform Circular Motion on a Diameter.

$$1) \quad CN = CP \sin \theta.$$

(1) may be abbreviated as (2) by designating the vertical displacement by  $y$  and the arm by  $R$ .

$$2) \quad y = R \sin \theta.$$

The greatest value of  $y$  equals  $R$  and corresponds to the displacement of  $N$  when the arm  $CP$  is in a vertical position. Then  $P$  coincides with  $N$  and the angle  $\theta = 90^\circ$ . These facts may be verified by substituting  $\theta = 90^\circ$  in (2) which becomes  $y = R \sin 90^\circ = R$ .

When the arm  $CP$  rotates beyond  $90^\circ$  the displacement decreases so that when  $\theta = 180^\circ$  the value of  $y$  is zero. The rotation of  $CP$  beyond  $180^\circ$  gives a displacement of  $N$  below  $C$  and the corresponding value of  $y$  is regarded with a negative sense. The greatest negative value of  $y$  equals  $-R$  and occurs when the arm  $CP$  has rotated through  $270^\circ$ . When the arm  $CP$  is rotated from  $270^\circ$  to  $360^\circ$  the value of  $y$  decreases and is numerically equal to zero when  $\theta = 360^\circ$ . If the rotation of  $CP$  were continued beyond  $360^\circ$ , then for each additional revolution of  $P$  the simple harmonic motion would be repeated in the same manner and in the same order as described above.

## 2. The Sine Curve or Record of Simple Harmonic Motion.

A continuous record of the simple harmonic motion of the projection of  $P$  is shown in the sine curve of Fig. 161. The whole arc of the circle  $OBFD$  is rectified by extending the horizontal diameter from  $O$  to  $E$ . Then  $OE$  corresponds to the circumference or  $360^\circ$  of arc. The ordinates of the sine curve represent the corresponding displacements of the projection of  $P$  at its successive positions 1, 2, 3, . . . 22, 23, 24. One complete rotation of  $CP$ , i.e., one complete revolution of  $P$ , produces one **cycle** of change extending from  $O$  to  $E$  or from  $0^\circ$  to  $360^\circ$  or from 0 to  $2\pi$  radians. In this way simple harmonic motion is unfolded into a rectilinear representation. This process is comparable to changing the rotation of a crank arm of an eccentric of an engine into the rectilinear motion of the slide of a valve.

**Ex. 1.** Construct the sine curve by projection as follows:

Construct a unit circle with center  $C$ . A unit circle is constructed with a radius of unit length. Draw the vertical diameter  $BD$  and the horizontal diameter  $FO$ . Extend  $FO$  to the right

of the circle so as to represent a horizontal axis. Construct a vertical axis through the origin  $O$ . Locate  $E$  on the horizontal axis so that  $OE$  equals the rectified arc of the circumference. This may be done by calculation making  $OE = 2\pi R$ . Therefore  $OE = 6.2832$  ( $R = 1$ ) units. It may also be laid off approximately by the following construction. Divide the circumference into any convenient number of equal divisions, say 24, i.e., at every  $\frac{\pi}{12}$  radian or  $15^\circ$  of arc. Beginning at 0 number these divisions consecutively from 0 to 24. Lay off 24 corresponding consecutive divisions to the right of 0, each being equal in length to the chords subtended by the divisions on the circle. Number these consecutively from 0 to 24. See Fig. (161).

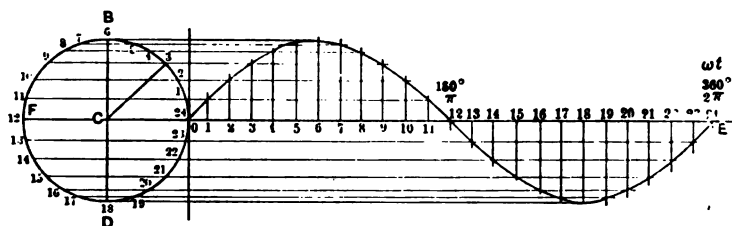


FIG. 161.—The Sine Curve or Record of Simple Harmonic Motion.

Through the points of division on the horizontal axis construct vertical lines and through the arc divisions of the circle draw lines parallel to the horizontal axis. These sets of lines intersect. Like numbered lines determine points of intersection through which we pass a smooth curve which is sinusoidal. This sine curve portrays harmonic motion continuously. The complete cycle is represented in the interval  $OE$ .

The curve is identical with the graph plotted from Ex. 8, Chapter XIV.

**3. Plotting the Fundamental Sine Curve.** The sine curve may be plotted from (2) which reduces to (3) when  $R = 1$ .

$$(3) \quad y = \sin \theta.$$

(3) is the equation of the **fundamental sine curve** to which all other sine curves are referred for comparison; (3) is plotted from Table XXIV, which has been prepared

by assigning values to  $\theta$  varying in gradations of  $30^\circ$  from  $0^\circ$  to  $360^\circ$ . The sines of multiples of  $30^\circ$  are the familiar values 0, .5, .866, and 1, and should be selected so as to minimize the labor of computation although the sines are easily read from a sine table or a slide rule.

TABLE XXIV. THE VARIATION OF THE ORDINATE AND THE ANGLE OF THE SINE CURVE

$y$		0	.5	.866	1	.866	.5	0	-.5	-.866	-1	-.866	-.5	0
$\theta = \omega t$	degrees	0	30	60	90	120	150	180	210	240	270	300	330	360
	$\pi$ -radians	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	$\pi$	$\frac{7\pi}{6}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{11\pi}{6}$	$2\pi$
	unit-radians	0	.52	1.05	1.57	2.09	3.13	3.14	3.66	4.18	4.71	5.23	5.75	6.28

The values of  $y$  and  $\theta$  are then plotted.  $y$  ranges from  $+1$  to  $-1$  and  $\theta$  ranges from  $0^\circ$  to  $360^\circ$ , i.e., from 0 to  $2\pi$  or 6.28 radians. The fundamental sine curve is shown as curve  $C$  in Fig. 163 and as  $y_1$  in Fig. 165. In both figures the vertical and horizontal scales are equal. The horizontal scale may be graduated in degrees,  $\pi$  or unit measure corresponding to the equivalent values of  $\theta$  in rows 2, 3, 4 of Table XXIV.

*Observation.* The sine curve is completed in  $360^\circ$  or  $2\pi$  or 6.28 radians and consists of two like loops or arches or arcs which are alternately placed on the two sides of the horizontal axis. The two loops together constitute a cycle and correspond to a cyclic rotation of an arm which moves through  $360^\circ$  of angle. In other words,  $\theta$  has changed from 0 to  $360^\circ$ . while  $y$  has increased from zero to  $+1$ , decreased from  $+1$  to zero and from zero to  $-1$  and increased from  $-1$  to zero. The maximum and minimum points of the sine curve occur at one-quarter and three-quarters of the cycle. The zero points occur at the beginning, at the half cycle, and at the end of the cycle.

**4. Angular Velocity, Period, Frequency.** The angular velocity ( $\omega$ ) of a rotating or revolving body is numerically

ual to the angle or arc ( $\theta$ ) swept through in a unit of time as expressed in (4), (5), and (6).

$$\begin{aligned} ) \quad & \omega = \frac{\theta}{t}, \\ ) \quad & \theta = \omega t, \\ ) \quad & t = \frac{\theta}{\omega}. \end{aligned}$$

If the moving arm  $OP$ , Fig. 160, rotates through  $360^\circ$  or  $2\pi$  radians in a unit of time, i.e., it makes one revolution per second, then  $\omega = 2\pi$ . If the arm makes 25 revolutions per second then  $\omega = 25 \times 2\pi = 50\pi = 50 \times 3.14 = 157.08$ . If the arm makes 60 revolutions per second, then  $\omega = 376.79$ . If the arm makes  $\frac{1}{2\pi}$  of one revolution per second then  $\omega = 1$ .

From (5) we can substitute in (3) and obtain (7).

$$) \quad y = \sin \omega t.$$

The interpretation of (7) states that the ordinate of a sine curve is the sine of the product  $\omega t$ . Therefore it depends upon the angular velocity of the rotating arm and the time in seconds which has elapsed since the arm, Fig. 160, occupied its initial position  $CO$ . In (7)  $\omega$  is constant and  $t$  is variable, therefore in plotting (7) the horizontal axis will be graduated in units of time.

For one complete cycle  $\theta$  attains the value  $2\pi$  and therefore one cycle  $\omega t$  ranges from 0 to  $2\pi$ . By substituting for  $\theta$  in (4) and (6) we obtain (8) and (9) respectively:

$$\begin{aligned} \omega &= \frac{2\pi}{T}, \\ T &= \frac{2\pi}{\omega}, \end{aligned}$$

The interpretation of (8) states that the angular velocity of the arm can be computed by dividing  $2\pi$  by the time required for a complete rotation of the arm. The numeric

value of  $\omega$  is the coefficient of  $t$  when the sine equation is given in the form (7).

The interpretation of (9) states that the time  $T$  of a complete rotation of the arm is computed by dividing  $2\pi$  by  $\omega$ , i.e., by dividing  $2\pi$  by the coefficient of  $t$  when the sine equation is given in the form (7). The time  $T$  for a complete rotation of the arm is a special value of  $t$  corresponding to the time which elapses for a complete cycle of the sine curve and is called the **period** of the curve. The horizontal scale for the sine curve may be expressed as a measure of time. The numeric value of the period of the curve is placed at the  $2\pi$  or  $360^\circ$  mark, i.e., at the end of the cycle and proportionate values of  $t$  are placed at the intermediate points of division.

The number of cycles per unit of time is called the **frequency** ( $f$ ) of the curve and is numerically the reciprocal of the period of the curve as expressed in (10).

$$(10) \quad f = \frac{1}{T} = \frac{\omega}{2\pi}.$$

The interpretation of (10) states that the frequency of a curve is numerically the same as the number of rotations per second of the moving arm, i.e., the ratio of the angular velocity  $\omega$  to  $2\pi$ .

**Ex. 2.** Make entries in the blank spaces in Table XXV.

**5. Comparison of Sine Curves of Different Frequency.** From Table XXV we observe that (16)  $y = \sin 2\pi t$  has a period of one second and a frequency of one cycle per second. If we plot this curve its cyclic length will be one unit, which is the numeric value of its period. Any convenient distance on the horizontal scale may be taken as a unit of time. In like manner (12) will have a cyclic length equal to 6.28 units, which is the numeric value of its period. (11) will have a cyclic length equal  $4\pi = 12.56$  units.

TABLE XXV. ANGULAR VELOCITY, PERIOD AND FREQUENCY

	Equation.	Angle $\omega t$ .	Angular Velocity $\omega$ .	Period in sec $\frac{2\pi}{\omega} = T$	Frequency per sec. $\frac{\omega}{2\pi} = f$
(11)	$y = \sin .5t$	$.5t$	0.5	$4\pi$	
(12)	$y = \sin t$	$t$	1.0	$2\pi$	
(13)	$y = \sin 2t$	$2t$	2.0	$\pi$	
(14)	$y = \sin 3t$	$3t$	3.0	$\frac{2\pi}{3}$	
(15)	$y = \sin \pi t$	$\pi t$	.....	.....	.5
(16)	$y = \sin 2\pi t$	$2\pi t$	.....	1	60
(17)	$y = \sin 50t$	.....	.....	.....	
(18)	.....	$120t$	.....	.....	
(19)	.....	.....	.....	.....	
(20)	.....	.....	.....	.04	
(21)	.....	.....	$100\pi$	.....	

Fig. 162 illustrates the graphs of (11), (13), and (14). (13) has four times the frequency of (11) and therefore

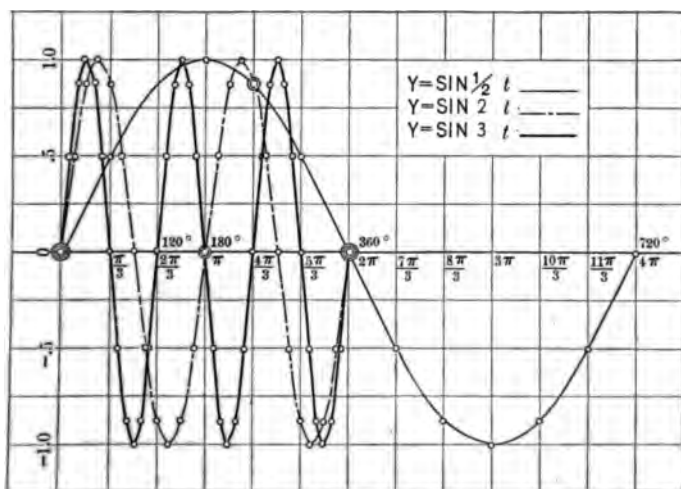


FIG. 162.—Sine Curves of Different Frequency.

there will be two completed cycles of (13) for a half cycle of (11). (14) has six times the frequency of (11) and therefore there will be three completed cycles of (14) for a half cycle of (11).  $\omega$  is called the **frequency factor**.

*Observation.* The frequency of a sine curve varies directly with  $\omega$ , i.e., as the coefficient of  $t$  in the angle  $\omega t$ , whereas the period varies inversely as  $\omega$ .

**Ex. 3.** On one sheet of cross-section paper plot the following groups of curves: (a)–(11), (12), (13); (b)–(11), (12), (14); (c)–(12), (13), (15); (d)–(12), (14), (15); (e)–(12), (15), (16); (f)–(14), (15), (16); (g)–(17), (18), (19); (h)–(18), (19), (20); (i)–(19), (20), (21).

**6. The Amplitude of a Sine Curve.** The graphs in Fig. 162 have an equal amplitude, i.e., an equal vertical displacement from the horizontal axis. Equations (11) to (21) inclusive are special forms of (7) in which special values have been assigned to  $\omega$ . (7) is a special form of (2) in which  $R=1$ .  $R$  is the radius of the generating circle, i.e., the length of the rotating arm. Suppose  $R \neq 1$ , then Eqs. (11) to (21) will be modified by the introduction of a multiplier  $R$  preceding  $\sin \omega t$ . The value of  $R$  determines the maximum height and minimum depth of the sine curve. Therefore,  $R$  is the factor which determines the amplitude of a curve and  $R$  is called the **amplitude factor**.

Fig. 163 represents the plotted graphs of (22), (23), (24), and (25).

$$(22) \quad y = \sin \theta,$$

$$(23) \quad y = \frac{1}{2} \sin \theta,$$

$$(24) \quad y = 3 \sin 5t,$$

$$(25) \quad y = \sin \left( \frac{\pi}{2} - \frac{\pi t}{4} \right) = \cos \frac{\pi t}{4}.$$

as  $C$  and  $D$  respectively.

The period of both (22) and (23) is 6.28, but their amplitudes are 1 and .5 respectively. Their plotted points are given in Table XXVI, in which it will be observed that for like values of  $\theta$  the corresponding ordinates are in the ratio of 2 : 1.  $C$  may be identified with (7) and therefore with (12). The horizontal scale of Fig. 163 may be interpreted as radians or time, since  $C$  has a period of  $2\pi$ .  $D$  has the same period as (12) and (22) but only

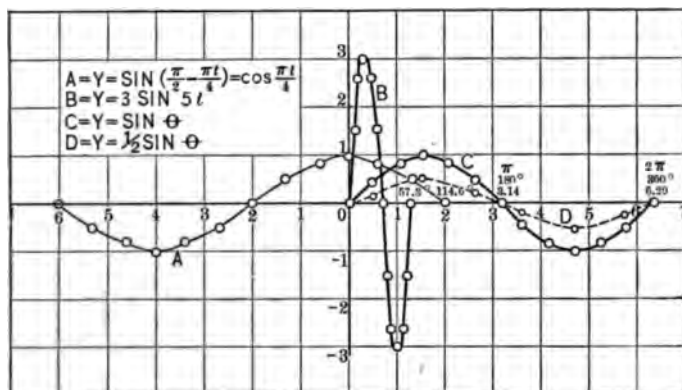


FIG. 163.—Sine Curves of Different Amplitude.

one-half the corresponding amplitude. The period of  $B$  is  $\frac{2\pi}{5} = 1.26$  but its amplitude is 3, which is three times the amplitude of  $C$  and six times the amplitude of  $D$ . Eq.  $A$  may be expressed as a sine curve in which the angle is  $\left(\frac{\pi}{2} - \frac{\pi}{4}\right)$  or as a cosine curve in which the angle  $\frac{\pi t}{4}$  is the complement to  $\left(\frac{\pi}{2} - \frac{\pi t}{4}\right)$ . The cosine curve does not intersect the  $Y$  axis at the origin, but the beginning of its cycle is located to the left of the  $Y$  axis. The maximum value

TABLE XXVI. CALCULATIONS FOR PLOTTING SINE CURVES OF DIFFERENT AMPLITUDES

C			D			B			A			
$\theta$	$v$	$\theta$	$v$	$5u$	$t$	$v$	$\frac{\pi}{2} - \frac{\pi t}{4}$	$\frac{\pi t}{4}$	$t$	$v$		
0	0	0	0	0	0	0	$2\pi$	$-\frac{3\pi}{2}$	-6	0		
$\frac{\pi}{6}$	.5	$\frac{\pi}{6}$	.25	$\frac{\pi}{6}$	.105	1.5	$\frac{11\pi}{6}$	$-\frac{4\pi}{3}$	-5.33	-.5		
$\frac{\pi}{3}$	.866	$\frac{\pi}{3}$	.433	$\frac{\pi}{3}$	.209	2.598	$\frac{5\pi}{3}$	$-\frac{7\pi}{6}$	-4.66	-.866		
$\frac{\pi}{2}$	1	$\frac{\pi}{2}$	.5	$\frac{\pi}{2}$	.314	3.0	$\frac{3\pi}{2}$	$-\pi$	-4	-1		
$\frac{2\pi}{3}$	.866	$\frac{2\pi}{3}$	.433	$\frac{2\pi}{3}$	.419	2.598	$\frac{4\pi}{3}$	$-\frac{5\pi}{6}$	-3.33	-.866		
$\frac{5\pi}{6}$	.5	$\frac{5\pi}{6}$	.25	$\frac{5\pi}{6}$	.523	1.5	$\frac{7\pi}{6}$	$-\frac{2\pi}{3}$	-2.66	-.5		
$180^\circ = \pi = 3.14$	0	$\pi$	0	$\pi$	.628	0	$\pi$	$-\frac{\pi}{2}$	-2	0		
$\frac{7\pi}{6}$	-.5	$\frac{7\pi}{6}$	-.25	$\frac{7\pi}{6}$	.733	-1.5	$\frac{5\pi}{6}$	$-\frac{\pi}{3}$	-1.33	.5		
$\frac{4\pi}{3}$	-.866	$\frac{4\pi}{3}$	-.433	$\frac{4\pi}{3}$	.838	-2.598	$\frac{2\pi}{3}$	$-\frac{\pi}{6}$	-.66	.866		
$\frac{3\pi}{2}$	-1	$\frac{3\pi}{2}$	-.5	$\frac{3\pi}{2}$	.942	-3.0	$\frac{\pi}{2}$	0	0	1		
$\frac{5\pi}{3}$	-.866	$\frac{5\pi}{3}$	-.433	$\frac{5\pi}{3}$	1.046	-2.598	$\frac{\pi}{3}$	$\frac{\pi}{6}$	+66	.866		
$\frac{11\pi}{6}$	-.5	$\frac{11\pi}{6}$	-.25	$\frac{11\pi}{6}$	1.151	-1.5	$\frac{\pi}{6}$	$\frac{\pi}{3}$	1.33	.5		
$360^\circ = 2\pi = 6.28$	0	$2\pi$	0	$2\pi$	1.256	0	0	$\frac{\pi}{2}$	2	0		

of the cosine curve  $D$  is numerically the same as its intercept on the  $Y$  axis.

The whole angle, which is designated as  $\theta$ ,  $5t$ ,  $\left(\frac{\pi}{2} - \frac{\pi t}{4}\right)$  in  $C$ ,  $B$ , and  $A$  respectively, must range through  $2\pi$  radians, whereas the range of values of  $t$  depend upon  $\omega$ , i.e., the coefficient of  $t$  in the whole angle. At the completion of the cycle of  $B$ ,  $5t = 2\pi$  and therefore  $t = \frac{2\pi}{5} = 1.256$  sec. At the completion of the half cycle,  $5t = \pi$  and therefore  $t = \frac{\pi}{5} = .628$ . In the same manner for  $A$  when  $\left(\frac{\pi}{2} - \frac{\pi t}{4}\right) = 0$ , then by transposition and dividing we obtain  $\frac{\pi t}{4} = \frac{\pi}{2}$  and  $t = +2$ . This statement means that the cosine curve begins its positive loop at 2 units to the left of the  $Y$  axis, i.e., two seconds earlier than curves  $B$ ,  $C$ , and  $D$ . The values of  $t$  for  $B$  and  $A$  in Table XXVI are determined from the corresponding values of  $5t$  and  $\left(\frac{\pi}{2} - \frac{\pi t}{4}\right)$  respectively.

*Observation.* The cosine curve is a sine curve which has been displaced along the horizontal axis an amount equal to one-quarter of a cycle.

**Ex. 4.** Make entries in the blank spaces in Table XXVII.

**Ex. 5.** Fig. (164) illustrates the graphs of (34) (35) and (36). The vertical scale is common to the three graphs. The horizontal scale of (36) is indicated in  $\pi$  measure. Determine the horizontal scales for (34) and (35).

*Observation.* The horizontal and vertical scales of sine curves may be made unequal unless the true shape of the curve is desired. Sine curves of different frequency may be plotted with different horizontal scales if they are to be considered independently.

TABLE XXVII. AMPLITUDE AND FREQUENCY

No.	Equation.	Period.	Frequency.	Amplitude.
(26)	$y = \sin 2\pi t$	1	1	1
(27)	$y = 2 \sin \frac{\pi t}{4}$	8	.....	2
(28)	$y = 1.5 \sin 50\pi t$	.....	25	1.5
(29)	$y = 140 \sin .2t$	.....	.....	1050
(30)	$y = 1050 \sin 4t$	.....	.....	1
(31)	$y = -\sin 2\pi t$	.....	.....	.....
(32)	$y = 7 \sin -\frac{\pi t}{4}$	.....	.....	.....
(33)	$y = -\sin -\frac{\pi t}{4}$	.....	.....	.....
(34)	$y = 2 \cos .7854t$	.....	.....	.....
(35)	$y = .6 \sin \frac{\pi t}{8}$	.....	.....	.....
(36)	$y = 2 \sin 1.5t$	.....	.....	.....
(37)	.....	.....	60	110
(38)	.....	.....	100	10
(39)	.....	.....	15	500
(40)	.....	.0005	.....	50

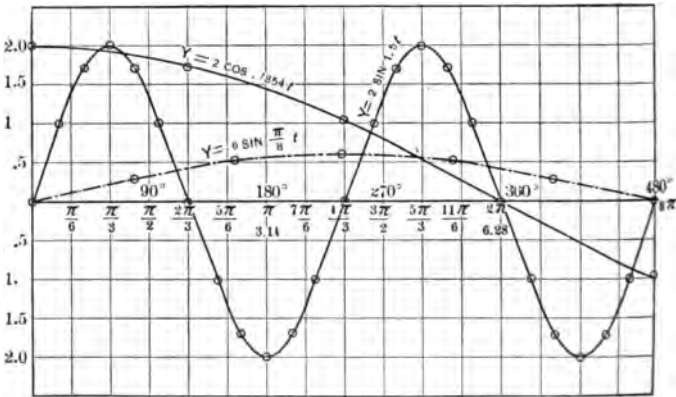


FIG. 164.—Sine Curves of Different Amplitudes.

*The scale of a sine curve is determined from its period. The cyclic length, i.e., period, is most conveniently divided into the four one-quarter cycle divisions and these are subdivided into thirds. The ordinates at the thirteen points including the initial and the terminal points are respectively:*

0, .5, .866, 1, .866, .5, 0, -.5, -.866, -1, -.866, -.5, 0.

*When the curve has an amplitude differing from 1 the above ordinates are multiplied by the amplitude.*

**Ex. 6.** Plot the following groups of curves, using the same horizontal and vertical scales: (a)-(26), (27), (28); (b)-(29), (37); (c)-(31), (33); (d)-(27), (34), (36); (e)-(34), (35), (36); (f)-(30), (39); (g)-(38), (40); (h)-(32), (38).

**7.** In practice it is customary to operate electrical machinery and regulate transmission circuits with alternating currents of standard frequency. This means that the alternating stresses in the circuit or net work are representable by sine curves of equal frequency in which  $y$  is replaced by  $e$  or  $i$  the instantaneous values of the E.M.F. and current respectively.

There is another factor, however, which is of considerable importance in commercial operation. Two sine curves may have equal frequencies yet when plotted they will be relatively displaced on the horizontal direction. When corresponding zero and maximum points are not coincident but are separated by a constant interval then the displaced curves are said to be out of phase. The constant interval of separation is measured in time or when measured in radians, or degrees it is called the phase angle. Fig. 165 represents two arms  $CP$  and  $CQ$  which rotate with like angular velocity and are constantly separated by the angle  $\alpha$ . The projections of their respective extremities  $P$  and  $Q$  give harmonic motions which are represented by the respective sine curves  $y_1$  and  $y_2$  which are formulated

in (41) and (42).  $CP$  and  $CQ$  are designated by  $R_1$  and  $R_2$  respectively.

$$(41) \quad y_1 = R_1 \sin \theta,$$

$$(42) \quad y_2 = R_2 \sin \phi.$$

When  $CP$  is in the initial position  $CO$ , its projection is zero, but at that instant  $CQ$  is in advance of its initial position by the angle  $\alpha$ . Therefore the sine curve  $y_2$  is advanced in its development before the beginning of  $y_1$ , i.e., when the ordinate of  $y_1$  equals zero the ordinate of  $y_2$  equals  $R_2 \sin \alpha$ . Maximum, minimum, and zero

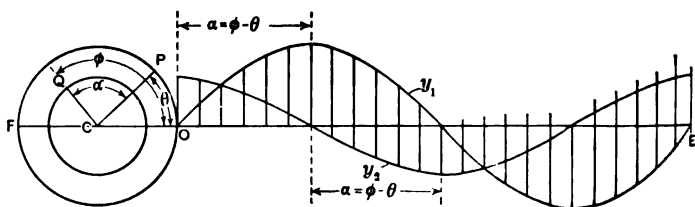


FIG. 165.—Sine Curves Out of Phase.

points are **displaced** by a constant distance representing the scaled value of  $\alpha$ . The angles  $\alpha$ ,  $\phi$ , and  $\theta$  are related as expressed in (43a) and (43b). Therefore (41) and (42) are rewritten as (44) and (45) respectively by substituting from (43a) and (43b).

$$(43a) \quad \phi = \theta + \alpha,$$

$$(43b) \quad \theta = \phi - \alpha,$$

$$(44) \quad y_1 = R_1 \sin (\phi - \alpha),$$

$$(45) \quad y_2 = R_2 \sin (\theta + \alpha).$$

Since the unequal arms  $CP$  and  $CQ$  rotate with like angular velocity, both corresponding sine curves will have

like frequency and therefore they are represented in Fig. 165 by  $y_1$  and  $y_2$ , which have an equal period but unequal amplitudes.

Comparing Eqs. (41) and (42) we observe that the coefficients of  $\theta$  and  $\phi$  are unity for both of these sine curves, which indicates an equal frequency and an equal period for  $y_1$  and  $y_2$ . The displacement of curve  $y_2$  in advance of  $y_1$  is indicated by the **added angle**  $\alpha$  in the total angle  $\theta + \alpha$  of (45) and therefore  $y_2$  is said to be out of phase with  $y_1$  or  $y_2$  leads  $y_1$  by the angle  $\alpha$ .

Comparing Eqs. (44) and (45) we observe that the coefficients of  $\phi$  and  $\theta$  are unity for both of the sine curves, which indicates an equal frequency and an equal period for  $y_2$  and  $y_1$ . The displacement of curve  $y_1$  in the rear of  $y_2$  is indicated by the **subtracted angle**  $\alpha$  in the total angle  $\phi - \alpha$  of (44) and therefore  $y_1$  is said to be out of phase with  $y_2$  or  $y_1$  lags behind  $y_2$  by the angle  $\alpha$ .

In the comparison of the graphs (41) and (45) we observe that when  $y_1 = 0$ , then  $\sin \theta = 0$ , since  $R_1$  is constant, but when  $\sin \theta = 0$ , then  $\theta = 0, \pi, 2\pi \dots n\pi$  where  $n$  is an integer. Accordingly (44) reduces to the following simultaneous values:

TABLE XXVIII. SIMULTANEOUS VALUES OF  $y_1$  AND  $y_2$  IN (41) AND (44)

$y_1$	$\theta$	$y_2$
0	0	$R_2 \sin (+\alpha) = +R_2 \sin \alpha$
0	$\pi$	$R_2 \sin (\pi + \alpha) = -R_2 \sin \alpha$
0	$2\pi$	$R_2 \sin (2\pi + \alpha) = +R_2 \sin \alpha$

The displacement between  $y_1$  and  $y_2$  is constantly equal to  $\alpha$  as shown by the above tabulated values. If a comparison be made between the sine curves  $y_1$  and  $y_2$ , regarding them as graphs of (42) and (44), we observe a like displacement for every set of pairs of simultaneous points.

The axial displacement of two sine curves of period  $T$  may be estimated as a decimal part of the period of either curve.

If  $\alpha$  represents the angle of lead or lag, then  $\frac{\alpha}{2\pi}$  represents the displacement as a decimal part of the length of the curve and  $\frac{\alpha}{2\pi}T$  is the unit measure of the displacement which reduces to  $\frac{\alpha}{\omega}$  as shown in (46).

$$(46) \quad \frac{\alpha}{2\pi}T = \frac{\alpha}{2\pi} \frac{2\pi}{\omega} = \frac{\alpha}{\omega} = \text{unit measure of displacement.}$$

The interpretation of (46) states that the unit displacement of two sine curves which are out of phase may be obtained by dividing the angle of lead or lag by the angular velocity  $\omega$ .

$$(47a) \quad y_1 = 115 \sin 50t,$$

$$(47b) \quad y_2 = 115 \sin (50t + 30^\circ) = 115 \sin \left( 50t + \frac{\pi}{6} \right).$$

In the plotting of (47a) and (47b) the amplitudes = 115, the periods  $\frac{2\pi}{50} = .1257$  units, and the displacement  $= \frac{\pi}{300} = \frac{.1257}{12} = .0105$  units.

**Ex. 7.** Group the following equations for the purpose of comparing their leads and lags and then tabulate their amplitudes, phase angle, frequency, period, and displacement. In each group assume one of the  $e$  curves as a standard for comparison.

$$(48) \quad e_1 = 115 \sin (120\pi t + 30^\circ),$$

$$(49) \quad i_1 = 50 \sin \left( 120\pi t + \frac{\pi}{12} \right),$$

$$(50) \quad i_2 = 25 \sin (120\pi t - .25),$$

$$(51) \quad e_2 = 120 \sin (120\pi t + .15),$$

$$(52) \quad e_3 = 118 \sin 120\pi t,$$

$$(53) \quad e_4 = 115 \sin (50\pi t + 30^\circ),$$

$$(54) \quad i_3 = 25 \sin \left( 50\pi t - \frac{\pi}{12} \right),$$

$$(55) \quad i_4 = 50 \sin (50\pi t + .25),$$

$$(56) \quad e_5 = 120 \sin (120\pi t + 45^\circ),$$

$$(57) \quad e_6 = 120 \sin (50\pi t + 45^\circ),$$

$$(58) \quad e_7 = 220 \sin (50\pi t - .15),$$

$$(59) \quad i_5 = 118 \sin 50\pi t.$$

**Ex. 8.** Plot the following groups of equations on the same cross-section paper: (a)-(52), (48); (b)-(52), (51); (c)-(52), (50); (d)-(52), (49); (e)-(52), (56); (f)-(48), (56); (g)-(56), (51); (h)-(56), (49); (i)-(56), (50); (j)-(59), (53); (k)-(59), (54); (l)-(59), (55); (m)-(59), (57); (n)-(59), (58); (o)-(58), (54); (p)-(58), (53); (q)-(58), (55); (r)-(58), (57); (s)-(48), (53); (t)-(49), (54); (u)-(50), (55); (v)-(51), (58); (w)-(48), (56); (x)-(56), (57); (y)-(48), (57); (z)-(48), (59).

### 8. The Resultant of Two Simple Harmonic Motions.

Two simple harmonic motions may be united by addition into a simple harmonic motion which is called their **resultant**. The resultant will have a period equal to that of its components. Fig. 166 represents two radii  $R_1$  and  $R_2$  which move uniformly with equal periods about the center  $C$ . Their respective harmonic formulas are expressed in (60) and (61).

$$(60) \quad y_1 = R_1 \sin (\omega t + b_1),$$

$$(61) \quad y_2 = R_2 \sin (\omega t + b_2).$$

The two distinct motions of  $S$  and  $K$  the respective extremities of  $R_1$  and  $R_2$ , contribute simple harmonic motions along the vertical diameter. These displacements are denoted in the usual manner by  $y_1$  and  $y_2$  respectively. The addition of  $y_1$  and  $y_2$  equals  $y$  the displacement of the

projection of a point  $E$  at the extremity of the arm  $B$ , and therefore (62), (63), and (64) follow:

$$(62) \quad y = y_1 + y_2,$$

$$(63) \quad y = R_1 \sin (\omega t + b_1) + R_2 \sin (\omega t + b_2),$$

$$(64) \quad y = R \sin (\omega t + b).$$

Each harmonic curve is representable by a sine curve and therefore the sum of the two simultaneous ordinates of curves  $y_1$  and  $y_2$  give the corresponding ordinate of  $y$ .

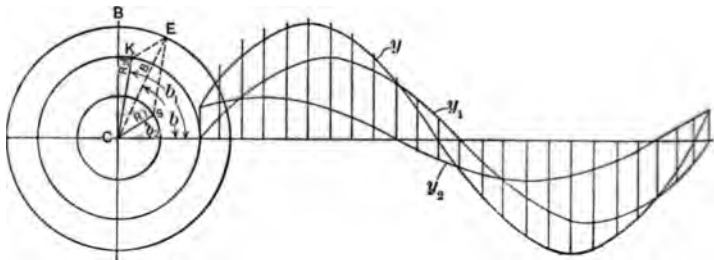


FIG. 166.—The Resultant of Two Simple Harmonic Motions.

*Observation.* Two or more sine curves of equal period are replaced by a single sine wave of like period. The resultant curve has its ordinates everywhere equal to the algebraic sum of the ordinates at simultaneous points on the component curves.

9. The resultant  $B$  in Fig. 166 is the diagonal of a parallelogram whose adjacent sides are numerically equal and parallel to  $R_1$  and  $R_2$ . By the law of the oblique triangle we write (65):

$$(65) \quad R = R_1^2 + R_2^2 + 2R_1R_2 \cos CSE,$$

but

$$(66) \quad CSE = \pi - (b_1 - b_2),$$

$$(67) \quad \therefore R = R_1^2 + R_2^2 + 2R_1R_2 \cos (\pi - b_1 + b_2),$$

and

$$(68) \quad R = R_1^2 + R_2^2 + 2R_1R_2 \cos (b_1 - b_2).$$

The interpretation of (68) gives the working formula for computing the amplitude of the resultant in terms of the amplitudes of the components and their phase difference.

The sum of the vertical projections of  $R_1$  and  $R_2$  equals the vertical projection of  $R$  and the sum of the horizontal projections of  $R_1$  and  $R_2$  equals the horizontal projection of  $R$ . The ratio of these sums of projections equals the tangent of the phase of the resultant as expressed in (69).

$$(69) \quad \tan b = \frac{R_1 \sin b_1 + R_2 \sin b_2}{R_1 \cos b_1 + R_2 \cos b_2}.$$

**Ex. 9.** Add the ordinates of the sine curves corresponding to the following groups of equations: (a)-(48), (51); (b)-(48), (52); (c)-(48), (56); (d)-(51), (52); (e)-(51), (56); (f)-(52), (56); (g)-(49), (50); (h)-(50), (51); (i)-(53), (57); (j)-(53), (58); (k)-(57), (58); (l)-(54), (55); (m)-(54), (59); (n)-(55), (59).

**10.** Periodic motions of unequal period may be united to produce a resultant periodic motion, but the latter does not follow the law of simple harmonic motion.

If the component harmonic motions are plotted as sine curves then the sum of any two simultaneous ordinates of the components is numerically equal to the simultaneous ordinate of the resultant curve.

If two component curves are represented by (70) and (71) then their resultant is expressed by (72) and (73).

$$(70) \quad y_1 = R_1 \sin (\theta + b_1),$$

$$(71) \quad y_2 = R_2 \sin (m\theta + b_2),$$

$$(72) \quad y = y_1 + y_2,$$

$$(73) \quad y = R_1 \sin (\theta + b_1) + R_2 \sin (m\theta + b_2),$$

Fig. 167 shows the component curves  $y_1$  and  $y_2$  and their resultant  $y_3$ .

**Observation.** *The period of the resultant of two or more sine curves is numerically equal to the period of the curve of least frequency.*

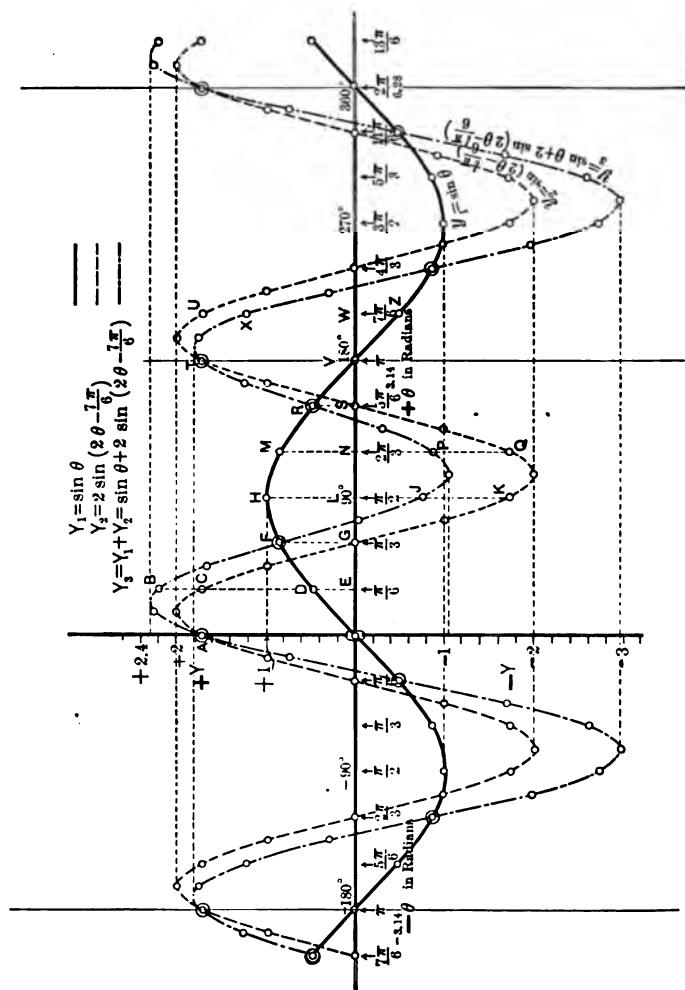


FIG. 107.—The Resultant of Two Harmonic Curves whose Amplitudes and Frequencies are in the Ratio 2 : 1.

11. Fig. 168 shows the effect of uniting two sine curves whose relative amplitudes and periods are both in the ratio of 2 : 1. The effect of the displacement of the (b) curve is shown in the formation of the hump whose shape is modified by further change in the relative displacement or relative amplitudes.

**Ex. 10.** Plot the component curves as well as the resultant for the following equations:

$$(74) \quad y = 2 \sin t + \frac{1}{2} \sin 3t,$$

$$(75) \quad y = 2 \sin (t + 1.0123) + \frac{1}{2} \sin 3t$$

$$(76) \quad y = \sin x + \frac{1}{2} \sin 3x + \frac{1}{2} \sin 5x + \frac{1}{2} \sin 7x,$$

$$(77) \quad y = 100 \sin x + 50 \sin (3x - 40^\circ),$$

$$(78) \quad y = 150 + 100 \sin x + 50 \sin (3x - 40^\circ),$$

$$(79) \quad y = 100 \sin 2\pi t + 60 \sin(6\pi t - 1.571) + 10 \sin(10\pi t - 3.14),$$

$$(80) \quad y = \frac{1}{2} - \frac{1}{2} \sin 2x,$$

$$(81) \quad y = 100 + 100 \sin x + 50 \sin (3x - 75^\circ),$$

What is the significance of the constant term in (78), (80), and (81)?

$$(82) \quad i = 140 \cos .2t - 80 \sin .2t,$$

$$(83) \quad i = 140 \cos (.2t + 15^\circ) - 80 \sin .2t,$$

$$(84) \quad i = 14 \cos (.2t - 15^\circ) - 8 \sin .2t,$$

$$(85) \quad e = 200 \cos .2t + 350 \sin .2t,$$

$$(86) \quad e = 200 \cos (.2t + 15^\circ) + 350 \sin .2t,$$

$$(87) \quad e = 200 \cos (.2t - 15^\circ) + 350 \sin .2t,$$

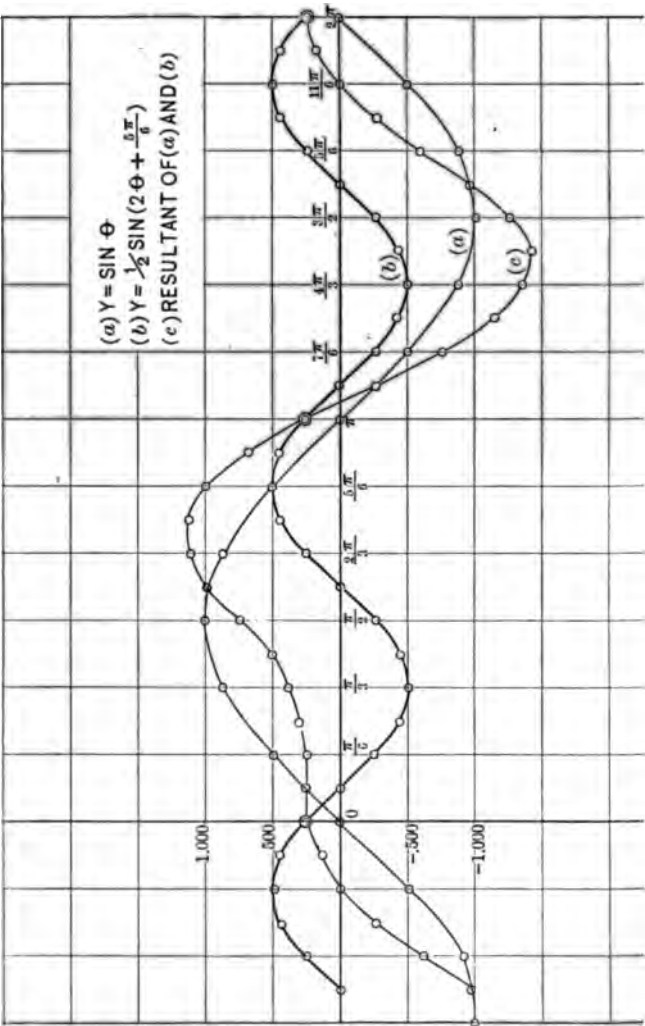
$$(88) \quad e = 200 \cos .2t + 350 \sin (.2t + 15^\circ),$$

$$(89) \quad e = 200 \cos .2t + 350 \sin (.2t - 15^\circ),$$

$$(90) \quad y = \sin .2t + \sin .6t,$$

$$(91) \quad y = \sin .2t - \sin .6t,$$

$$(92) \quad y = \cos 120\pi t + \cos 360\pi t,$$



$$y = \cos 120\pi t - \cos 360\pi t,$$

$$y = 5 \sin .2t + 10 \sin .6t,$$

$$y = 5 \cos 120\pi t + 15 \cos 360\pi t,$$

$$y = 1.5 \sin .2t - 2 \sin .6t,$$

$$y = 25 \sin 50\pi t - 10 \sin 150\pi t,$$

$$y = 25 \sin 50\pi t + 25 \sin 150\pi t,$$

$$y = 25 \cos 50\pi t - 10 \cos 150\pi t.$$

**The Product of Two Sine Curves.** The product of ordinates of two sine curves equals the ordinate of a third or product curve which is also periodic. In all applications the two constituent curves are of the same frequency and their product is a curve of twice

(1) CURRENT IN PHASE WITH E.M.F.

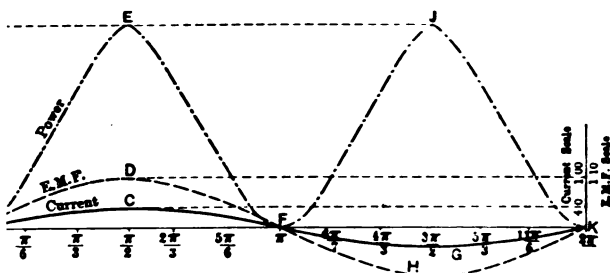


FIG. 169.—Power Curve.

frequency. If one of the constituent curves represents a harmonically varying E.M.F. and the other constituent represents a harmonically varying current then their product is a harmonically varying power curve. In Fig. 169 curves *D* and *C* are sine curves representing the voltage E.M.F. and current in a circuit and correspond to (100) and (101). Their product is represented by the power curve *E*. *OK* is the axis of the *D* and *C* curves.

The power curve *E* is also a sine curve whose

axis lies midway between  $EJ$  and  $OK$  as expressed in Eqs. (102) and (103).  $E$  is a maximum point of the power curve and occurs simultaneously with the maximum values of  $D$  and  $C$  of the E.M.F. and current curve respectively. The maximum ordinate of the  $D$  curve is 110 and the maximum ordinate of the  $C$  curve is 44 and therefore the product  $110 \times 40 = 4400$  which is the maximum ordinate of the  $E$  curve. The ordinate at  $J = 4400 = (-110) \times (-40)$ , i.e., the product of the minimum values  $(-110)$  and  $(-40)$  of the  $D$  and  $C$  curves respectively.

$$(100) \quad e = 110 \sin \theta,$$

$$(101) \quad i = 40 \sin \theta,$$

$$(102) \quad p = ei = 4400 \sin^2 \theta,$$

$$(103) \quad p = 4400\left(\frac{1}{2} - \frac{1}{2} \cos 2\theta\right) = 2200 - 2200 \cos 2\theta.$$

The interpretation of (103) states that the power curve is a cosine curve of twice the frequency of (100) and (101) and that its axis is located 2200 units above  $OK$ . The cosine curve has an amplitude of 2200, which is one-half the product of the amplitudes of the constituents, and being negative in sign its position is displaced  $90^\circ$  behind a sine curve of like frequency. These facts are verified in Fig. 169. All of the ordinates of the power curve extend above the line  $OK$  and therefore are positive. The power although periodic is not alternating but pulsating.

13. In Fig. 170 the E.M.F. and current curves  $MSW$  and  $LNTB$  respectively are represented out of phase as expressed in Eqs. (104) and (105). The power curve is represented by a cosine curve  $LPQARUWB$  corresponding to (112). The maximum values of the three curves do not occur simultaneously.

$$(104) \quad e = 110 \sin \theta,$$

$$(105) \quad i = 50 \sin \left( \theta - \frac{\pi}{6} \right),$$

$$(106) \quad p = ei = 5500 \sin \theta \sin \left( \theta - \frac{\pi}{6} \right),$$

$$(107) \quad p = 5500 \sin \theta \left\{ \sin \theta \cos \frac{\pi}{6} - \cos \theta \sin \frac{\pi}{6} \right\},$$

$$(108) \quad p = 5500 \left\{ \sin^2 \theta \cos \frac{\pi}{6} - \cos \theta \sin \theta \sin \frac{\pi}{6} \right\},$$

$$(109) \quad p = 5500 \left\{ \frac{1 - \cos 2\theta}{2} \cos \frac{\pi}{6} - \frac{\sin 2\theta}{2} \sin \frac{\pi}{6} \right\},$$

$$(110) \quad p = 5500 \left\{ \frac{1}{2} \cos \frac{\pi}{6} - \frac{1}{2} \cos 2\theta \cos \frac{\pi}{6} - \frac{1}{2} \sin 2\theta \sin \frac{\pi}{6} \right\},$$

$$(111) \quad p = 2750 \cos \frac{\pi}{6} - 2750 \cos \left( 2\theta - \frac{\pi}{6} \right),$$

$$(112) \quad p = 2349 - 2750 \cos \left( 2\theta - \frac{\pi}{6} \right).$$

(II) CURRENT 30° OUT OF PHASE WITH E.M.F.

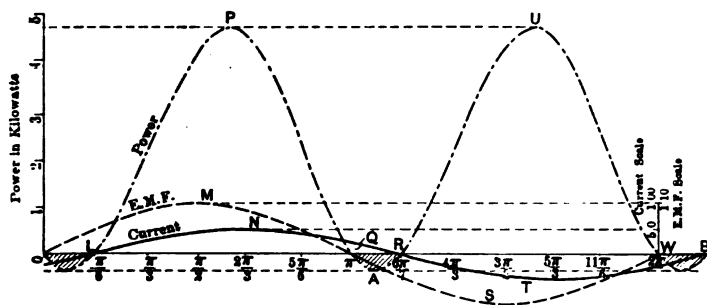


FIG. 170.—Power Curve.

The interpretation of (112) states that the power curve is a cosine curve of twice the frequency of (104) and (105) and that its axis is located 2349 units above the line  $LB$ . The cosine curve has an amplitude of 2750, which is one-half the product of the amplitudes of the constituents.

It is displaced  $90^\circ + \frac{\pi}{6}$ , i.e.,  $120^\circ$  behind a sine curve of

the positive and negative loops of

The increase of the phase  $\epsilon$  and current decreases the displacement and negative loops of the power angle equals  $90^\circ$  the positive and negative power curves are equal, and they are in its normal position as a sine wave to the axes of the E.M.F. and current.

**Ex. 11.** Plot (113)  $y = \sin^2 \theta$  and represents a power curve.

**Ex. 12.** Plot the power curve of constituent curves whose numbers

(a)-(82), (85); (b)-(83), (85); (c)-(83), (86); (f)-(84), (86); (g)-(84), (87); (j)-(82), (88); (k)-(82), (89); (n)-(83), (89); (o)-(

*Observation.* A power curve is a cosine curve whose amplitude is  $\frac{1}{2}$  of the constituent E.M.F. and current. The power curve is removed above the constituent curves by an amount which is the product of the constituents times the cosine of an

#### 14. The Curve of Damped or Decaying Oscillation.

One of the most important curves which is met with in the study of transient phenomena, such as the transmission of intelligence through space, is known as the curve of damped or decaying oscillation and is represented in Fig. 171.

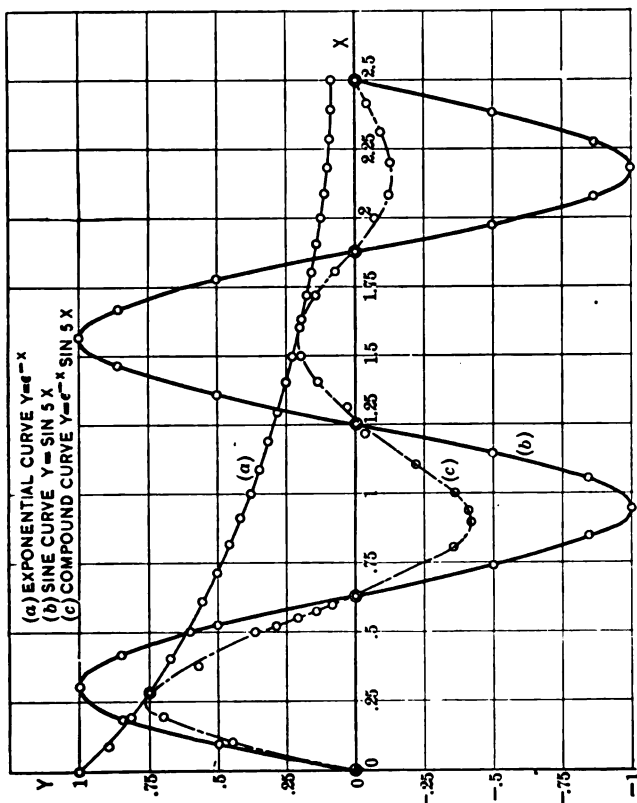


Fig. 171.—The Curve of Damped or Decaying Oscillation.

$$(114) \quad y = R e^{-Kt} \sin (\omega t + \theta),$$

$$(115) \quad y_1 = e^{-Kt},$$

$$y_2 = R \sin (\omega t + \theta).$$

the special value  $K=1$ , (b) re-  
for which  $R=1$ ,  $\theta=0$ , and  $\omega=5$   
represents time and has been repl  
curve (c) has the same frequency  
diminishes continuously. In a ver  
flattens out so as to be scarcely d  
axis. For practical purposes this  
terpreted to indicate a cessation  
curve although theoretically it wo

**Ex. 13.** Plot (114), (115), (116)  
checking the graphic work by calcul  
sets of values given in Table XXIX.

TABLE XXIX. CONSTANTS FOR 1  
CURVE

No.	$R$	$K$	$\omega$	$\theta$	No.
<i>a</i>	1	.02	1	0	<i>i</i>
<i>b</i>	1	.02	1	30°	<i>j</i>
<i>c</i>	1	.02	1	60°	<i>k</i>
<i>d</i>	1	.02	1	90°	<i>l</i>
<i>e</i>	2	.02	1	30°	<i>m</i>
<i>f</i>	2	.02	1	60°	<i>n</i>

In Fig. 172 construct  $AC = P_2 + P_1$ ,  $BC = P_2 - P_1$ ,  $CD = 1$  unit  $= DE$ . Erect  $GD \perp BE$  making  $BG = GE = 2$  nits. Then  $GD = \sqrt{3}$ . Erect  $CF \perp BD$ , then

$$CF = \sqrt{3} (P_2 - P_1).$$

Therefore  $\angle FAC = \phi$ .

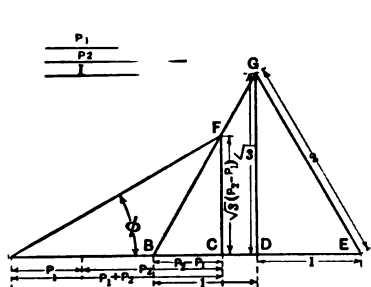


FIG. 172.

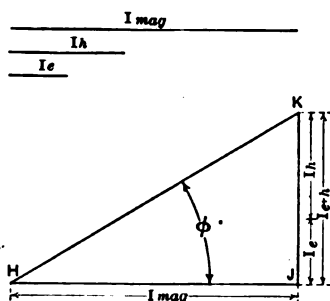


FIG 173.

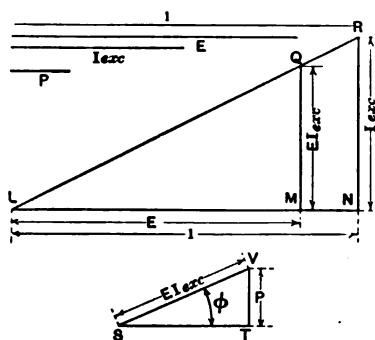


FIG. 174.

Verify the constructions of Figs. 173 and 174 from Eqs. (118) and (119) respectively:

$$(118) \quad \phi = \tan^{-1} \frac{I_e + I_h}{I_{\text{mag}}},$$

$$(119) \quad \phi = \sin^{-1} \frac{P}{EI_{\text{exc}}}.$$

## CHAPTER XXIII

### RATES, DERIVATIVES, AND INTEGRALS

**1. Speed.** We refer to the speed of a train, trolley-car, motor-car, or other moving body, with the definite idea that a precise distance has been traversed in a specified time. **Speed** is the name given to the **ratio** between **distance** and **time**.

A train which traverses a distance of 300 miles in 6 hours has an average speed  $= \frac{300}{6} = 50$  **miles per hour**. At the same rate, i.e., speed, the train would travel 20 miles in 24 minutes.

**Ex. 1.** What is the distance traversed by a car which is propelled for 15 minutes at a speed of 30 miles per hour?

**Ex. 2.** Construct a speed chart from the following time-table XXX. Plot distances vertically and time horizontally. The horizontal axis should provide for 24 hours.

**Ex. 3.** Determine the average rate of travel for each train by dividing the total distance by the total time. Calculate the rate of travel between the stations at which the train stops and verify these results by showing that the slope of the speed curve is a measure of the rate of travel between station stops. Draw a straight line between the first and last point on each speed curve and show that its slope is the average rate of travel for the entire distance. The following train numbers are for investigation: (a) 51; (b) 59; (c) 63; (d) 65; (e) 67; (f) 69; (g) 71; (h) 73; (i) 75; (j) 77; (k) 79; (l) 81; (m) 85; (n) 89; (o) 91; (p) 93.

**2. Velocity.** The word velocity is used often synonymously for speed. Strictly interpreted velocity involves the notion of direction as well as speed. In the discussion of the velocity of a moving body it is necessary to consider

### TIME TABLE XXX

Train Number and Time of Departure.																	
Dis- tance.	Station.	71	77	75	73	85	51	93	67	81	59	69	03	65	91	89	79
		A.M.	A.M.	A.M.	A.M.	A.M.	A.M.	P.M.	P.M.	P.M.	P.M.	P.M.	P.M.	P.M.	P.M.	P.M.	A.M.
.....	NY.....	8.08	10.08	8.30	10.16	11.08	12.38	1.08	2.38	3.34	3.38	4.38	5.08	6.00	9.30	12.30	
.....	MHT.....	8.28	10.27	8.48	10.35	11.27	12.56	1.27	2.56	3.52	3.57	4.56	5.26	6.18	9.49	12.49	
10.1	NA.....	8.31	10.31	8.52	10.39	11.31	1.00	1.31	3.00	..	4.01	5.00	...	...	9.53	12.53	
15.5	E.....	8.40	...	9.01	10.47	...	1.10	...	3.09	...	...	...	...	...	10.02	1.03	
58.2	T.....	5.28	9.31	11.23	10.04	11.40	12.30	2.03	2.25	4.02	4.46	4.53	6.43	...	10.58	2.24	
86.2	NP.....	6.10	10.08	11.57	10.44	12.20	1.08	2.43	3.00	4.42	..	5.32	6.38	7.24	7.52	11.42	3.15
91.7	P.....	6.33	10.23	12.10	11.10	12.33	1.20	2.57	3.15	4.55	..	5.44	6.52	7.36	8.24	12.19	3.30
105.1	C.....	...	...	11.27	...	...	...	...	...	...	6.03	...	7.53	8.41	12.39	...	
118.4	WMT.....	7.10	10.58	12.45	11.46	1.08	1.50	3.32	3.50	5.35	6.10	6.26	7.31	8.12	9.00	1.00	4.12
130.4	N.....	...	...	12.02	...	...	...	...	...	...	...	...	...	...	...	...	5.02
150.7	PRV.....	...	...	12.39	...	...	...	...	...	...	...	...	...	8.54	...	...	5.06
151.9	H.....	...	...	12.43	...	...	...	...	...	...	...	...	...	8.58	...	...	5.06
186.9	B.....	8.41	12.28	2.10	1.50	2.39	3.20	5.05	5.15	7.13	7.33	8.15	9.14	9.50	10.27	2.46	6.01
226.9	W (arrival)...	9.45	1.33	3.15	3.00	3.50	4.20	6.10	6.16	8.16	8.34	9.25	10.25	10.55	11.30	3.55	7.12

the **uniformity** or **non-uniformity** of the speed, i.e., the **constancy** or **variability** of the quantity of motion in a unit of time. A unit of time may be an hour, a minute, or a second or a lesser interval which depends upon the nature of the problem and the refinements of measurement and observation.

We form a very good understanding of these facts by considering the vicissitudes of a journey between New York and Chicago. Straight, level roadbeds are interspersed by circuitous winding stretches of track. The ascent of an incline in one interval is followed by a descent

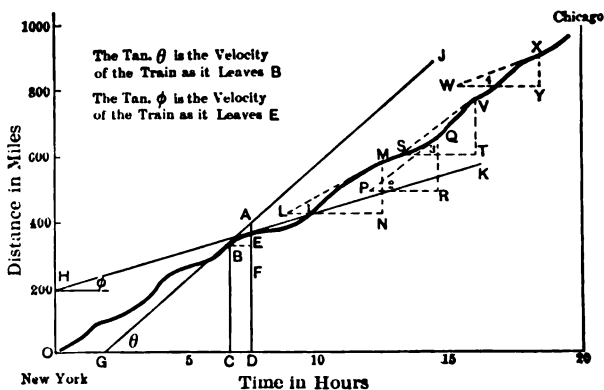


FIG. 175.—Chart of Variable Speed.

of a decline in the next interval. In consecutive intervals of time both variable and constant velocities will be observed. Such a record of performances is shown in Fig. 175. Time is plotted horizontally and distance vertically, so that the slope of the heavy black line *OBEMQVX* is the measure of the speed of the train, i.e., its rate per hour. At every instant the speed is changing and it is by considering short lapses of time that we can approximate to the actual rate at any instant.

**3. Slope.** The slope of a curve is measured by the slope of the lines which are tangent to the curve. *T*

slope of a line is proportional to the tangent of its angle of inclination with the horizontal. The slope of the curve  $OBEMVX$  at the respective points  $B$ ,  $E$ ,  $M$ ,  $V$ , and  $X$  is numerically the same as the respective slopes of the tangents  $GB$ ,  $HE$ ,  $LM$ ,  $SV$ , and  $WX$ . Therefore the rate of the train at  $B = R \tan \theta = \frac{BC}{GC}$ . The slope at  $E = R \tan \phi$ .  $R$  is the ratio of the vertical to the horizontal scale.

**Ex. 4.** Show that the slopes at  $M$ ,  $Q$ ,  $V$ , and  $X$  in Fig. 175 are expressed by the respective ratios:  $\frac{MN}{LN}$ ,  $\frac{QR}{PR}$ ,  $\frac{VT}{ST}$ , and  $\frac{XY}{WY}$ .

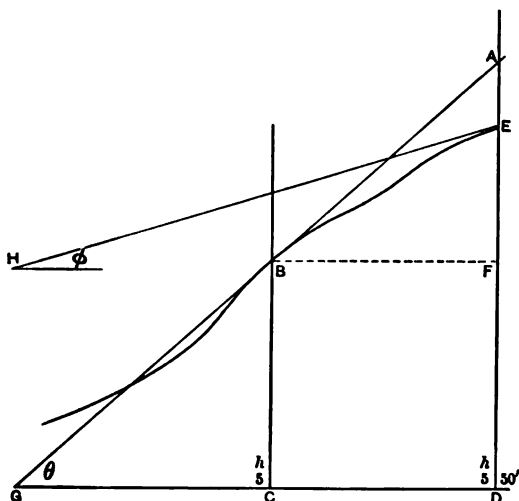


FIG. 176.—Initial, Average, and Final Velocities.

**4. The Scale of Velocity.** In Fig. 176 the  $\tan \theta$  measures the velocity of the train as it passes station  $B$ , whereas the  $\tan \phi$  measures the velocity of the train as it passes station  $E$ . These rates are expressed as miles per hour, because the vertical scale is in miles and the horizontal scale is in hours and therefore their ratio gives a rate in miles per hour.

=35 miles. Therefore the average velocity  $B$  and  $E$  is expressed in (1) and (2) abbreviated by  $\Delta$ , so that  $\Delta t$  is an increment or interval of time and  $\Delta s$  the increment of distance or space  $t$

$$(1) \quad V_a = \frac{\text{increment of distance}}{\text{increment of time}}$$

$$(2) \quad V_a = \frac{\Delta s}{\Delta t} = R \tan EBF.$$

The initial velocity ( $V_i$ ) is the velocity at the beginning of the time interval, and ( $V_f$ ) is the velocity at  $E$ , i.e., at the end of the time interval.

$$(3) \quad V_i = R \tan \theta. \quad (4)$$

$$(5) \quad V_i > V_a > V_f.$$

$$(6) \quad \therefore \tan \theta > \tan EBF$$

The interpretation of (5) and (6) is that the average velocity is less than the initial velocity and greater than the final velocity.

THEOREM 1. The average velocity of a body moving with constant acceleration is equal to the arithmetic mean of the initial and final velocities.

In the above illustration the increment of time is 50 minutes; in other words, the station  $E$  is reached 50 minutes after  $B$  and the velocity has changed correspondingly from  $V_i$  to  $V_f$ . If stations are noted successively closer to  $B$  the corresponding time interval will decrease and the average velocity will approach its limiting value, which is the initial velocity. If the slope corresponding to the successive average velocities be designated by  $\tan \alpha$ , then (9) follows from (8).

$$(8) \quad \text{The limit of } V_a = V_i.$$

$$\text{The limit of } \alpha = \theta.$$

$$(9) \quad \therefore \text{ The limit of } \tan \alpha = \tan \theta,$$

but the

$$(10) \quad \tan \alpha = \frac{\Delta s}{\Delta t}.$$

$$(11) \quad \therefore \lim_{\Delta t \rightarrow 0} \frac{\Delta s}{\Delta t} = \tan \theta.$$

$\Delta t \rightarrow 0$  means as  $\Delta t$  approaches ( $\rightarrow$ ) zero. The symbol  $\frac{ds}{dt}$  is an adopted abbreviation for the left-hand member of (11) and is known variously as a **derivative** of space, **differential coefficient** of space, or a **time rate of change** of space. Substituting the abbreviation in (11) we obtain (12).

$$(12) \quad \frac{ds}{dt} = \tan \theta = \text{instantaneous velocity} = V.$$

The interpretation of (12) states that  $V$ , the velocity at any instant, is the limiting value of the average speed measured to or from that instant. In other words, we approach more nearly to the value of the velocity of a train at each successive instant as we shorten the time interval which elapses during an observation. Hence arises the definition that **velocity is the time rate of change of space or distance.**

These facts which we have observed from a record of the speed of a moving train are applied in an identical manner to the graphical records of every changing action or phenomenon. The rate of change of any plotted action or phenomenon is indicated by the slope of its tangent lines.

*Observation.* When a body moves uniformly then in equal consecutive increments of time it traverses equal distances. This fact is illustrated by a graph for which the coordinate axes correspond to distance and time. Linear graphs are the only graphs which have a constant slope and therefore uniform motion is represented graphically by straight lines and non-uniform motion is represented by non-linear graphs.

**6. Linear and Angular Velocity.** When an electric motor maintains a constant speed then every conductor on the periphery of the armature has a constant linear velocity, i.e., the distance or length of arc traversed will be equal for equal intervals of time. The path of the wire is a circle. If the motor maintains a constant speed of  $n$  revolutions per second then the linear velocity  $v$  of a conductor is the length of the circumference ( $l$ ) times the number ( $n$ ) of revolutions. The **linear velocity** of a rotating body is by the interpretation of (13) the **rate of lengthening of the arc per second**.

$$(13) \quad v = l n = \frac{dl}{dt}.$$

The coil containing the conductor generates equal angles in equal intervals of time. In the same circle equal arcs subtend equal central angles. The rate of change of an angle expressed in radians is called angular velocity ( $\omega$ ).

If the angle is designated as  $\theta$ , then  $\omega = \frac{d\theta}{dt}$ .

$$(14) \quad l = 2\pi R.$$

$$(15) \quad v = 2\pi R n = R 2\pi n. \text{---Sub (14) in (13)}$$

In (15)  $2\pi n$  is the radian change per second and is therefore replaceable by  $\omega$  as shown in (16).

$$(16) \quad v = R\omega = R \frac{d\theta}{dt}. \quad \therefore v \propto \omega.$$

The interpretation of (16) states that the linear velocity of a rotating body is directly proportional to its angular velocity and that the proportionality factor  $R$  is the radius of the circular path.

**Ex. 5.** Fill in the blank spaces in Table XXXI.  $n$  is expressed as revolutions per minute, whereas  $v$  and  $\omega$  are the respective velocities in feet and radians per second.  $R$  is the radius in inches.

TABLE XXXI

No.	$R$	$n$	$v$	$\omega$	No.	$R$	$n$	$v$	$\frac{d\theta}{dt}$	No.	$R$	$n$	$\frac{d\theta}{dt}$	$\frac{d\theta}{dt}$
1	10	50	..	..	8	..	200	800	...	15	..	..	...	$120\pi$
2	25	..	..	300	9	..	..	...	500	16	..	..	...	$50\pi$
3	..	7	60	..	10	..	11	77	...	17	81	18	...	...
4	..	..	..	80	11	..	..	...	62.8	18	12	..	5000	...
5	..	..	..	$2\pi$	12	..	..	...	39	19	12	650	...	...
6	36	..	100	..	13	..	..	...	$100\pi$	20	..	500	3000	...
7	50	50	..	..	14	..	42	360	...	21	..	200	2500	...

*Observation.* All points in a crank arm or coil rotate with equal angular velocity, but their respective linear velocities are directly proportional to the radii of their circular paths.

**7. The Derivative.** Every linear formula represents a constant or uniform ratio of change between two variables, i.e., a constant rate of change between two variables. The slope of a curve for any of its points is the ratio between the rates of change of two variables at that instant. When one of the variables is time, the slope is the time rate of change of the other variable. The graph *NMTFP* in Fig. 177 represents a non-uniform motion and has been constructed from a non-linear formula. At each successive

instant the rate of change of the variables is different from the rate at the preceding and following instants. At the rate of change is measured by  $\tan \alpha$ .

$$(17) \quad \tan \alpha = \frac{\text{rate of the ordinate}}{\text{rate of the abscissa}}.$$

$$(18) \quad \tan \alpha = \frac{\text{rate of } y}{\text{rate of } x}.$$

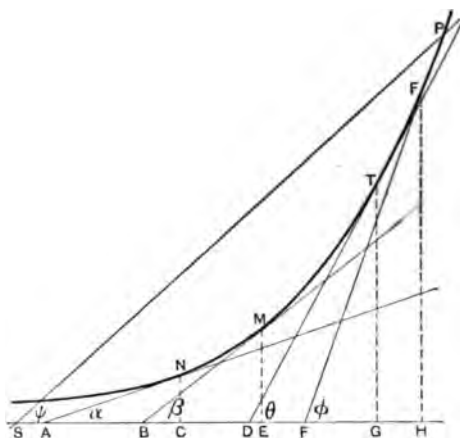


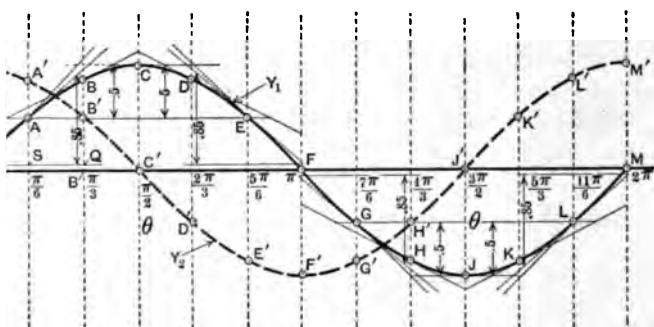
FIG. 177.—Representation of Non-uniform Motion.

The symbol for abbreviating the right-hand member of (18) is  $\frac{dy}{dx}$  and hence  $d$  is not a multiplier but a symbol of operation.  $dx$  is an abbreviation for the rate of  $x$ , i.e.  $dx$  is a contraction for the time rate  $\frac{dx}{dt}$ .

$$(19) \quad \therefore \tan \alpha = \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{\text{time rate of } y}{\text{time rate of } x}.$$

$$(20) \quad \tan \alpha = \frac{d}{dx}(y).$$

The interpretation of (19) states that the slope to a curve at any point is the ratio of the time rate of change of the ordinate to the time rate of change of the abscissa. (20) is another form of (19) in which  $\frac{d}{dx}$  shows itself more distinctly as a symbol of operation which acts upon  $y$ . The  $d$  in the numerator and the  $x$  in the denominator are not factors or multipliers. The entire symbol made of the two  $d$ 's and  $x$  is an abbreviation for a definite operation, just as  $\sin$ ,  $\cos$ ,  $\tan$ , and  $\csc$  are three-lettered abbreviations for definite functions.



178.—The Graphic Determination of the Derivative of a Curve.

**The Graphic Determination of the Derivative of a Curve.** The graphic determination of the derivative of a curve is the operation of measuring the slope of a curve at successive points. In Fig. 178  $y_1$  is a sine curve, whose ordinates are instantaneous values of current ( $i$ ) whose abscissas are radians and whose equation is  $i = \sin \theta$ . Tangents have been drawn at the lettered points on the curve and their measured slopes express the derivative of the curve  $\frac{dy}{d\theta}$ , i.e., the rate of change of the ordinate  $y$  with respect to the angle  $\theta$ , according to Table XXXII.

TABLE XXXII

Point on the curve.....	O	A	B	C	D	E	F	G	H	I	K	L	M
Slope of tangent $= \frac{dy}{d\theta}$	1	.866	.5	0	-.5	-.866	-1	-.866	-.5	0	.5	.866	1
Ordinate of the curve, $y$	0	.5	.866	1	.866	.5	0	-.5	-.866	-1	-.866	-.5	0
Abscissa of the curve, $\theta$	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	$\pi$	$\frac{7\pi}{6}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{11\pi}{6}$	$2\pi$

It is observed that the numeric values of slopes follow a cycle of changes corresponding to the ordinates of the curve, excepting that the initial values are displaced. The successive values of the derivatives are plotted as ordinates of a new curve labeled  $y_2$  which is also a sine curve but which leads  $y_1$  by  $90^\circ$ . The equation for  $y_2$  is given in (21).

$$(21) \quad y_2 = \sin(\theta + 90^\circ) = \cos \theta,$$

but

$$(22) \quad y_2 = \frac{d(y)}{d\theta} = \frac{d}{d\theta} (\sin \theta).$$

$$(23) \quad \therefore \frac{d}{d\theta} (\sin \theta) = \cos \theta.$$

The interpretation of (23) states that a new curve called a **derivative** may be derived from a given curve which is called its **primitive**. The ordinates of the derivative curve are the respective rates of change of the simultaneous ordinates of the primitive curve. (23) also states that the derivative of a sine is a cosine.

9. In Fig. 179  $y = I \sin \omega t$  is a primitive curve and its corresponding derivative is the curve  $y_t = \omega I \cos \omega t$ . Both curves have a like frequency but the amplitude of the derivative is  $\omega$  times the amplitude of the primitive. The derivative may be represented by the dotted cosine curve.

fig. 179, in which case its scale is  $\omega$  times the scale of the primitive.

$$y = I \sin \omega t.$$

$$\frac{d}{dt}(y) = y_t = \omega I \cos \omega t.$$

$$\therefore \frac{d}{dt}(I \sin \omega t) = \omega I \cos \omega t.$$

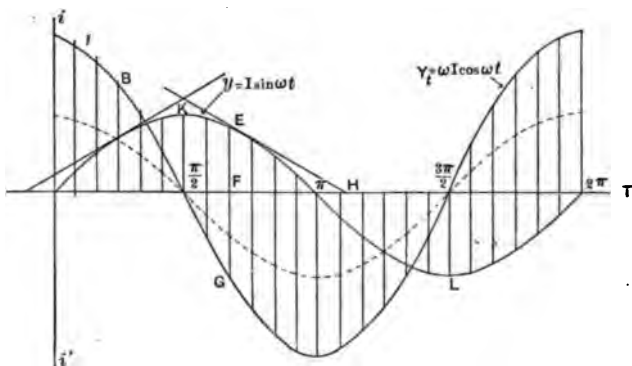


FIG. 179.—A Primitive and its Derivative.

The interpretation of (26) states that the derivative of the curve is a cosine curve and that the ratio of the amplitude of the derivative to the amplitude of the primitive is  $2\pi$  times the frequency of the curves.

**Ex. 6.** Plot  $y = x^2$  and determine its slope at ten distinct points.

Use the slopes as ordinates of a derivative curve which should produce a linear graph. Determine the equation of the derivative.

The tangents may be most readily constructed as follows: Draw a chord between any two points. Construct a perpendicular bisector to the chord and where the perpendicular intersects the curve draw a tangent parallel to the chord. The work is made easier if a bisected chord of fixed length is moved about the curve.

**Ex. 7.** Plot the derivative curves for the following primitives: (a)  $y = 2x^2$ ; (b)  $y = 2.5x^2$ ; (c)  $y = .2x^2$ ; (d)  $y = 3x^2$ ; (e)  $y = 2.4x^2$ ; (f)  $y = 2.8x^2$ .

**Ex. 8.** Plot the derivative curves for the following primitives: (a)  $y = x^3$ ; (b)  $y = 2x^3$ ; (c)  $y = .5x^3$ ; (d)  $y = x^4$ ; (e)  $y = .8x^4$ ; (f)  $y = .5x^4$ .

**Ex. 9.** Construct a sine curve for the equation  $i = I \sin \theta$  and determine its derivative curve.

**Ex. 10.** Construct the primitive and its derivative for the following equations: (a)  $y = \sin 2t$ ; (b)  $y = .5 \sin 2\pi t$ ; (c)  $i = 10 \sin 50\pi t$ ; (d)  $e = 110 \sin 120\pi t$ .

**Ex. 11.** Show that the derivative of a negative sine curve is a negative cosine curve.

**Ex. 12.** Show that the derivative of a negative cosine curve is a sine curve.

**Ex. 13.** Show that the derivative of a cosine curve is a negative sine curve.

**Ex. 14.** Plot the sum of the (a) equations in Exs. 7 and 8, and show that the derivative of the sum is equal to the sum of the derivatives.

**10. Differentiation of Formulas.** The differentiation of formulas is the mathematical process by means of which a primitive equation is transformed into a derivative equation. By comparing the primitive with the derivative a formal statement or law of differentiation may be framed for different types of primitives. The primitive (27) transforms into the derivative (28).

(27)	<div style="display: inline-block; vertical-align: middle; text-align: right; padding-right: 5px;">Ordinate</div> <div style="display: inline-block; vertical-align: middle; text-align: right; padding-right: 5px;">Coefficient</div> <div style="display: inline-block; vertical-align: middle; text-align: right;">Power of abscissa</div>	$y = a x^n.$
(28)		$\frac{dy}{dx} = anx^{n-1}$

Comparing (28) with (27) we observe the following law for writing the derivative of a power of a variable. Multiply the coefficient of the primitive by the exponent of its variable and decrease the exponent of the variable by unity.

**Ex. 15.** Verify the following derivatives by applying the law of a power to the primitives. The derivative of an algebraic sum equals the sum of the derivatives of its several terms. The deriv-

ive of a constant term is zero. Fill in the blank spaces in Table XXIII.

TABLE XXXIII

Primitive.	Derivative.
$y = ax^3$	$\frac{dy}{dx} = 3ax^2$
$y = 5x$	$\frac{dy}{dx} = 5x^0 = 5$
$y = \frac{ax^2}{2}$	$\frac{dy}{dx} = ax$
$y = 5x^{\frac{1}{2}} = 5\sqrt{x}$	$\frac{dy}{dx} = 7.5x^{-\frac{1}{2}} = 7.5\sqrt{x}$
$y = ax^{-1.25}$	$\frac{dy}{dx} = -1.25ax^{-2.25}$
$y = -5x^{-\frac{1}{2}}$	$\frac{dy}{dx} = \frac{5}{2}x^{-\frac{3}{2}}$
$y = \frac{a}{x} = ax^{-1}$	$\frac{dy}{dx} =$
$y = ax^3 + 5x$	$\frac{dy}{dx} = 3ax^2 + 5$
$y = 5x^3 + 2x^2 + 3x + 5$	$\frac{dy}{dx} = 15x^2 + 4x + 3$
$s = .5gt^2 + bt + c$	$\frac{ds}{dt} = v = gt + b$
$y = x^2 + \frac{a^2}{x^2}$	$\frac{dy}{dx} =$
$p = Rtv^{-1}$	$\frac{dp}{dv} =$
$E = \frac{1}{2}ax^2 + d$	$\frac{dE}{dx} =$
$s = 16.1t^2 + 16.1t$	$\frac{ds}{dt} =$

TABLE XXXIV.—NUMERIC VAL  
(29) AND DERIVAT

Values of the abscissa $x$ . .	0	.5	1	1.5	2
Values of the ordinate $y$ . .	0	.5	2	4.5	8
Values of the ordinate $\frac{dy}{dx}$	0	2	4	6	8

*Observation. The derivative of an algebraic equation and the degree less in unit value than the degree of it*

### 11. The Proof of the Law of the

The law for the derivative of a power follows: Construct the graph of (31)

$$(31) \quad y = ax^n.$$

Select any point  $P$  on the curve, its ordinate and abscissa  $y$  and  $x$  respectively. Designate a neighboring point  $Q$ . Designate the abscissa of  $Q$  by  $y + \Delta y$  and  $x + \Delta x$ . The increments indicate the growths or increments between the respective ordinates. By substituting the coordinates of  $Q$

Subtracting (31) from (32) we obtain (33),

$$(33) \quad \Delta y = a(x + \Delta x)^n - ax^n = a\{(x + \Delta x)^n - x^n\}.$$

The parenthetical quantity,  $(x + \Delta x)^n$ , may be expanded by means of the Binomial Theorem given on page 240, and (33) becomes (34) which simplifies by cancellation into (35):

$$(34) \quad \Delta y = a \left\{ \left( x^n + nx^{n-1}\Delta x + \frac{n(n-1)x^{n-2}}{2!}\Delta x^2 + \frac{n(n-1)(n-2)x^{n-3}}{3!}\Delta x^3 + \dots + \Delta x^n \right) - x^n \right\}.$$

$$(35) \quad \Delta y = a \left\{ nx^{n-1}\Delta x + \frac{n(n-1)x^{n-2}}{2!}\Delta x^2 + \frac{n(n-1)(n-2)x^{n-3}}{3!}\Delta x^3 + \dots + \Delta x^n \right\}.$$

$$(36) \quad \therefore \frac{\Delta y}{\Delta x} = a \left\{ nx^{n-1} + \frac{n(n-1)x^{n-2}}{2!}\Delta x + \dots + \Delta x^{n-1} \right\}.$$

(36) results from dividing (35) by  $\Delta x$ . (36) is an expression between the two always equal variables  $\frac{\Delta y}{\Delta x}$  and the terms of the right-hand member. By the Fundamental Theorem on limits given on page 227 the limits of the two members of (36) are equal and therefore (37), (38), and (39) follow from (36). Since the limit of  $\Delta x = 0$ , then the limit of every term containing  $\Delta x$  is zero. The constant term  $ax^{n-1}$  remains unaltered in the operation of obtaining limits.

$$(37) \quad \lim \frac{\Delta y}{\Delta x} = \lim a \left\{ nx^{n-1} + \frac{n(n-1)x^{n-2}\Delta x}{2!} + \dots + \Delta x^{n-1} \right\}.$$

$$(38) \quad \therefore \frac{dy}{dx} = anx^{n-1} + \text{zero} + \text{zero} + \dots + \text{zero}.$$

$$(39) \quad \therefore \frac{dy}{dx} = anx^{n-1}.$$

Therefore to write the derivative of a power of a variable multiply the exponent into the constant coefficient and decrease the exponent of the variable by unity.

**12.** The operation of obtaining a derivative is exceedingly laborious, but a law may be formulated for each type of primitive. A list of primitives and their corresponding derivatives is given in Table XXXV and may be referred to for use or verification. Table XXXV contains the differential form (40) in which the rate of the dependent variable is expressed in terms of the rate of the independent variable.

DERIVATIVE FORM	DIFFERENTIAL FORM
(39) $\frac{dy}{dx} = anx^{n-1}.$	(40) $dy = anx^{n-1}dx.$

(40) expresses the rate of change of  $y$  in terms of the rate of change of  $x$ . In other words, the ordinate changes  $anx^{n-1}$  times as fast as the abscissa. Suppose  $a=3$  and  $n=2$ , then (31) becomes (41), (39) becomes (42), and (40) becomes (42a):

(41) $y = 3x^2,$	(42) $\frac{dy}{dx} = 6x,$	(42a) $dy = 6xdx.$
------------------	----------------------------	--------------------

The following are obtained from (41), (42), and (42a):

When  $x=1$ , then the ordinate = 3,  
                   the slope = 6,  
                   the rate of the ordinate = 6 times the rate of  
                   the abscissa.

When  $x=2$ , then the ordinate = 12,  
                   the slope = 12,  
                   the rate of the ordinate = 12 times the rate of  
                   the abscissa.

When  $x=3$ , the ordinate = 27,  
                   the slope = 18,  
                   the rate of the ordinate = 18 times the rate of  
                   the abscissa.

**13.** A primitive curve is transformed into a derivative curve which we call the derivative by the operation of differentiation of the rates trans-

the derivative into the differential form. The reverse operation which transforms the differential equation into the primitive is called **antidifferentiation** or **integration**. Differentiation is therefore the process of analyzing or disintegration of an algebraic expression into its tangents, whereas integration is the process of synthesizing or summing and building a curve from its tangents. The symbol for integration is  $\int$  which was originally an old-fashioned S, suggested by the summing process. It also resembles the familiar scroll figures on musical instruments. Placing the integral symbol in (40) we obtain (43) which is an instruction to perform the operation of antidifferentiation, i.e., integration:

$$(43) \quad \int dy = \int ax^{n-1} dx.$$

$$(43a) \quad \int dy = \cancel{\int} dy = y.$$

In the left-hand member of (43)  $y$  is subject to two opposite operations and since the one follows immediately after the other  $y$  remains unaffected. The two operations are in juxtaposition and are therefore in the nature of a command and its countercommand and therefore they destroy each other and are struck out as is shown in (43a). An indicated operation in an equation can be removed either by performing the operation or subjecting the equation to the antioperation. The latter method was applied to the left member of (43) and the former method was applied to the right-hand member of (43). To integrate the right-hand side of (43) reverse the law of differentiation and obtain the following law for the integration of the power of a variable. Increase the exponent by unity and divide the coefficient by the new exponent, as shown in (43b).

$$\int ax^{n-1} dx = \frac{ax^{n-1+1}}{n-1+1} = \frac{ax^n}{n} = ax^n.$$

$$\therefore y = ax^n.$$

*Observation. Every equation is the derivative of some primitive equation. The former may be obtained from the latter by graphic methods and by the laws of differentiation.*

*Every equation is also the primitive of some derivative equation. The primitive is also called the integral and may be obtained by graphic methods or by the laws of anti-differentiation or integration.*

**14.** In paragraph 8 it was shown that the derivative of a sine curve is represented by a cosine curve. Therefore the primitive of a cosine curve is a sine curve.

## DIFFERENTIATION.

$$(44) \quad y = \sin x.$$

$$(45) \quad \frac{dy}{dx} = \cos x$$

$$(46) \quad \therefore dy = \cos x dx.$$

## INTEGRATION.

$$(46) \quad dy = \cos x dx.$$

$$(47) \quad \int dy = \int \cos x dx.$$

$$(44) \quad \therefore y = \sin x.$$

The process of integration is simplified by recognizing the primitive forms which are associated with definite derivatives or differential forms according to Table XXXV. The process of integration is performed automatically by various instruments known as water-meters, gas-meters, integrating wattmeters, integrators, and planimeters.

**Ex. 17.** Consult Table XXXV and write the derivatives for the following primitives:

$$(a) i = 5 \sin \theta; \quad (b) i = 50 \sin \theta; \quad (c) i = I \sin 50\pi t;$$

$$(d) i = .5 \sin t; \quad (e) i = .5 \sin 120\pi t; \quad (f) y = 5 \sin (2x + 5).$$

**Ex. 18.** Consult Table XXXV and write the primitives for the following derivatives:

$$(a) \quad \frac{di}{d\theta} = 5 \cos \theta;$$

$$(b) \quad \frac{di}{d\theta} = -50 \sin \theta;$$

$$(c) \quad \frac{di}{dt} = 50\pi I \cos 50\pi t;$$

$$(d) \quad \frac{di}{dt} = -500\pi \cos 120\pi t;$$

$$(f) \quad \frac{dy}{dx} = 60 \cos \left( 2x + \frac{\pi}{2} \right);$$

$$(g) \quad \frac{di}{dt} = 5n \sin (nt + \pi).$$

**Ex. 19.** Consult Table XXXV and write the primitives for the following derivatives.

$$(a) \frac{dy}{dx} = 5x^3; \quad (b) \frac{d\theta}{dT} = 5T + K; \quad (c) \frac{dL}{dT} = aT^2 + bT + c.$$

$$(d) \frac{dy}{dx} = -ax^{-2}; \quad (e) \frac{dy}{dx} = -1.25x^{-2.25}; \quad (f) \frac{dy}{dx} = -\frac{5a}{x^4}.$$

**15.** Logarithmic and Exponential Formulas are intimately related, as shown by the various transformations which follow, and also by consulting Table XXXV.

$$(48) \quad y = c^x.$$

$$(49) \quad \log_e y = x \log_e c.$$

$$(50) \quad \log_e c = \text{a constant} = K.$$

$$(51) \quad \therefore \log_e y = xK.$$

$$(52) \quad \therefore y = e^{xK}.$$

#### DIFFERENTIATION

$$(52) \quad y = e^{xK}.$$

$$(53) \quad \frac{dy}{dx} = K e^{xK} = Ky.$$

$$(54) \quad dy = K e^{xK} dx = Ky dx.$$

$$(51) \quad \log_e y = xK.$$

$$(56) \quad \frac{dy}{dx} = Ky.$$

$$(57) \quad \frac{dy}{y} = K dx.$$

#### INTEGRATION

$$(54) \quad dy = K e^{xK} dx.$$

$$(55) \quad \int dy = \int K e^{xK} dx.$$

$$(52) \quad y = \frac{K e^{xK}}{K} = e^{xK}.$$

$$(57) \quad \frac{dy}{y} = K dx.$$

$$(58) \quad \int \frac{dy}{y} = \int K dx.$$

$$(51) \quad \log_e y = Kx.$$

TABLE XXXV.—AN ABRIDGED LIST OF INTEGRAL FORMS

Number.	Primitive. $y =$	Derivative. $\frac{dy}{dx} =$	Differential. $dy =$	Integral. $y =$
1	$ax^m$	$mx^{m-1}$	$mx^{m-1}dx$	$\int mx^{m-1}dx$
2	$ax$	$a$	$adx$	$\int adx = \int ax^0dx$
3	$a+bx+cx^2$	$b+cx$	$bdx+cx^2dx$	$\int (b+cx)dx = \int bdx + \int cx^2dx$
4	$\log x$	$\frac{1}{x}$	$\frac{dx}{x}$	$\int \frac{dx}{x} = \int x^{-1}dx$
5	$e^x$	$e^x$	$e^x dx$	$\int e^x dx$
6	$a^x$	$a^x \log_e a$	$a^x \log_e a dx$	$\int a^x \log_e a dx$
7	$e^{mx}$	$me^{mx}$	$me^{mx} dx$	$\int me^{mx} dx$
8	$\sin x$	$\cos x$	$\cos x dx$	$\int \cos x dx$



number	$y =$	$\frac{dy}{dx} =$	$\frac{dy}{dx} =$	$y =$
9.	$\sin \omega x$	$\omega \cos \omega x$	$\omega \cos \omega t$	$\int \omega \cos \omega t dt$
10	$a \sin (\omega t + \phi)$	$a \omega \cos (\omega t + \phi)$	$a \omega \cos (\omega t + \phi) dt$	$\int a \omega \cos (\omega t + \phi) dt \phi$
11	$\frac{x}{2} - \frac{\sin 2x}{4}$	$\sin^2 x$	$\sin^2 x dx$	$\int \sin^2 x dx$
12	$\cos x$	$-\sin x$	$-\sin x dx$	$\int -\sin x dx$
13	$-\sin x$	$-\cos x$	$-\cos x dx$	$\int -\cos x dx$
14	$-\cos x$	$\sin x$	$\sin x dx$	$\int \sin x dx$
15	$\log \sin x$	$\cot x$	$\cot x dx$	$\int \cot x dx$
16	$-\log (\cos x)$	$\tan x$	$\tan x dx$	$\int \tan x dx$
17	$\tan x$	$\sec^2 x$	$\sec^2 x dx$	$\int \sec^2 x dx$

TABLE XXXV.—AN ABRIDGED LIST OF INTEGRAL FORMS—Continued

Number.	Primitive. $y =$	Derivative. $\frac{dy}{dx}$	Differential, $dy =$	Integral. $y =$
18	$\cot x$	$-\csc^2 x$	$-\csc^2 x dx$	$\int -\csc^2 x dx$
19	$\sec x$	$\sec x \tan x$	$\sec x \tan x dx$	$\int \sec x \tan x dx$
20	$\csc x$	$-\csc x \cot x$	$-\csc x \cot x dx$	$\int -\csc x \cot x dx$
21	$\sin^{-1} \frac{x}{a}$	$\frac{1}{\sqrt{a^2 - x^2}}$	$\frac{dx}{\sqrt{a^2 - x^2}}$	$\int \frac{dx}{\sqrt{a^2 - x^2}}$
22	$\frac{1}{a} \tan^{-1} \frac{x}{a}$	$\frac{1}{a^2 + x^2}$	$\frac{dx}{a^2 + x^2}$	$\int \frac{dx}{a^2 + x^2}$
23	$\frac{1}{a} \sec^{-1} \frac{x}{a}$	$\frac{1}{x\sqrt{x^2 - a^2}}$	$\frac{dx}{x\sqrt{x^2 - a^2}}$	$\int \frac{dx}{x\sqrt{x^2 - a^2}}$
24	$\log \{x + \sqrt{x^2 + a^2}\}$	$\frac{1}{\sqrt{a^2 + x^2}}$	$\frac{dx}{\sqrt{a^2 + x^2}}$	$\int \frac{dx}{\sqrt{a^2 + x^2}}$
25	$\log \{x + \sqrt{x^2 - a^2}\}$	$\frac{1}{\sqrt{x^2 - a^2}}$	$\frac{dx}{\sqrt{x^2 - a^2}}$	$\int \frac{dx}{\sqrt{x^2 - a^2}}$

Number.	Primitive. $y =$	Derivative. $\frac{dy}{dx} =$	Differential. $dy =$	Integral. $y =$
26	$\frac{1}{2a} \log \frac{a+x}{a-x}$	$\frac{1}{a^2-x^2}$	$\frac{dx}{a^2-x^2}$	$\int \frac{dx}{a^2-x^2}$
27	$\frac{1}{a-b} \log \frac{x-a}{x-b}$	$\frac{1}{(x-a)(x-b)}$	$\frac{dx}{(x-a)(x-b)}$	$\int \frac{dx}{(x-a)(x-b)}$
28	$\frac{1}{a} \sin^{-1} \frac{x-a}{x}$	$\frac{1}{\sqrt{2ax-x^2}}$	$\frac{dx}{\sqrt{2ax-x^2}}$	$\int \frac{dx}{\sqrt{2ax-x^2}}$
29	$\frac{1}{2} x \sqrt{x^2+a^2} + \frac{1}{2} a^2 \sin^{-1} \frac{x}{a}$	$\sqrt{a^2-x^2}$	$\sqrt{a^2-x^2} dx$	$\int \sqrt{a^2-x^2} dx$
30	$\frac{x \sqrt{x^2+a^2} + a^2 \log \{x + \sqrt{x^2+a^2}\}}{2}$	$\sqrt{x^2+a^2}$	$\sqrt{x^2+a^2} dx$	$\int \sqrt{x^2+a^2} dx$
31	$\frac{x \sqrt{x^2-a^2} - a^2 \log \{x + \sqrt{x^2-a^2}\}}{2}$	$\sqrt{x^2-a^2}$	$\sqrt{x^2-a^2} dx$	$\int \sqrt{x^2-a^2} dx$
32	$\frac{1}{a} \cos^{-1} \frac{a}{x}$	$\frac{1}{x \sqrt{x^2-a^2}}$	$\frac{dx}{x \sqrt{x^2-a^2}}$	$\int \frac{dx}{x \sqrt{x^2-a^2}}$
33	$\frac{1}{a} \log \frac{x}{a + \sqrt{a^2 \pm x^2}}$	$\frac{1}{x \sqrt{a^2 \pm x^2}}$	$\frac{dx}{x \sqrt{a^2 \pm x^2}}$	$\int \frac{dx}{x \sqrt{a^2 \pm x^2}}$

TABLE XXXV.—AN ABRIDGED LIST OF INTEGRAL FORMS—Continued

Number.	Primitive. $y =$	Derivative. $\frac{dy}{dx} =$	Differential. $dy =$	Integral. $y =$
34	$\pm \sqrt{a^2 \pm x^2}$	$\frac{x}{\sqrt{a^2 \pm x^2}}$	$\frac{xdx}{\sqrt{a^2 \pm x^2}}$	$\int \frac{xdx}{\sqrt{a^2 \pm x^2}}$
35	$\sqrt{x^2 - a^2}$	$\frac{x}{\sqrt{x^2 - a^2}}$	$\frac{xdx}{\sqrt{x^2 - a^2}}$	$\int \frac{xdx}{\sqrt{x^2 - a^2}}$
36	$\frac{x-a}{2} \sqrt{2ax-x^2} + \frac{a^2}{2} \sin^{-1} \frac{x-a}{a}$	$\sqrt{2ax-x^2}$	$\sqrt{2ax-x^2} dx$	$\int \sqrt{2ax-x^2} dx$
37	$\frac{\sqrt{x-1}}{\sqrt{x+1}}$	$\frac{1}{(x+1)\sqrt{x^2-1}}$	$\frac{dx}{(x+1)\sqrt{x^2-1}}$	$\int \frac{dx}{(x+1)\sqrt{x^2-1}}$
38	$-\frac{\sqrt{x+1}}{\sqrt{x-1}}$	$\frac{1}{(x-1)\sqrt{x^2-1}}$	$\frac{dx}{(x-1)\sqrt{x^2-1}}$	$\int \frac{dx}{(x-1)\sqrt{x^2-1}}$
39	$\sin^{-1} x - \sqrt{1-x^2}$	$\frac{1+x}{\sqrt{1-x}}$	$\sqrt{\frac{1+x}{1-x}} dx$	$\int \sqrt{\frac{1+x}{1-x}} dx$
40	$\sqrt{\frac{(x+a)(x+b)}{(a-b) \log(\sqrt{x+a} + \sqrt{x+b})}}$	$\sqrt{\frac{x+a}{x+b}}$	$\sqrt{\frac{x+a}{x+b}} dx$	$\int \sqrt{\frac{x+a}{x+b}} dx$

**Graphic Integration.** In paragraph 8 it was shown from Fig. 178 that a primitive which is represented by a sine curve has a derivative which is a cosine i.e., the derivative is also a sine curve which leads the primitive by  $90^\circ$ . The slope of the primitive at any instant is determined by the tangent to the curve at the point whose abscissa corresponds to that instant. The numeric values of the slopes are equal to the simultaneous ordinates of the derivative. The reversal of graphic differentiation is called **antidifferentiation** or **graphic integration** and the building of a primitive from a given derivative.

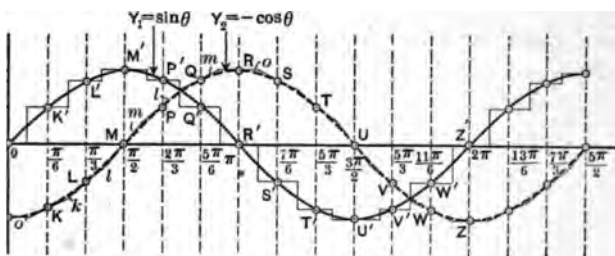


FIG. 180.—Graphic Integration.

180  $y_1 = \sin \theta$  is a given derivative curve and  $y_2 = -\cos \theta$  is its corresponding integral. The numeric values of the ordinates of  $y_1$  are equal to the slopes of the tangents to  $y_2$ . The ordinates at  $J', K', L', M'$  of  $y_1$  are 0, .5, .866, and 1 respectively. The lines  $o, k, l, m$  on the left of the  $Y$  axis have slopes equal to 0, .5, .866, respectively. At the points  $J, K, L$ , and  $M$  of  $y_2$  the tangents are respectively parallel to  $o, k, l, m$  and therefore their respective slopes are 0, .5, .866, and 1. The ordinates of  $P', Q',$  and  $R'$  on  $y_1$  are .866, .5, and 0 respectively and therefore at the points  $P, Q,$  and  $R$  on  $y_2$ , the tangents are parallel to  $l, k,$  and  $o$  respectively. The integral curve is therefore a sine curve which touches the given curve at the points  $J, K, L, M, P, Q, R$ .

The integral curve is determined as follows: The small triangular areas which have been added outside the derivative curve are constructed so as to be equal to the corresponding triangular areas subtracted inside the derivative curve. The eye can adjust these pairs of triangular areas very accurately after a little practice so that the total error is negligible. The result is that the area under the derivative curve  $y_1$  is replaced by the area under the **stepped figure** which is bounded by **risers and levels**, i.e., by vertical and horizontal lines. The ordinates of the stepped figure change only at the levels and are constant between the risers. The ordinates of the levels through  $J'$ ,  $K'$ ,  $L'$ ,  $M'$ ,  $P'$ ,  $Q'$ ,  $R'$  are 0, .5, .866, 1, .866, .5, and 0 respectively, and the same numeric values of the slopes of the tangents of  $y_2$  are observed at the respective points  $J$ ,  $K$ ,  $L$ ,  $M$ ,  $P$ ,  $Q$ , and  $R$ .

The construction of the integral curve follows: Extend the risers and also the ordinates passing through the points  $J'$ ,  $K'$ ,  $L'$ ,  $M'$ ,  $P'$ ,  $Q'$ , and  $R'$ . Locate a pole, i.e., a fixed point, at a unit's distance to the left of the origin. Extend the levels to the left and join their points of intersection on the  $Y$  axis with the pole and label these oblique lines  $o$ ,  $k$ ,  $l$ , and  $m$ . Locate  $J$  at a unit's distance below the origin. Draw a tangent parallel to  $o$  extending from  $J$  to the first riser and from its extremity draw a tangent parallel to  $k$  extending to the second riser. From the last point draw a tangent parallel to  $l$  extending to the third riser. Continue to draw tangents between consecutive risers. The steps on the right half of the loop were constructed to have the same levels as the steps on the left half of the loop. The tangents for the loop therefore began with  $o$  and were followed in order by  $k$ ,  $l$ ,  $m$ ,  $l$ ,  $k$ , and  $o$ . The integral curve is then drawn so as to touch the tangent lines at  $J$ ,  $K$ ,  $L$ ,  $M$ ,  $P$ ,  $Q$ , and  $R$ , which are the respective points of intersections of the tangents with the extended ordinates through  $J'$ ,  $K'$ ,  $L'$ ,  $M'$ ,  $P'$ ,  $Q'$ ,  $R'$ . The tangents to the positive

loop of  $y_1$  have positive slopes, whereas the tangents to the negative loop of  $y_1$  have negative slopes. The tangents to the lower loop are determined by constructing parallels to another set of oblique lines drawn from the pole to those points on the  $Y$  axis which are crossed by the lower levels, as shown in Figs. 181 and 182. The integral of a sine curve is a sine curve lagging  $90^\circ$  behind the derivative,

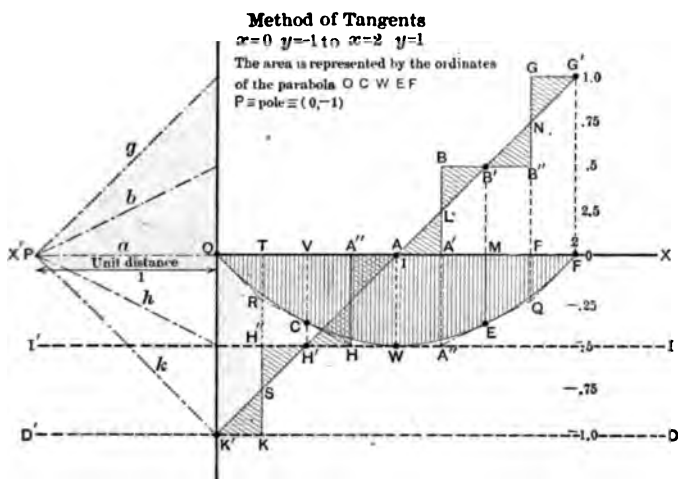


FIG. 181.—The Graphic Integration of a Linear Graph.

as expressed in (59) and (60), and which is also substantiated by form 14 in Table XXXV.

DERIVATIVE	INTEGRAL
(59) $y_1 = \sin \theta = \frac{dy_2}{d\theta}$	(60) $y_2 = -\cos \theta = \sin\left(\theta - \frac{\pi}{2}\right)$

**17. The Graphic Integration of a Linear Equation.** In Fig. 181 the linear graph  $K'G'$  is represented by Eq. (61) and its integral by the parabola  $OCWEF$ , whose equation is expressed by (62):

DERIVATIVE	INTEGRAL
(61) $y_1 = -1 + x$	(62) $y_2 = -x + \frac{1}{2}x^2$

The area of the stepped figure  $K'KH''HA''A$  replaces the negative area between the  $X$  axis and the line  $K'A$ , whereas the stepped figure  $AA'BB''GG'$  replaces the positive area between the  $X$  axis and the line  $AG'$ . The levels are projected over to the  $Y$  axis and the intersections of the former on the latter are joined with the pole  $P$ , which is located at a unit's distance to the left of the origin. The tangent  $OR$  is parallel to  $k$  and extends from the origin to the first riser  $TK$ . The tangent  $RH$  is parallel to  $h$  and extends between the first and second risers. The tangent  $HA''$  is parallel to  $a$  and extends between the second and third risers. The tangent  $A''Q$  is parallel to  $b$  and extends between the third and fourth risers. The tangent  $QF$  is parallel to  $g$  and extends from the fourth riser to  $F$ , which is the last point on the parabola. The parabola, i.e., the integral curve, touches the tangents at  $O, C, W, E$ , and  $F$ , which are obtained by projecting  $K', H', A, B', G'$  from the derivative curve. The **ordinates** of the integral curve **measure the area** bounded by the line  $K'G'$ , the  $X$  axis, the  $Y$  axis, and the extended ordinate. Thus  $VC$  measures the area  $K'H'VO$ .  $VC = -.375$  and the area of

$$K'H'VO = -\frac{OV}{2}(OK' + VH') = -\frac{.5}{2}(1 + .5) = -.375.$$

The ordinates  $VC$ ,  $OK'$ , and  $VH$ , as well as the area of  $K'H'VO$  are negative, since they extend below the  $X$  axis. The ordinate

$$AW = -.5 = \text{area } OK'A = \frac{-OK' \times OA}{2} = -\frac{1 \times 1}{2}.$$

The ordinate  $ME = -.375$  = the excess of the negative area  $OK'A$  over the positive area  $AMB'$

$$OK'A - AMB' = -.5 - .5 \times .5 \times .5 = -.375.$$

The ordinate at  $F$  equals zero, which indicates that the excess of the negative area  $OK'A$  over the positive area  $AFG'$  is zero.

**18. Double Integration.** Any curve may be regarded either a derivative or a primitive. Thus the parabola  $CEI'Z'G$  in Fig. 182, is the integral of the linear graph 41. The cubic curve  $VHJKLMN$  is the integral of

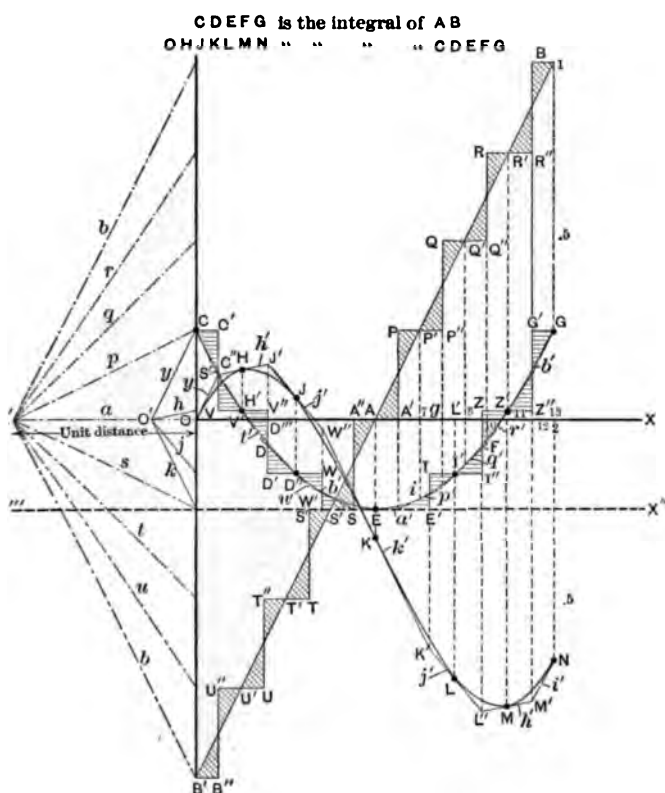


FIG. 182.—Double Integration.

the parabola  $CEI'Z'G$ . In this sense the parabola is the first derivative of the cubic and the linear graph is a second derivative of the cubic, since it is a derivative of a derivative. The symbol for the second derivative is  $\frac{d}{dx} \frac{d}{dx}$ .

which contracts into  $\frac{d^2}{dx^2}$  but is not the same as  $\left(\frac{d}{dx}\right)^2$ , since the latter is obtained by squaring the first derivative.

## LINEAR GRAPH

## A DERIVATIVE

$$(63) \quad y_1 = -1 + x, \quad (63a) \quad \frac{d(y_1)}{dx} = -1 + x,$$

## A SECOND DERIVATIVE

$$(63b) \quad \frac{d^2}{dx^2}(y_1) = -1 + x.$$

## THE PARABOLA

## A DERIVATIVE

$$(64) \quad y_2 = .25 - x + .5x^2, \quad (64a) \quad \frac{d}{dx}(y_2) = .25 - x + .5x^2.$$

## THE CUBIC

$$(65) \quad y_3 = .25x - .5x^2 + \frac{1}{6}x^3.$$

The symbol of double integration is  $\int \int$ , as shown in (66). If one integration is performed, as shown in (67), then double integration reduces to single integration and (66) becomes (68) and the latter reduces to the cubic (65).

The innermost integral symbol operates upon the expression as far as and including the innermost differential, as shown by the brace in (66).

$$(66) \quad y_3 = \int \int \underbrace{(-1 + x) dx dx}.$$

$$(67) \quad \int (-1 + x) dx = - \int dx + \int x dx + \text{a constant} \\ = .25 - x + .5x^2.$$

The derivative of a constant is zero. Therefore an arbitrary constant may be added to the integral after performing an integration. The value of the constant corresponds to the  $Y$  intercept of the integral graph. The parabola for convenience and clarity was constructed from the point  $C$  on the  $Y$  axis and therefore the constant

of integration = .25, which is the numeric value of the  $Y$  intercept.

$$68) \quad y_3 = \int (.25 - x + .5x^2) dx = \int .25 dx - \int x dx + \int .5x^2 dx.$$

$$65) \quad y_3 = .25x - .5x^2 + \frac{x^3}{6}.$$

*Observation.* The degree of the integral is one greater than the degree of its derivative when both represent algebraic equations. When an integral curve passes through the origin then its ordinate measures that area under its derivative curve, which is bounded by the simultaneous ordinate of the derivative, the derivative curve, and the two axes.

When the integral curve does not pass through the origin then the area of the derivative is measured by subtracting the ordinate of the integral curve from the numeric value of the latter's  $Y$  intercept. In other words, a horizontal line is drawn through the  $Y$  intercept and the distance between it and the integral curve measures the area of the derivative curve at a simultaneous ordinate.

**19. The Scales of Integral Curves.** If a derivative curve is plotted so that its  $X$  and  $Y$  scales are equal then the same scales apply to the integral curve. This is shown in Figs. 180 and 181. In Fig. 182 the vertical scale of the linear graph is one-half of its horizontal scale. In such cases when the unit polar distance  $X'O$  is constructed according to the horizontal scale, then the scales of the integral curve (parabola) will be like the scales of its derivative. The polar distance  $OO'$  for integrating the parabola in Fig. 182 is one-fourth of the horizontal scale, and therefore the vertical scale of the cubic is reduced to one-fourth of the vertical scale of the parabola. Therefore the vertical scale of the cubic equals one-eighth of its horizontal scale. The ratio of the polar distance to a unit distance on the horizontal axis is called the **polar scale factor**.

*Observation. In the process of graphic integration the horizontal scales of both integral and derivative curves are identical. The vertical scales of the integral and derivative curves are in the same ratio as the polar scale factor.*

**Ex. 20.** Integrate the sine curve  $y_1 = \sin \theta$  as shown in Fig. 180, wherein  $y_2$  is the integral of  $y_1$ . Construct a straight line  $JZ$  through  $U'$  from which measure the height of the points of  $y_1$ . The latter give the areas between the sine curve  $y_1$  and the  $x$  axis from the origin to the simultaneous ordinate. Thus  $L$  is .5 of a unit above the line  $JZ$ , and therefore the area of  $J'L'\frac{\pi}{3} = .5$  square units. In other words, the ordinate of the sine curve has swept through or generated an area of  $\frac{1}{2}$  square unit. Determine (a) the area of half a sine loop; (b) the area of a loop; (c) divide the area of a loop by  $\pi$  its length and determine the mean value of its ordinates; (d) the area for one cycle. Why do the ordinates of  $y_2$  decrease after passing  $R$ .

**Ex. 21.** In Fig. 182 draw a parallel  $CG$  to  $XX'$  through  $C$ . Measure the segments of five perpendiculars extending between the parabola and  $CG$  and show that their numeric values express the area between  $B'B$  and  $XX'$  and from the measured perpendicular to the origin.

**Ex. 22.** Integrate the following graphs:

- (a)  $y = .5x$ ;      (b)  $y = .75x$ ;      (c)  $y = x$ ;      (d)  $y = 1.25x$ ;  
 (e)  $y = 1.5x$ ;      (f)  $y = 2x$ ;      (g)  $y = 3x$ ;      (h)  $y = 4x$ ;  
 (i)  $y = 5x$ ;      (j)  $y = 6x$ ;      (k)  $y = 7x$ ;      (l)  $y = 10x$ .

In each case begin the integral curve at the origin, but verify the work by checking the ordinate of the integral with the area of the triangle.

**Ex. 23.** Integrate the following graphs:

- (a)  $y = -1 + x$ ;      (b)  $y = -.75 + x$ ;      (c)  $y = -.5 + x$ ;  
 (d)  $y = -.25 + x$ ;      (e)  $y = .25 + x$ ;      (f)  $y = .5 + x$ ;  
 (g)  $y = .75 + x$ ;      (h)  $y = -1 + 2x$ ;      (i)  $y = -1 + .5x$ ;  
 (j)  $y = .5 - x$ ;      (k)  $y = 1 - 2x$ ;      (l)  $y = -.1 - .5x$ ;  
 (m)  $y = -.5 - .1x$ .

n each case draw two integral curves so that one integral intercepts the  $Y$  axis at the origin and so that the other intercepts the  $Y$  axis .25 above the origin. Use a polar factor of .5 for the second of the two integral curves. Write the equations for the integral curves.

**Ex. 24.** Perform graphic differentiation upon the two integral curves in Ex. 23, showing that both integral curves give the same derivative.

**Ex. 25.** Integrate the integral curves obtained in Ex. 22.

**Ex. 26.** Integrate the integral curves obtained in Ex. 23.

**Ex. 27.** Construct a sine curve  $y = \sin \theta$  as shown in Fig. 183, and measure its ordinates at the points  $O, A'', B'', C'', D'', E'', F'', G'', H'',$  etc. Square the respective ordinates. Plot the

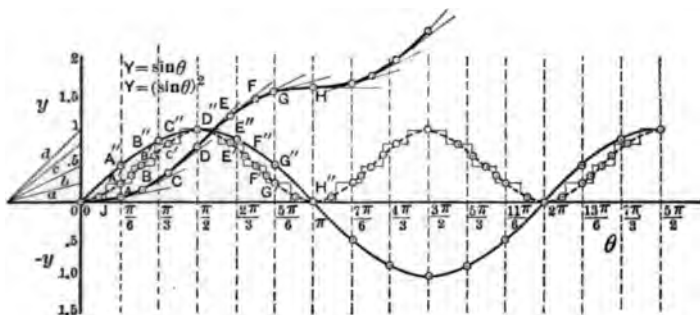


FIG. 183.—The Integral of  $Y = \sin^2 \theta$ .

respective squares on the simultaneous ordinates obtaining the points  $A'B'C'D'E'F'G'H''$ , etc., which lie upon the  $\sin^2$  curve whose equation is expressed in (68).

$$(68) \quad y = \sin^2 \theta = (\sin \theta)^2.$$

The graph of (68) will be readily identified with the power curves, shown in Figs. 169 and 170. Integrate the graph of (68), which is shown as  $ABCDEFGH$ , etc., in Fig. 183. Determine the area of the  $\sin^2 \theta$  curve under one loop and divide this value by  $\pi$  the length of the loop. The quotient is the average height of the sine curve, and the square root of the latter is defined as the mean effective value of the sine curve. The graph of (68) is also a sine curve whose axis lies midway between  $D$  and the axis of the sine curve and therefore the area of (68) may be obtained from (a) and (b) in Ex. 20.

### 20. The Integration Formula for the Area of a Curve.

In the preceding paragraphs we have observed that the successive ordinates of the derivative curve represent the successive rates of change of the integral curve and inversely the successive ordinates of the integral curve represent the successive areas under the derivative curve. In all such cases if we designate the ordinates of the integral and derivative curves by  $y_1$  and  $y$  respectively, we have the relations expressed in (69), (70), and (71).

$$(69) \quad y = \frac{dy_1}{dx} \qquad (70) \quad dy_1 = y dx.$$

$$(71) \quad y_1 = \int y dx.$$

(71) is called the **areal formula**, i.e., the integration formula for obtaining the area of any curve whose ordinate is represented by  $y$ . The area (74) under the curve  $ODB$  in Fig. 184, whose equation is expressed in (72), is obtained by integrating (73) after substituting the value of  $y$  in the areal formula (71).

$$(72) \quad y_1 = ax^n.$$

$$(73) \quad y_1 = \text{area under } ODB = \int ax^n dx.$$

$$(74) \quad y_1 = \frac{ax^{n+1}}{n+1}.$$

(73) may be interpreted as the limit of the sum of all the **elementary strips**, such as  $D$  in Fig. 184, whose dimensions are  $y$  and  $dx$ . An elementary strip represents the rate of change of the area of the derivative curve. These strips extend from the origin indefinitely to the right and therefore (73) expresses an indefinite area and accordingly formulas (71) and (73) are called **indefinite integrals**.

If it is desired to limit the area between the origin and ordinate whose abscissa is  $c$ , this fact defines an area prescribes a corresponding definite integral. The limits are written adjacent to the integral symbol, as shown (75).

$$y_1 = \int_0^c y dx = \left[ \frac{ax^{n+1}}{n+1} \right]_0^c.$$

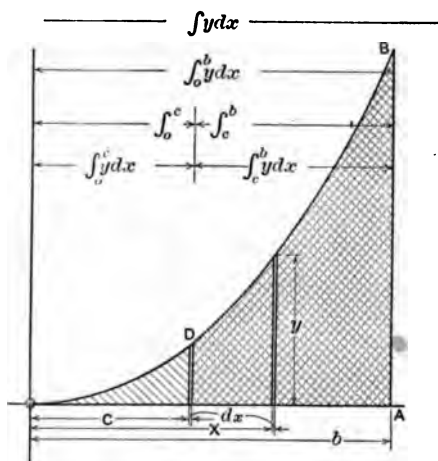


FIG. 184.—The Areal Integral Formula.

The right-hand member of (75) is **evaluated**, i.e., its value is determined by substituting the limiting value  $c$  in place of  $x$ , and therefore (75) becomes (76):

$$y_1 = \text{area (from 0 to } c) = \frac{ac^{n+1}}{n+1}.$$

For the special equation  $y=2x^3$  the area between the curve and an ordinate whose distance = 1.5 units is expressed (77):

$$y_1 = \frac{2(1.5)^4}{4} = 2.53 + \text{square units.}$$

If the area extends from the origin to an ordinate  $b$  whose abscissa = 4 units, these facts are symbolized in (78), which evaluates to (79):

$$(78) \quad y_1 = \int_0^b y dx = \left. \frac{ax^{n+1}}{n+1} \right]_0^b = \frac{ab^{n+1}}{n+1}.$$

$$(79) \quad y_1 = \frac{2(4)^4}{4} = 128 \text{ square units.}$$

The area between the two ordinates whose respective abscissas are  $b$  and  $c$  is indicated in (80):

$$(80) \quad y_1 = \int_c^b y dx = \left. \frac{ax^{n+1}}{n+1} \right]_c^b.$$

The evaluation of (80) gives us the difference between the values obtained in (76) and (78). Therefore (80) becomes (81), which reduces to (82):

$$(81) \quad y_1 = \left. \frac{ax^{n+1}}{n+1} \right]_c^b = \frac{a(b)^{n+1}}{n+1} - \frac{a(c)^{n+1}}{n+1}.$$

$$(82) \quad y_1 = 128 - 2.53 = 125.47 \text{ square units.}$$

*Observation.* A definite integral is a prescribed summation and is pictured by a definite area under a curve. The limits of the integral indicate the extent of the summation process. The **integrand** or quantity affected by the integral sign is the product of the ordinate of a curve times the rate of change of its abscissa. The limiting values of the abscissas of the curve are placed to the right of the integral symbol, so that the greater or superior limit appears above and the lesser or inferior limit appears below the integrand. The integration is performed by substituting the equivalent form from the table of integrals. The evaluation of the form is the numeric excess of the two results which are obtained by substituting first the superior limit and secondly, the inferior limit.

21. The area of the sine loop (84), as shown in Fig. 185, may be considered as the integral, i.e., limit of the sum of the elementary strips of dimension  $y$  and  $d\theta$ , as expressed in (83). The limits of  $\theta$  are zero and  $\pi$ , which indicate the extent of the sine loop along the horizontal or  $\theta$  axis.

$$(83) \quad y_1 = \text{area sine loop} = \int_0^{\pi} y d\theta.$$

$$(84) \quad y = \sin \theta.$$

$$(85) \quad \therefore y_1 = \int_0^{\pi} \sin \theta d\theta = -\cos \theta \Big|_0^{\pi}.$$

$$(86) \quad \therefore y_1 = (-\cos \pi) - (-\cos 0) = -(-1) - (-1) = 2 \text{ square units.}$$

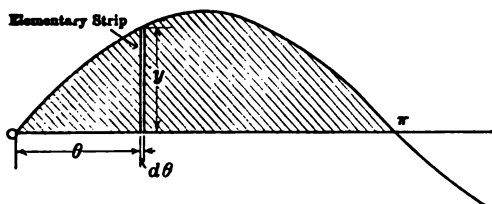


FIG. 185.—The Area of the Sine Loop.

The average value ( $y_{\text{aver}}$ ) of the ordinates of the sine loop whose equation is given in (84) is obtained by dividing the area of the sine loop by its length as expressed in (87):

$$(87) \quad y_{\text{aver}} = \frac{y_1}{\pi} = \frac{\int_0^{\pi} \sin \theta d\theta}{\pi} = \frac{2}{\pi} = .636.$$

The maximum ordinate of the current curve, i.e., sine loop in Fig. 186 is  $I$ , and therefore its average value =  $.636I$ . In other words, the average value of a current which follows the sine law is .636 times its maximum value.

**Ex. 23.** Determine the areas under the following curves, between the origin and the abscissa 5, between the origin and the abscissa 1, between the

abscissas 5 and 10: (a), (b), (c), (d), (e), (f), (g), (h), (i), (j), in Ex. 22; (a), (b), (c), (d), (e), (f), (g), (h), (i), (j), (k), (l), (m), in Ex. 23.

**Ex. 29.** Determine the area, under the following curves, between the origin and abscissa 1, between the origin and abscissa 2, between the abscissas 1 and 2: (a), (b), (c), (d), (e), (f), in Ex. 7; (a), (b), (c), (d), (e), (f) in Ex. 8; (d), (e), (f), in Ex. 19.

**Ex. 30.** Determine the area under one loop of the following curves: (a), (b), (c), (d), in Ex. 10; (a), (b), (c), (d), (e), (f), in Ex. 17; (a), (b), (c), (d), (e), (f), (g), in Ex. 18.

**22. Mean Effective Value of a Harmonic E.M.F. or Current.** The heating effect representing the useful energy spent in a circuit varies as the square of the current. In a direct-current circuit we recognize the familiar expression

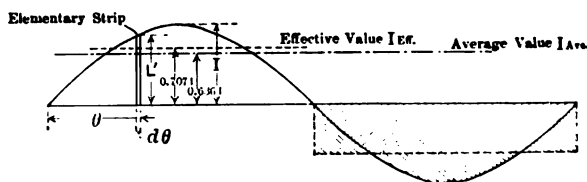


FIG. 186.—The Distinction between Average and Effective Values.

$I^2R$  loss. An equivalent loss may take place with an alternating current. There is this distinction, however, that with the direct current,  $I$  remains constant and therefore  $I^2$  remains constant, whereas with alternating current  $I$  is variable and therefore  $I^2$  is different at each instant. In order to produce the same energy effect the average of the ( $I^2_{alt}$ ) of the alternating current must equal the ( $I^2_{dir}$ ) of the direct current. The square root of the average of the squares of the successive instantaneous values of the alternating current is called its **effective value** ( $I_{eff}$ ).

$$(88) \quad I^2_{dir} = I^2_{aver}.$$

$$(89) \quad I_{eff} = \sqrt{\frac{\text{sum of squares of ordinates}}{\text{number of ordinates}}} = \sqrt{\frac{\text{area } \sin^2 \text{ curve}}{\text{length } \sin^2 \text{ curve}}}.$$

The average of the squares may be obtained by integrating the area under the curve of squares of current and dividing the area by the length of the loop.

(90) is the equation of an alternating current and (91) the corresponding equation of its square. These curves are illustrated in Fig. 183. The area under the  $\sin^2$  curve is given in (92):

$$(90) \quad i = I_m \sin \theta.$$

$$(91) \quad i^2 = I_m^2 \sin^2 \theta.$$

$$\begin{aligned} (92) \quad \text{Area} &= \int_0^\pi I_m^2 \sin^2 \theta d\theta = I_m^2 \int_0^\pi \sin^2 \theta d\theta. \\ &= \left( \frac{I_m^2 \theta}{2} - \frac{I_m^2}{4} \cos 2\theta \right) \Big|_0^\pi \\ &= \left( \frac{I_m^2 \pi}{2} - 0 \right) - \left( \frac{I_m^2}{4} \cos 2\pi - \frac{I_m^2}{4} \cos 0 \right) \\ &= \frac{I_m^2 \pi}{2} - \frac{I_m^2}{4} + \frac{I_m^2}{4} = \frac{I_m^2 \pi}{2}. \end{aligned}$$

$$(93) \quad I_{\text{eff}} = \sqrt{\frac{\int_0^\pi I_m^2 \sin^2 \theta d\theta}{\pi}} =$$

$$(94) \quad I_{\text{eff}} = \sqrt{\frac{\frac{I_m^2 \pi}{2}}{\pi}} = \sqrt{\frac{I_m^2}{2}} = \frac{I_m}{\sqrt{2}} = \frac{I_m}{1.414} = .707 I_m.$$

(94) results from substituting (92) and (93). The interpretation of (94) states that the effective value of an alternating current equals .707 times its maximum value. The effective value  $E_{\text{eff}}$  of an E.M.F. is the square root of the average of the squares of the successive instantaneous values of the alternating E.M.F. It may be derived in the manner described for obtaining  $I_{\text{eff}}$  or by considering

the energy loss equal to  $\frac{E_{\text{eff}}^2}{R}$ , as shown in (95), (96), and (97):

$$(95) \quad I_{\text{eff}}^2 R = \frac{E_{\text{eff}}^2}{R}.$$

$$(96) \quad I_{\text{eff}}^2 R^2 = E_{\text{eff}}^2.$$

$$(97) \quad E_{\text{eff}} = I_{\text{eff}} R.$$

The average and effective values of a sine wave are shown in Fig. 186.

**23.** The ordinates of the sine curve are represented in Fig. 187 by the two sets of perpendiculars  $P_1R$ ,  $PR$ ,  $P_2R_2$ ,  $P_3R_3$ , etc., and  $Q_1S_1$ ,  $QS$ ,  $Q_2S_2$ ,  $Q_3S_3$ , etc. If the total number of ordinates is designated by  $n$ , then the ordinates

may be arranged in  $\frac{n}{2}$  pairs of ordinates, such as  $PR$  and

$QS$ , whose corresponding radii vectors are  $90^\circ$  apart and therefore  $PR = I \sin \theta$  and  $QS = I \cos \theta$ . The sum of the squares of each pair of ordinates multiplied by the number of pairs of ordinates, is the sum  $\Sigma$  of the squares of all the ordinates as expressed in (98).  $\Sigma$  is the symbol of summation of a number of quantities which are similarly constituted. In this case each pair of ordinates is of the type  $PR$  and  $QS$ .

$$(98) \quad \frac{\text{Sum of } \frac{n}{2} \text{ pairs of squares of ordinates}}{\text{total number of ordinates}} = \frac{\frac{\Sigma PR^2 + QS^2}{2}}{n} \\ = \frac{\Sigma I_m^2 (\sin^2 \theta + \cos^2 \theta)}{2n} = \frac{\Sigma I_m^2}{2n} = \frac{n I_m^2}{2n} = \frac{I_m^2}{2}.$$

$$(99) \quad I_{\text{eff}} = \sqrt{\frac{I^2}{2}} = .707 I_m.$$

**24. The Moment of Inertia Formula for a Structural Section.** The moment of inertia of a plane figure is expressed in (100), which is the integral of the products obtained by multiplying each elementary area by the square of its distance from the neutral axis.

In Fig. 188 the rectangular section has a width  $b$  and a height  $\left(\frac{D}{2}\right)$  above the neutral axis and a negative height  $\left(-\frac{D}{2}\right)$  below the neutral axis.

$$(100) \quad I = \int \overline{\text{distance}}^2 \times \text{elementary area} = \int y^2 x dy.$$

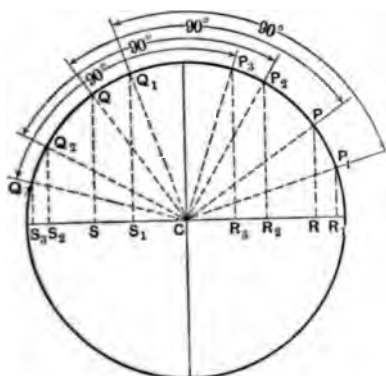


FIG. 187.—The Geometric Representation of the Effective Value of a Sine Loop.

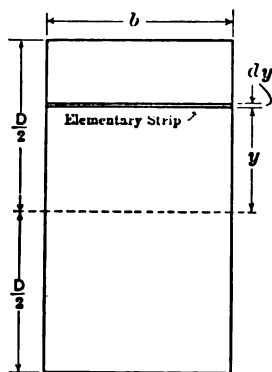


FIG. 188.—The Determination of the Moment of Inertia of a Rectangle.

The elementary strip in Fig. 188 has dimensions  $b$  and  $dy$  and therefore an area  $b dy$ . The distance of the strip from the axis is  $y$ . The strips extend from the superior limit  $\left(\frac{D}{2}\right)$  to the inferior limit  $\left(-\frac{D}{2}\right)$ . Therefore the moment of inertia  $I_{\text{rect}}$  of a rectangular section of width  $b$  and depth  $D$  about its neutral axis is expressed in (101):

$$(101) \quad I_{\text{rect}} = \int_{-\frac{D}{2}}^{\frac{D}{2}} y^2 b dy = b \int_{-\frac{D}{2}}^{\frac{D}{2}} y^2 dy$$

$$= \left[ \frac{by^3}{3} \right]_{-\frac{D}{2}}^{\frac{D}{2}} = \frac{b}{3} \left( \frac{D}{2} \right)^3 - \frac{b}{3} \left( -\frac{D}{2} \right)^3.$$

$$(102) \quad I_{\text{rect}} = \frac{bD^3}{24} + \frac{bD^3}{24} = \frac{bD^3}{12}.$$

**25. The Static Moment Formula for Any Figure.** The static moment of a plane figure is expressed in (103), which is the integral of the products obtained by multiplying each elementary area by its distance from an axis of reference called a moment axis:

$$(103) \quad M = \int \text{distance} \times \text{elementary area} = \int yx dy.$$

For any plane figure the distance  $z$  of the neutral axis from the moment axis may be computed by (104), which is the ratio of the static moment of the figure to its area.

$$(104) \quad z = \frac{\int yx dy}{\int y dx} = \frac{\text{static moment of figure}}{\text{area of figure}}.$$

**26. Graphic Determination of Static Moments, Moments of Inertia, and Centers of Gravity of Sections.** In Fig. 189,  $y_1$  is the graph of a sine curve (105) and  $A_1(y)$  is its integral (106). Therefore the ordinates of  $A_1(y)$  represent the areas under  $y_1$  between the origin and the simultaneous ordinate of  $A_1(y)$ .

The static moment of a sine curve about the  $Y$  axis is expressed in (107), which is obtained from (103) by substituting  $\theta$  for the distance of the elementary strip from the  $Y$  axis and  $y_1 d\theta$  for its area:

$$(105) \quad y_1 = \sin \theta. \quad (106) \quad A_1(y) = \int \sin \theta d\theta.$$

$$(107) \quad M = \int \theta y_1 d\theta.$$

$$(108) \quad M = \int m d\theta = A_2(m).$$

In (107) the product  $\theta y_1$  is replaced by  $m$ , as expressed in (108). If each ordinate of  $y_1$  is multiplied by its

$\theta$  and the respective products plotted on the measured ordinates we shall obtain the curve  $m$ , which is called the moment curve. (108) indicates that the area or integral of the  $m$  curve will give the static moment of the sine curve. The integral of the  $m$  curve is represented by  $A_2(m)$ , and therefore any measured ordinate of  $A_2(m)$  is the numeric value of the moment of that much of the area of the sine

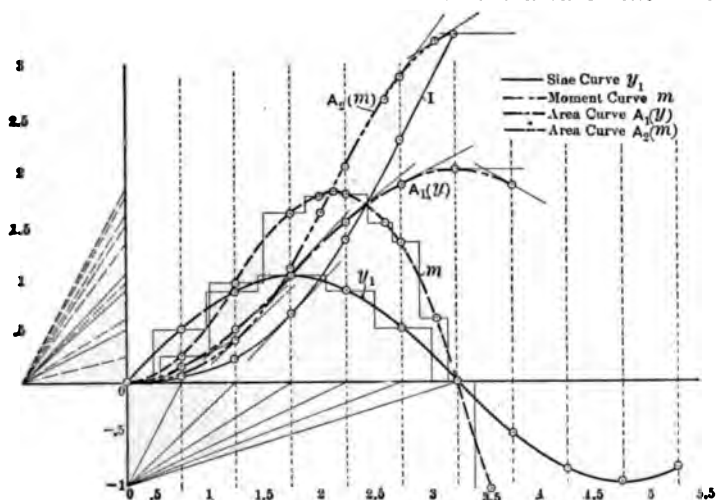


FIG. 189.—Graphic Determination of Static Moment, Moment of Inertia, and Center of Gravity.

curve which extends from the origin to the simultaneous ordinate.

The distance of the center of gravity from the  $Y$  axis for any area of the sine curve extending from the origin to an ordinate is equal to the ratio of the simultaneous ordinates of  $A_2(m)$  and  $A_1(y)$ :

$$(109) \text{ Center of gravity } y_1 = \frac{\text{simultaneous ordinate } A_2(m)}{\text{simultaneous ordinate } A_1(y)}.$$

The center of gravity may be determined by producing the last tangent of the second integral curve until it

by integrating the given  
 Twice the ordinate of the third i  
 value of the moment of inertia  
 may express the moment of in  
 about the  $Y$  axis by (110) and  
 the  $X$  axis by (111):

$$(110) \quad I_y = \int x^2 y dx. \quad ($$

Let  $y$  be the last ordinate o  
 simultaneous ordinate  $y_1$  of t  
 the area of  $y$ :

$$(112) \quad y_1 = \int y dx.$$

The area of the  $y_1$  curve  
 curve (113), whose ordinate is :

$$(113) \quad y_2 = \int y_1 dx = \iint$$

The area of the  $y_2$  curve  
 curve whose ordinate is  $y_3$ , as  $\epsilon$

would be obtained by plotting (110). The ordinate of  $y_2$  in (113) may be compared with (103).

*Observation.* If a given curve be integrated three times in succession then the simultaneous ordinates of the four curves determine the following properties: The ordinate of the first integral curve measures the area of the given curve, the ordinate of the second integral curve measures the static moment of the area, and the ordinate of the third integral curve measures one-half of the moment of inertia of the area. The area is measured from the origin to the simultaneous ordinate and the  $Y$  axis is the axis of reference. When the  $X$  axis is used as the axis of reference then the corresponding pole is located on the  $Y$  axis at a unit's distance below the origin.

**Ex. 31.** Draw the half section of a T-rail and by graphic integration determine its area, the static moment, center of gravity, and moment of inertia from the neutral axis.

**Ex. 32.** Draw the load diagram  $L$  of a beam by constructing ordinates above the  $X$  axis to represent the loads and then join the ends of the ordinates. Integrate  $L$  which gives a curve  $S$  whose simultaneous ordinates express the shear on the beam at that section. Integrate  $S$  obtaining the curve  $B$ . Any simultaneous ordinate of  $B$  expresses the bending moment of the beam at that section. Join the last point  $J$  of  $B$  with the origin  $O$ . Through the pole draw a line parallel to  $OJ$  intersecting the  $Y$  axis at  $H$ . Then  $HO$  is the force acting at the left support. How can the force at the other support be determined from the diagram?

*Observation.* At maximum and minimum points on a primitive curve, the tangents are horizontal and therefore at the corresponding points on the derivative curve the ordinates are zero. Likewise the integral curve will have its maximum and minimum points on the simultaneous ordinates which pass through the zero points of a derivative.

**27. Nomenclature and Interpretation of Differential and Integral Expressions.** Velocity is the time rate of change of linear or angular space. Therefore linear velocity is defined by (116) and angular velocity by (117):

ACCELERATION ( $\alpha$ ) IS THE TIME  
as expressed in (118):

$$(118) \quad \alpha = \frac{dv}{dt} =$$

Substituting the equivalent  
(119):

$$(119) \quad \alpha = \frac{d}{dt} \frac{ds}{dt}$$

The successive derivative :  
but are abbreviated as a sec  
means the operation of differe  
twice in succession. (119) sta  
rate of change of space per secc

**Ex. 33.** A linear velocity is 1  
What is the value of  $\frac{ds}{dt}$  when exp  
(b) in feet per second?

**Ex. 34.** The value of  $\frac{ds}{dt}$  is giv

(a) Temperature coefficient is a rate of change of resistance ( $R$ ) with respect to temperature ( $T$ ). The primitive is

$$R = R_0(1 + .0042T).$$

(b) The coefficient of linear expansion is a rate of change of length ( $L$ ) with respect to temperature ( $T$ ). The primitive for hard-drawn copper wire is

$$L = L_0(1 + .0000094T).$$

(c) The coefficient of expansion is the rate of change of volume ( $V$ ) with temperature ( $T$ ). The primitive for mercury is

$$V = .000181792T + 175 \times 10^{-9}T^2 + 35116 \times 10^{-10}T^3.$$

**Ex. 37.** Interpret (120) in which  $e$ =instantaneous counter E.M.F.,  $L$  the coefficient of self-induction, and  $i$ =instantaneous current.

$$(120) \quad e = -L \frac{di}{dt}.$$

**Ex. 38.** Interpret (121) in which  $i$ =instantaneous current flowing into a condenser,  $C$ =capacity of condenser,  $e$ =instantaneous E.M.F.

$$(121) \quad i = C \frac{de}{dt}.$$

**Ex. 39.** Interpret (122).

$$(122) \quad i = \frac{dq}{dt}.$$

**Ex. 40.** Substitute (122) in (121) and give the authority for writing (123).

$$(123) \quad q = Ce.$$

**Ex. 41.** Interpret (124) in which  $\Phi$  is the magnetic flux,  $E$  is the induced E.M.F. in volts, and  $Z$  the number of turns of wire. What is the meaning of the minus sign?

$$(124) \quad E = -\frac{Z}{10} \frac{d\Phi}{dt}.$$

**Construct** a circle about a center  $O$  and draw its the radius  $I$  of the circle about 1.5 ins. and

velocity of the point  $A$ . The rate of projection of  $DA$  on a vertical diameter is equal by construction. Prove (125) is a definition of trigonometric function

$$(125) \quad i = I \sin \omega t$$

$$(126) \quad \frac{di}{dt} = \omega I \cos \omega t$$

**Ex. 43.** The hysteresis loop,  $ABCD$ , is used to determine the work done in magnetizing a volume of iron. The working formula is expressed in (127). The area of the loop is the work done in one cycle of magnetization. The volume is  $V$ . The iron is subjected to a magnetic field  $H$  and the corresponding number  $B$  of lines of magnetic flux. These values of  $H$  and  $B$  are plotted on a graph. At the beginning of the magnetization, the curve starts at the origin, shown by the extra curve starting at  $A$ .

$$(127) \quad W = \frac{V}{4\pi} \oint H dB$$

The magnetic flux is reversed by changing the direction of the current. The negative limiting values are designated by  $H$  and  $B$ . The integral is identified with the area of the loop. The  $dx$  is replaced by  $H$  and  $dx$  is replaced by  $B$ .

cept of the loop serves as a convenient reference line from which to measure the areas. Trace the hysteresis loop on squared paper and check the area by counting squares.

**Ex. 44.** Construct a sine curve and label it  $A$ . Construct a sine curve  $B$  which leads  $A$  by  $90^\circ$  and whose amplitude equals  $\pi$  times the amplitude of  $A$ . Construct a sine curve  $D$  lagging  $90^\circ$  behind  $A$  and whose amplitude is  $\frac{1}{\pi}$  times the amplitude of  $A$ .

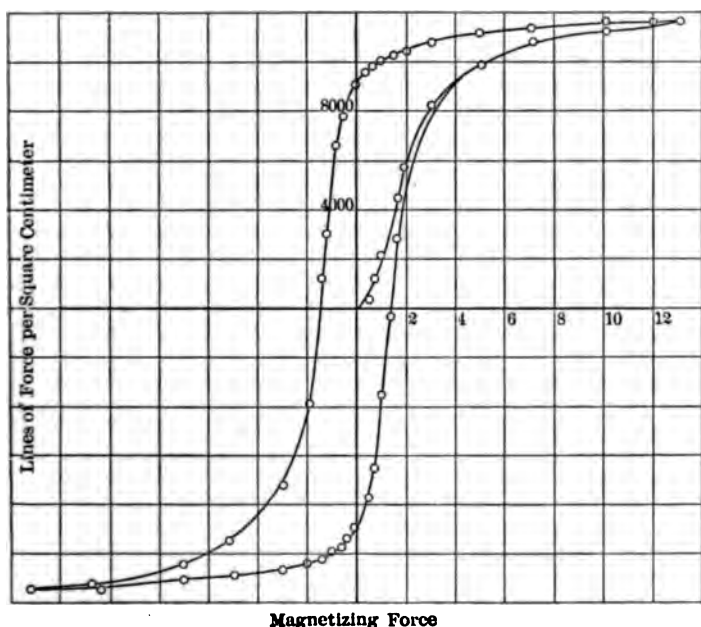


FIG. 190.—A Hysteresis Loop.

Show that equations (128), (129), and (130) correspond to  $B$ ,  $A$ , and  $D$  respectively.

$$(128) \quad \frac{di}{dt} = \pi I \cos \pi t.$$

$$(129) \quad i = I \sin \pi t = \frac{dq}{dt}.$$

$$(130) \quad q = -\frac{I}{\pi} \cos \pi t.$$

Construct  $U$  in phase with  $A$  that of  $A$ . Show that (133) corre

$$(133) \quad e = R$$

Construct  $V$  in phase with  $D$  that of  $D$ . Construct  $X$  the ne and (135) correspond to  $V$  and  $X$

$$(134) \quad e_e = \frac{q}{C}$$

$$(135) \quad e_s' = -\frac{q}{C}$$

$$(136) \quad i = \frac{dq}{dt} =$$

Show that (136) may be obt show that (136) corresponds to  $A$ .

**Ex. 45.** Construct the result corresponding equation. Label the value be represented by  $e_1$ .

$$(137) \quad i = I e^{-\frac{Rt}{L}}.$$

$$(138) \quad \log_e i = \log_e I - \frac{Rt}{L}.$$

$$(139) \quad \frac{di}{dt} = -\frac{Ri}{L}.$$

DATA.

$$R = 3 \, \Omega, \quad L = .04 \text{ henry.}$$

$$I = 36.7 \text{ amps.}$$

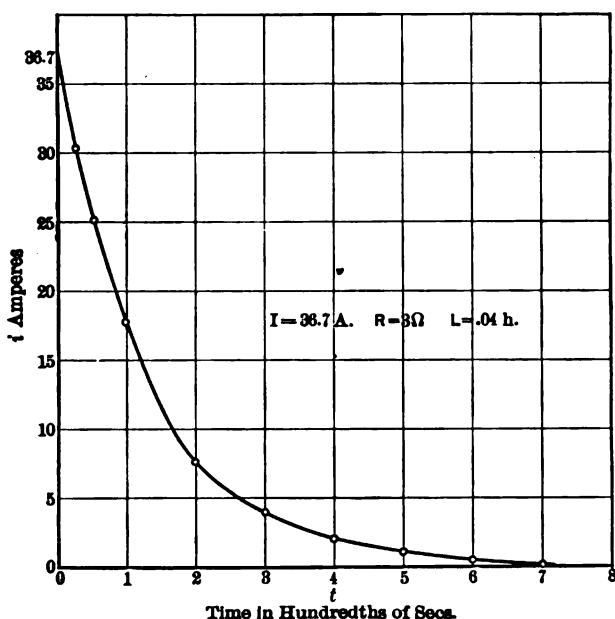


FIG. 191.—Decaying Current Curve.

**Ex. 49.** Construct the graph for growing currents from (140) and show that its derivative is expressed by (141). Construct the derivative and verify its ordinates by substitution in (141). Show that the ordinates of (140) may be obtained by subtracting the ordinates of (137) from  $I$ . Use the data given in Ex. 48.

$$(140) \quad i = I \left( 1 - e^{-\frac{Rt}{L}} \right).$$

$$(141) \quad \frac{di}{dt} = \frac{Ri}{L}.$$

**Ex. 50.** The growth and decay of current in an inductive circuit is expressed in (142) in which  $e$  and  $i$  are instantaneous values of E.M.F. and current respectively,  $R$  is the resistance, and  $L$  the inductance of the circuit. The impressed E.M.F.  $e$  is used for two purposes. State which term represents the part

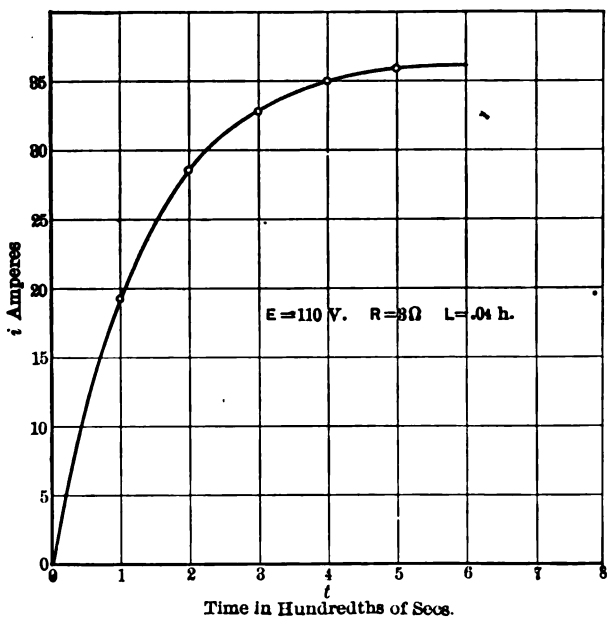


FIG. 192.—Growing Current Curve.

of the E.M.F., corresponding to the resistance drop and which term represents the part of the E.M.F. which causes the current to increase. Let  $e$  become zero, then show that the self-induced E.M.F. is expressed in (141)

$$(142) \quad e = iR + L \frac{di}{dt}.$$

## CHAPTER XXIV

### EXPERIMENTAL CURVES

1. In the study of the operation of electrical machinery and devices it is customary to determine their characteristics by observing their performance during a series of tests. Simultaneous measurements are made of speed, voltage, armature and field current, torque, mechanical output, and a series of **characteristic curves** are plotted from the data. Owing to the inaccuracy of observation, errors of instruments, and fluctuation of power, the plotted data may show slight deviations. A smooth curve is drawn through the average of the points, thus minimizing the inaccuracies.

2. **The Effect of the Variation of Load and Terminal Voltage on the Operation of a Shunt Motor.** In Fig. 193 curve I shows the relation between the total amperes and speed of a shunt motor which is operated with a constant impressed voltage but with variable load. Curve I shows that the speed of a shunt motor decreases with an increase of load.

In Fig. 193 Curve II shows the relation between the total amperes and counter E.M.F. The counter E.M.F. is the difference between the impressed voltage and the  $RI$  drop in the armature. Curve II should have about the same shape as I, since the counter E.M.F. decreases with decrease in speed and increase of  $RI$  in the armature.

In Fig. 194 Curve I shows the relation between terminal volts and speed, curve II shows the relation between ter-

minal volts and field current, and curve III show the relation between terminal volts and current in the armature.

In Fig. (194) Curve I shows that the speed decreases with decrease in terminal voltage. Curve II shows that the field current varies directly with the terminal voltage.

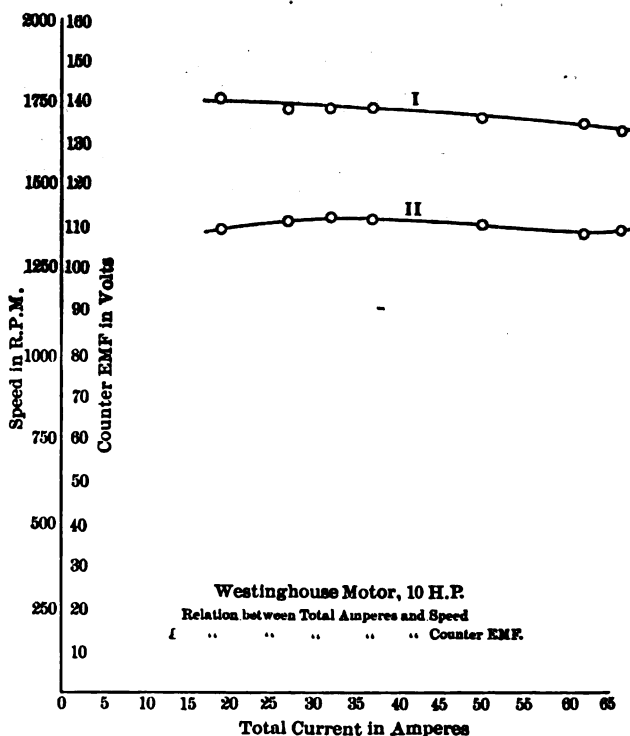


FIG. 193.—The Effect of a Varying Load on the Operation of a Shunt Motor.

Curve III shows that the armature current decreases with the lowering of the terminal voltage until the  $k_1$  curve is reached beyond which the current increases, owing to the counter E.M.F.

3. Characteristic Curves of a Compound Generator. Curves I, II, III, in Fig. 195, show the results of a preliminary test on a compound generator. I was obtained when the machine was operated with the series field cut

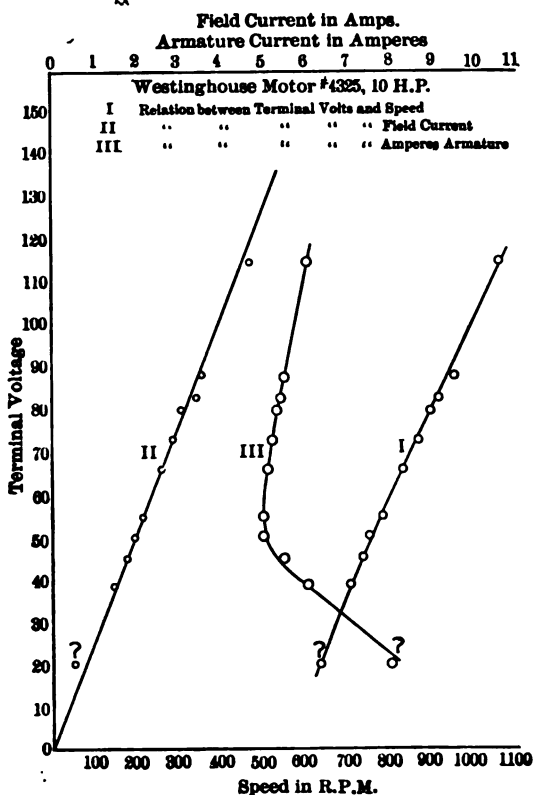


FIG. 194.—The Effect of Variable Terminal Voltage on the Operation of a Shunt Motor.

out and therefore I shows the usual characteristics of a shunt generator. II was obtained when the series coil was connected in **cumulatively**, i.e., so that the magnetic field of the **series** coil was added to the magnetic field of the

shunt coil. III was obtained when the series coil was connected differentially, i.e., so that the magnetic field of the series coil was subtracted from the magnetic field of the shunt coil. Explain the causes for the changing voltages under I, II, and III in Fig. 195.

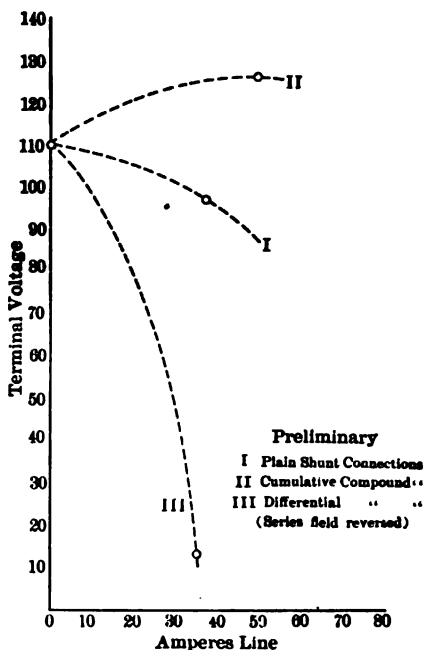


FIG. 195.—The Preliminary Test of a Compound Generator.

In Fig. 196 *a*, *b*, *c* are characteristic curves of the compound generator, corresponding to II in Fig. 195. They show the relation between terminal voltage and line current when starting with normal voltage, 90 per cent of normal voltage and 10 per cent above normal voltage respectively. Explain by means of the different flux densities why the voltage is nearly equal at full load in the three cases.

**4. Characteristic Curves of a Series Generator.** The external characteristic of a generator is the curve which expresses the relation of the terminal voltage to the line current and is represented by I in Fig. 197. The drop in the armature plus the drop in the field is represented

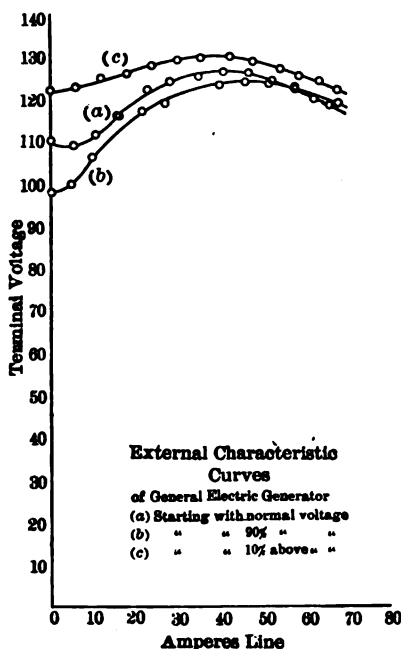


FIG. 196.—The Effect of Normal, Subnormal, and Abnormal Initial Terminal Voltage of a Compound Generator.

by the line  $Ob$  whose slope  $\tan \phi$  equals the sum of the armature and field resistances. Refer to paragraph 4 on page 305 and Fig. 115. The total characteristic curve which expresses the relation between the total generated voltage and total current is shown by curve II, Fig. 197, whose ordinates represent the sum of the simultaneous ordinates of I and  $Ob$ . The line  $Oa$  has a slope  $\tan \theta$  which equals

the sum of the armature field and external resistances. When the external resistance reaches a critical value  $Oa$  will intersect I in the neighborhood marked  $m$ . Under this condition the machine is unstable and if the external resistance exceeds the critical value, the machine cannot

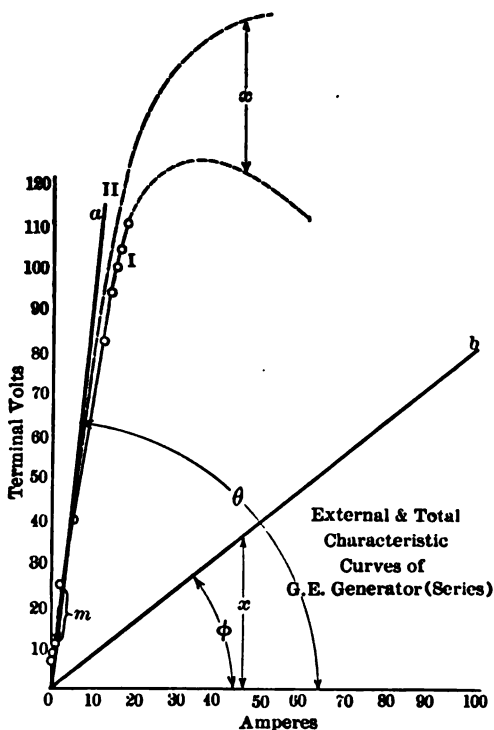


FIG. 197.—The Characteristics of a Series Generator.

build up. Why does I rise to the knee and then fall off in value.

5. The Changes in the Current, Voltage, and Density of the Electrolyte of a Storage Battery during Discharge. The light line shows the change in current, the medium line

shows the change in voltage, and the heavy line shows the change in the density of the electrolyte at successive intervals of time. The rapid initial fall in potential is due to the  $IR$  drop of the battery, showing that on open circuit a storage battery will indicate a higher voltage than will be indicated immediately after closing the circuit.

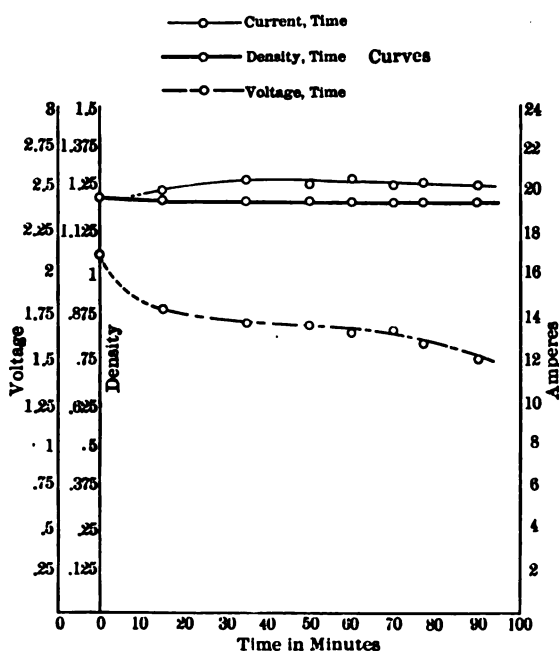


FIG. 198.—The Discharge Characteristics of a Storage Battery.

**6. The Efficiency Test of a Shunt Motor.** In Fig. 199 curve I shows the relation between per cent of no-load speed and horse-power output. The speed drops off slightly as the output of the motor increases. Curve II shows the relation between per cent efficiency and horse-power output. The efficiency is a maximum at full load.

the torque of shunt motors w

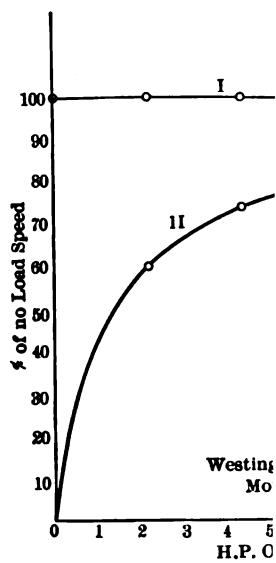


FIG. 199.—The Efficiency

**Ex. 1.** A shunt motor at rest shows a field current of 4.1 amps. When 112 volts are applied to its terminals and the armature shows a drop of .15 volt under the + and - brushes when a current of 5 amps. flows through it. Determine the field resistance and the armature resistance of the motor.

**Ex. 2.** Determine the power curve for the discharge of the storage batteries whose performances are illustrated in Fig. 198. Calculate the power curve and obtain the watt hour curve.

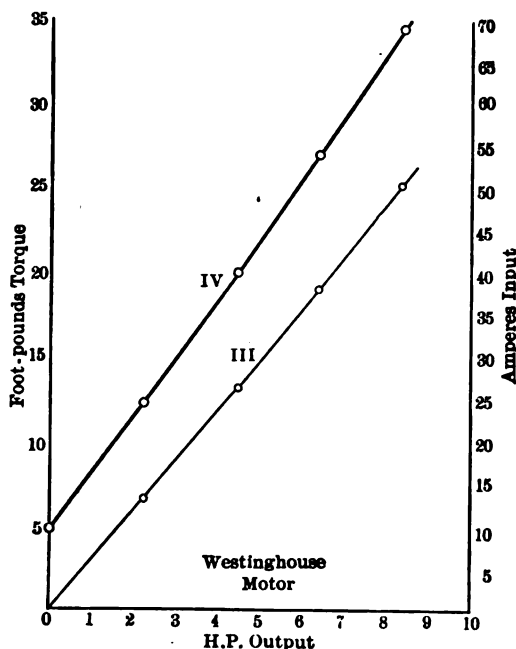


FIG. 200.—Efficiency Test of a Shunt Motor.

**Ex. 3** Determine the electrical efficiency curve of the shunt motor whose performance is illustrated in Fig. 194.

**Ex. 4.** Fig. 203 represents a transverse section through the field poles of a multipolar shunt generator. Fig. 204 represents a longitudinal section through the shaft and a pair of opposite poles of the same machine. Study a similar generator enumerating the various numbered parts and add their respective dimensions together with a description of the kind of materials used in their

construction. Determine the weight of each detachable part wherever possible and estimate the parts which cannot be disassembled.

Weigh the entire machine. The following list identifies numbered parts: (1) base; (2) field frame or yoke; (3) field

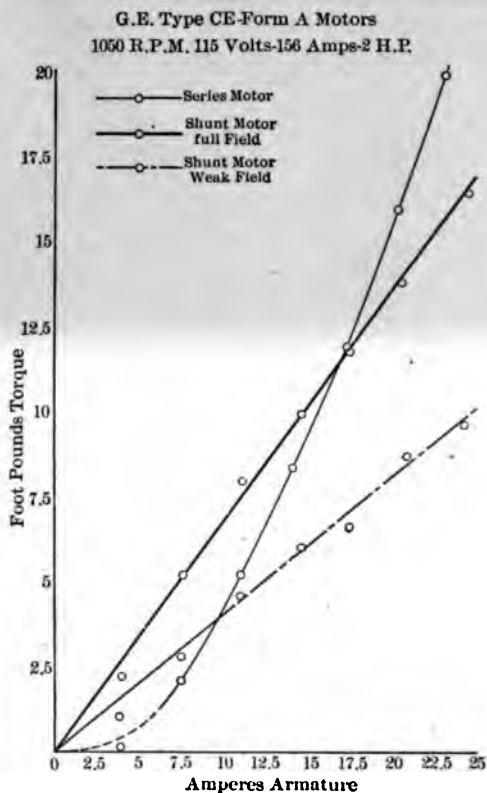


FIG. 201.—Starting Torques of Series and Shunt Motors.

cores; (4) series field coils; (5) shunt field coils; (6) pole shoes; (7) bearing pedestals; (8) armature; (9) armature coils; (10) armature core; (11) armature spider; (12) air-gap; (13) air duct; (14) shaft; (15) commutator; (16) insulation; (17) brush holders; (18) holders; (19) rocker arm; (20) pulley. Supplement

ve data with complete information regarding the manufacturer the type of the machine; its rated capacity, volts, amperes, number of poles, speed, per cent of polar span, both maximum minimum, size of wire, weight per spool, and number of turns both shunt and field coils, length and diameter, and volume of mature core, thickness of laminations, number of ventilating

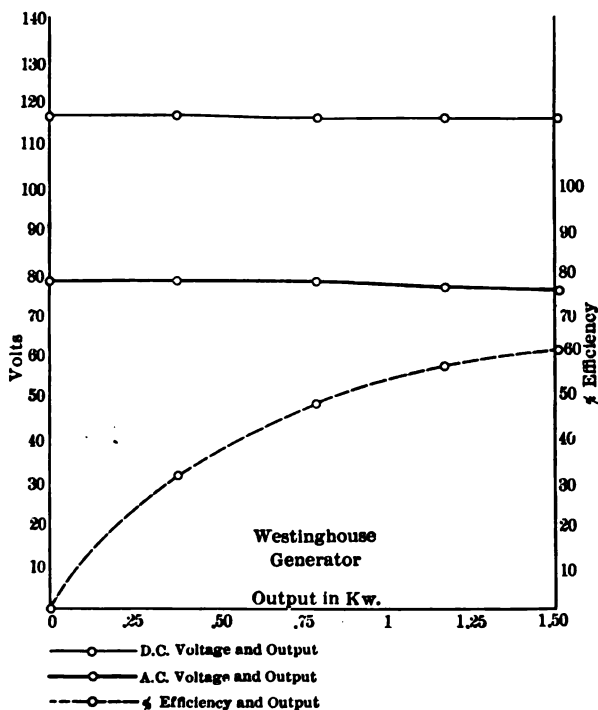


FIG. 202.—The Performance of a Rotary Converter.

ts, number and width of slots, number, width, depth, length, active length of commutator bars, thickness of mica insulation, as per bar, style of armature winding, number of coils, coils slot, turns per coil, length of one turn, weight of wire on mature, number of brush-holder studs, number of brushes per 1, amperes per square inch of contact surfaces, brush pressure. With the above data as a suggestive guide make a complete ~~have~~ machine, showing in detail all calculations.

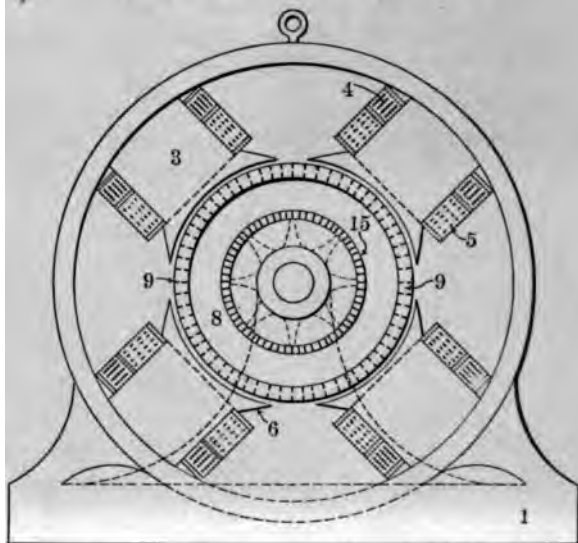


FIG. 203.—Transverse Section of a Multipolar Generator.

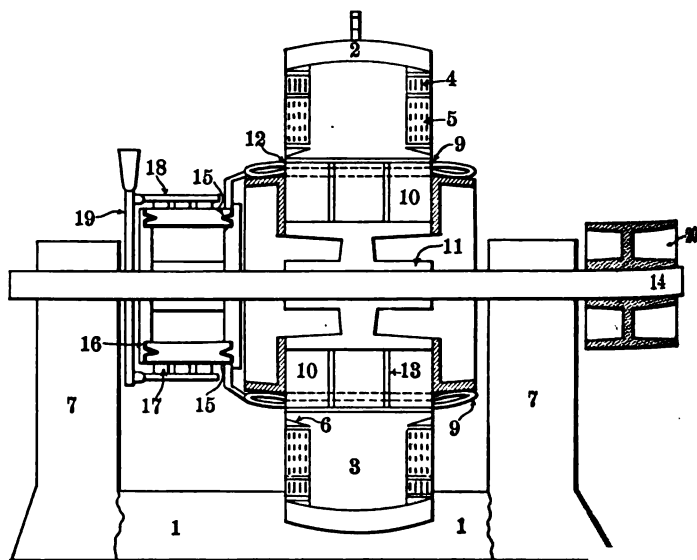


FIG. 204.—Longitudinal Section of a Multinod Generator.

Give an approximate estimate of its operation under various conditions of load. Make a sketch of the machine in cross-section.

**9. Electrical Measuring Instruments.** Station and laboratory measuring instruments may be classified under one of the following heads according to the principle of their construction: (a) Electrodynamic; (b) electromagnetic; (c) magnetodynamic; (d) electrothermic; (e) electrostatic; (f) resonating. These instruments are used for measuring current, voltage, resistance, conductivity, power, frequency, power factor, and also the wattless component of the current. The magnetodynamic type of instrument can be used for direct current measurements only, whereas all other types may be used both for direct and alternating current measurements. Instruments are further classified as indicating, recording, or integrating types. Indicating instruments possess an index or movable pointer whose position on a graduated scale indicates the magnitude of the quantity which is measured. A recording instrument has its index or movable pointer adjusted with a stylus or inking point under which a strip or disc of paper is moved uniformly so as to produce a continuous record. Integrating instruments possess a clock mechanism which propels a series of pointers or indexes over graduated dials. The dials are similar to the dials of gas meters and water meters.

**10. Siemens Dynamometer.** The construction of a Siemens dynamometer is shown in Fig. 205. It is an electrodynamic type and measures the current strength by the reaction between the current flowing through the fixed coils  $C_1$  and  $C_2$  and the movable coil  $MC$ . The movable coil  $MC$  contains a few turns of wire and is suspended by the thread  $F$ . Its terminals dip into the mercury cups  $X$  and  $Y$ . A helical spring  $S$  is attached to the upper end of  $MC$  and also to the movable knob  $K$ . A pointer  $P_1$  is attached to the movable coil  $MC$  and its motion is limited by two stops placed on the graduated dial  $D$ . A second

pointer  $P$  is attached to the knob  $S$ , so that when  $S$  is turned  $P$  indicates a reading on the circular scale of  $D$ .

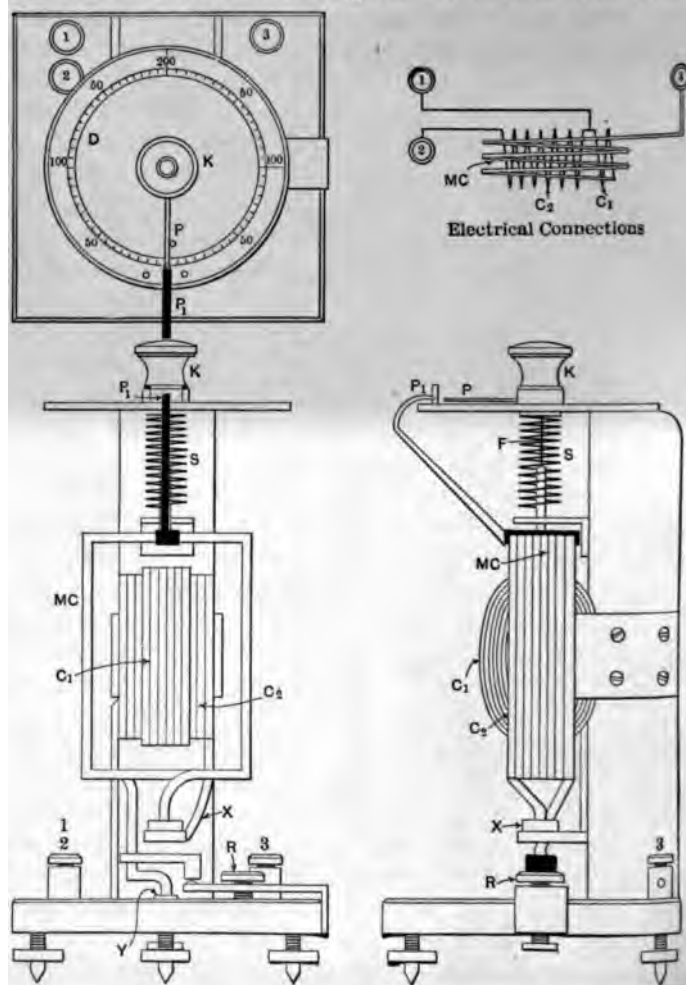


FIG. 205.—A Siemens Dynamometer.

The fixed  
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coil. The fixed coil has two windings designated by  $C_1$  and  $C_2$  which are used for strong and weak currents respectively. The dial is graduated into 400 divisions. The electrical connections show that in operation the movable and fixed coils are connected in series so that when current flows through them the movable coil is deflected. The knob is turned to bring the movable coil back to its normal position at right angles to the fixed coil. The force  $F$  exerted by the spring is proportional to the angle  $\theta$  of torsion, which is read from the position of  $P$ . The force exerted on the movable coil is proportional to the square of the current  $I$ . Therefore

$$F \propto \theta. \quad (1) \quad F = K_1 \theta.$$

$$F \propto I^2. \quad (2) \quad F = K_2 I^2.$$

$$\theta \propto I^2. \quad (3) \quad \therefore \theta = \frac{K_2}{K_1} I^2 = K I^2.$$

$$4) \quad \therefore \frac{\theta_1}{\theta_2} = \frac{I_1^2}{I_2^2}. \quad (5) \quad \frac{I_1}{I_2} = \frac{\sqrt{\theta_1}}{\sqrt{\theta_2}}.$$

The interpretation of (5) states that the current is proportional to the square root of the angle through which the pointer  $P$  has been turned.

**Ex. 5.** What are the constants of two Siemens dynamometers, one of which indicates  $220^\circ$  of angular twist when a current of 20 amperes flows through it and the other indicates  $175^\circ$  of angular twist when 17 amperes flows through it.

11. The Thomson ammeter, shown in Fig. 206, and the Queen ammeter, shown in Fig. 205, are instruments of the electromagnetic type. In such instruments the forces act between the parts in such a way as to reduce the reluctance of the system to a minimum. In the Thomson the current which is to be measured is passed through an inclined cylindric coil  $EE$ . The movable part consists of a vertical shaft mounted between jewel bearings.

A pointer *B*, a soft iron vane *V*, a damper *K*, and one end of the spiral spring are attached to the shaft. The other end of the spring is fixed to the support *F* and in this manner serves as a controlling force to oppose the electromagnetic

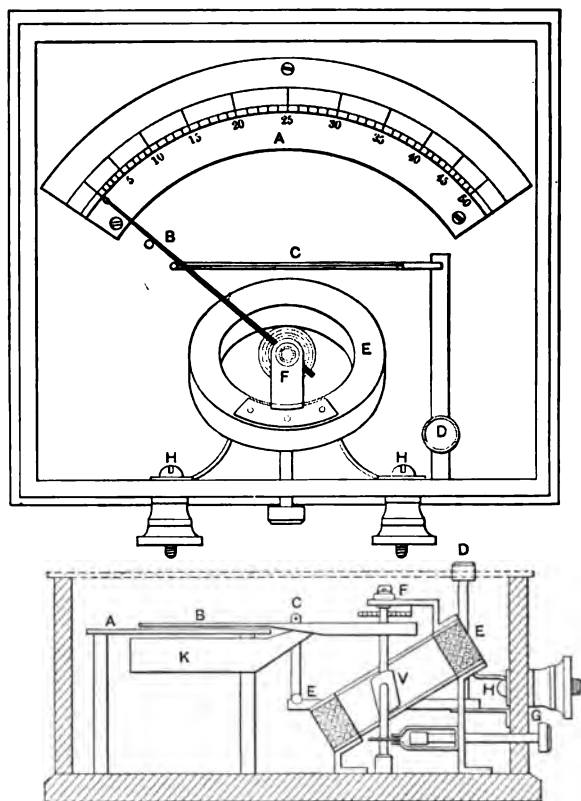


FIG. 206.—Thomson Ammeter.

force between the coil *EE* and the vane *V*, and also serves to restore the pointer to its zero point upon the cessation of current. In the Queen instrument the current which is to be measured is passed through the link-shaped coil *CC*.

The movable part consists of a vertical shaft mounted between jewel bearings. A V-shaped piece of soft iron  $V$ , which has been bent into a semicylindric shape, is attached to the shaft together with a pointer  $P$ , a damper  $F$ , and a spiral spring  $S$ . The other end of the spring is attached to the support  $J$  and serves as a controlling force to oppose the electromagnetic force between the coil  $CC$  and the vane  $V$ . The field in the ends of the coil is a maximum. The iron vane  $V$  tends to turn in such a position that more lines of force may pass through it.

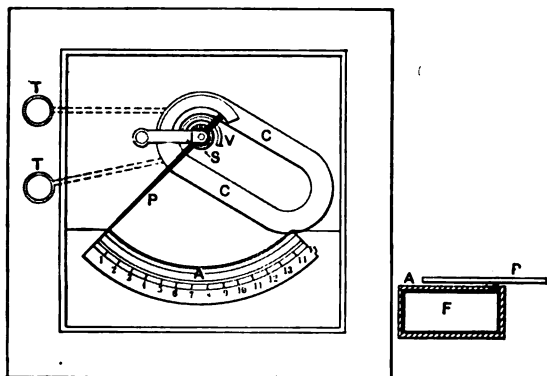


FIG. 207.—Queen Ammeter.

In the Thomson and Queen instruments the force is proportional to the square of the flux density, and therefore if the permeability of the iron is constant the force is proportional to the square of the current. An instrument which measures the effective value of an alternating current, i.e., the square root of the average square of the instantaneous currents must exert a torque which is proportional to the square of the current.

$$F \propto \phi^2. \quad (6) \quad F = K_1 \phi^2.$$

$$\phi \propto I. \quad (7) \quad \phi = K_2 I.$$

$$F \propto I^2. \quad (8) \quad \therefore F = K_1 K_2^2 I^2 = K I^2.$$

The modification of the electromagnetic type is represented by the Whitney D.C. voltmeter, shown in Fig. 208. The current which is to be measured passes through the solenoid *CC* which produces a magnetic field at right angles to the field of the permanent magnet *MM*. The moving part is a jewel-pivoted shaft to which is attached the soft

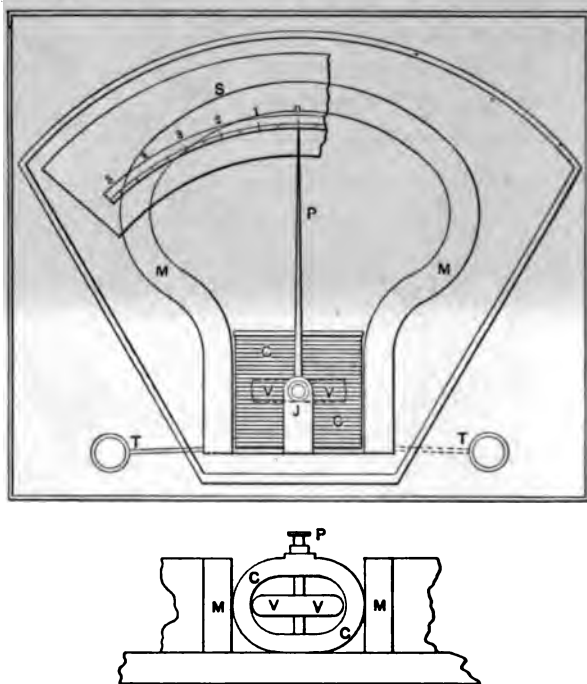


FIG. 208.—Whitney D.C. Voltmeter.

iron vane *V*. The permanent magnet serves as the controlling force to bring the pointer to zero after the cessation of current. The position of the vane when the instrument is operative depends upon the resultant of the magnetic field due to both *MM* and *CC*.

The Weston D.C. ammeter shows

in F

presents the magnetoelectro type and is better known as a permanent magnet instrument. The magnetic field is provided by the permanent magnet to which a pair of soft iron pole pieces *BB* are attached. A soft iron cylindric core is centrally placed between the shaped pole pieces so as to secure a very strong and uniform field. The movable part consists of a jewel-pivoted aluminum frame wound with a coil of fine wire. One spiral control spring is mounted above

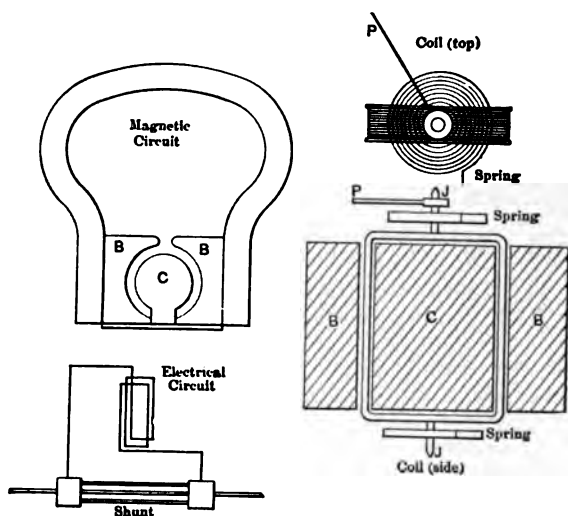


FIG. 209.—Detail of Weston Ammeter.

the coil and another spiral control spring is mounted below the coil and in addition these springs serve to convey the current from the moving coil of the instrument. A pointer is attached to the movement. This instrument has a uniform scale. Owing to its simplicity, its superior mechanical construction, its freedom from errors and its electrical efficiency, the Weston instrument is universally recognized as a precision standard for the calibration of all types of meters. The Weston instrument is a millivoltmeter

and when it is provided with adjustable internal resistances which are placed in series with its movable coil it becomes a voltmeter. When the moving coil is connected in parallel with a low-resistance conductor called a **shunt**, the instrument becomes an ammeter. Shunts are adjusted or tapped, so that they give a drop of 50 millivolts when the maximum current flows through them. Since all ammeters are adjusted so as to be operative over a full-scale deflection with 50 millivolts any ammeter may be used interchangeably on any shunt. Extra external resistances called **multipliers** are used in series with a voltmeter for the purpose of increasing the range of its measurement. The force at each position of the coil is proportional to the current flowing through the coil, since the flux and number of turns remain constant.

**Ex. 6.** Determine the resistance of a 1000-ampere shunt to give a drop of 50 millivolts.

**Ex. 7.** The resistance of a 150-volt Weston voltmeter is 14,000 ohms. Determine the resistance of a suitable multiplier which when used in series with the voltmeter will make it possible to measure a maximum voltage of 600.

**Ex. 8.** A direct reading voltmeter has 150 uniform scale divisions, and is calibrated for 150 volts. The observer can approximate to .1 of a volt throughout the entire range. What is the percentage error due to the error of reading, (a) when the instrument indicates 100 volts; (b) at 150 volts; (c) at 10 volts; at 1 volt. Draw a curve showing the per cent of error plotted vertically and the voltage horizontally.

**Ex. 9.** A direct reading ammeter calibrated with 100 divisions reads 100 amperes. Allow that the observer can approximate to .1 of a scale division. Plot the curve between per cent error and current.

**Ex. 10.** A portable voltmeter is checked with a standard instrument. The former reads respectively: 59.8, 69.7, 79.8, 89.8, 99.6, when the latter reads correspondingly 60, 70, 80, 90, 100 volts. What is the per cent error in the portable instrument for the corresponding standard readings?

**Ex. 11.** A Siemens dynamometer has a uniform scale and it is assumed that the observer can approximate to  $\frac{1}{2}$  scale division. Plot the curve for per cent error using the data in Ex. 5.

**Ex. 12.** In a direct reading A.C. instrument (voltmeter or ammeter) the value of each scale division is the same throughout the scale, although the scale is not uniformly spaced. Suppose each scale division represents one ampere, then in approximating to tenths of a division the error is inversely proportional to the width of the division. What is the relative error in reading 100 amperes compared to 10 amperes?

**Ex. 13.** D.C. instruments give deflections which are proportional to the current, whereas A.C. instruments intended for both D.C. and A.C. measurements, give deflections which are proportional to the square of the current. Considering a 150-amp. instrument with scale divisions corresponding to 1 amp. each what is the relative per cent error for an error made at 10 amp. and 100-amp. divisions respectively?

## CHAPTER XXV

### INDUCTANCE AND CAPACITY

**1. Counter E.M.Fs.** Whenever an E.M.F.,  $e$ , is applied to a circuit it has two functions to perform, i.e., a part  $e_1$  is used to overcome the resistance of the circuit and a second part  $e_2$  is used to overcome any counter E.M.F. which may be set up in the circuit. In a D.C. circuit counter E.M.Fs. are transient and are observed for a brief moment immediately after closing and opening a circuit. In an A.C. circuit counter E.M.Fs. are continuously present and constitute an appreciable element in the consideration of the E.M.F., which is applied to or impressed upon a circuit. This relation is expressed in (1).

$$(1) \qquad e = e_1 + e_2.$$

(1a) Applied E.M.F. = E.M.F. to overcome resistance +  
E.M.F. to overcome counter E.M.F.

Every electric circuit possesses three distinct properties, i.e., **resistance, inductance and capacity**, whose respective magnitudes may range from negligible values to infinite values. These three properties are comparable to the friction, inertia, and elasticity of a mechanical circuit, which may be illustrated by a fluid in motion or by any moving mechanism.

An electric conductor is surrounded by a magnetic field of force and also by an electrostatic field of force owing to its proximity to other conductors and substances in nature. The passage of an electric current in a conductor sets up a disturbance in the surrounding fields of force which requires

a definite expenditure of electric energy in the circuit. Any change in the current which is supplied to a circuit will cause a simultaneous change in the energy which is stored in the surrounding fields. The energy in the fields surrounding a conductor carrying a direct current will persist unchanged as long as the current in the conductor is maintained uniform, i.e., at a constant amperage. A change in the intensity of the current flowing through a conductor produces a simultaneous change in the intensity of the surrounding fields. Again, any change in the intensity of the surrounding field reacts upon the conductor and induces or produces therein a counter E.M.F. of inductance when it is due to the magnetic field and a counter E.M.F. of capacity when it is due to the electrostatic field.

**2. Counter E.M.F. of Self-induction.** The counter E.M.F. of self-induction impedes the introduction, variation, and extinction of a current passing through a conductor and in a D.C. circuit gives rise to the phenomena of growing and decaying currents as illustrated in Figs. 192 and 191 respectively. Therefore a momentary but definite time elapses in a D.C. circuit until an increasing or growing current reaches its maximum value and a momentary but definite time elapses in a D.C. circuit until a decaying or falling current vanishes. These transient effects are proportional to the counter E.M.F.,  $e_L$  of self-induction which is in turn proportional both to the rate of change of the magnetic field set up by the conductor, and to the number  $N$  of turns of conductor linked with the changing field as expressed in (2).

$$(2) \quad e_L = N \frac{d\phi}{dt}.$$

*Observation.* A counter E.M.F. of one volt is induced in a circuit when the number of lines of force passing through or linked with it changes at the rate of  $10^8$  per second.

In a D.C. circuit after the transient effect has disappeared the E.M.F. impressed upon the circuit overcomes resistance alone and therefore the resulting continuous current is expressed in (3), but more precisely by (3a):

$$(3) \quad i = \frac{e_1}{R} \qquad (3a) \quad I = \frac{E}{R}.$$

If an alternating E.M.F.,  $e_1$  be impressed upon the above circuit the resulting current cannot attain the same maximum value  $I$  of the direct circuit, as the delayed current would tend to fall as soon as the E.M.F. is reversed. If the frequency is increased the rate of change of current is increased which means a more rapid change of flux and consequently a greater counter E.M.F. and therefore a reduced maximum value which can be attained by the alternating current.

*Observation.* Inductance is a property of an electric circuit which causes a counter E.M.F. to be set up in it whenever a change of current takes place in it.

**3. Unit of Inductance.** A circuit or piece of apparatus has one unit of inductance called a **henry** ( $L$ ) when a counter E.M.F. of one volt is set up in it by a current which changes at the rate of one ampere per second:

$$(4) \quad L = - \frac{e_2}{\frac{di}{dt}} \qquad (5) \quad e_2 = -L \frac{di}{dt}.$$

In a D.C. circuit the impressed E.M.F. is expressed in (6), which results from substituting in (1) the value of  $e_1$  from (3) and the value of  $e_2$  from (5). The positive sign is placed before  $e_2$ , since it overcomes a counter E.M.F. which is represented by a negative sign, as shown in (5).

$$(6) \quad e = iR + L \frac{di}{dt}.$$

$$(7) \quad e - iR = L \frac{di}{dt}.$$

$$(8) \quad -\frac{R}{L} dt = \frac{di}{i - \frac{e}{R}}.$$

$$(9) \quad -\frac{R}{L} \int_0^t dt = \int_0^i \frac{di}{i - \frac{e}{R}}.$$

$$(10) \quad -\frac{Rt}{L} \Big|_0^t = \log \left( i - \frac{e}{R} \right) \Big|_0^i.$$

$$(11) \quad i = \frac{e}{R} \left( 1 - e^{-\frac{Rt}{L}} \right).$$

In (9) time extends from zero to  $t$  seconds and the current ranges from 0 to  $i$  amperes. Substituting these limiting values in (10) and writing the result with exponential notation we obtain (11), which is the equation of a growing current. The ratio  $\frac{L}{R}$  is called the **time constant of the circuit**.

The equation for a decaying current may be obtained from (1) by considering the condition of the circuit at the instant of removing the impressed E.M.F., i.e., when  $e=0$ , as expressed in (12):

$$(12) \quad e=0=iR+L \frac{di}{dt}.$$

$$(13) \quad -\frac{R}{L} dt = \frac{di}{i}.$$

$$(14) \quad -\frac{R}{L} \int_0^t dt = \int_{\frac{e}{R}}^i \frac{di}{i}.$$

$$(15) \quad -\frac{Rt}{L} \Big|_0^t = \log i \Big|_{\frac{e}{R}}^i.$$

$$(16) \quad i = \frac{e}{R} e^{-\frac{Rt}{L}}.$$

(16) is recognized as the equation of a decaying current.

**Ex. 1.** Determine the inductance of the following mirror galvanometers: (a) resistance = 5000  $\Omega$  and time constant = .0004 sec.; (b) resistance = 2700  $\Omega$  and time constant = .001 sec.; (c) resistance = 1,000,000  $\Omega$  and time constant = .0007 sec.

**4.** In an alternating current circuit the counter E.M.F.  $e_2$  may be expressed in (18) by differentiating (17) and substituting in (12)

$$(17) \quad i = I \sin \omega t.$$

$$(18) \quad e_2 = -\omega IL \cos \omega t.$$

$e_2$  is a maximum when  $\omega t = 0$ , as shown in (19), but when  $\omega t = 0$ , then  $i = 0$ :

$$(19) \quad e_2 = -\omega IL \text{ (maximum).}$$

Therefore the counter E.M.F.  $e_2$  of self-induction lags  $90^\circ$  behind the current and in order to overcome the counter E.M.F.  $e_2$  the impressed E.M.F.  $e$  must have a component  $e'_2$  equal and opposite to  $e_2$  which leads the current by  $90^\circ$ :

$$(20) \quad e'_2 = L \frac{di}{dt} = \omega IL \cos \omega t.$$

$$(21) \quad e'_2 = \omega IL \text{ (maximum).}$$

$$(22) \quad \frac{e'_2}{I} = \omega L = X_L.$$

The maximum value of  $e'_2$  is expressed in (21) which occurs when  $\omega t = 0$  or when  $i = 0$ . (22) results from dividing (21) by  $I$ . The right-hand member of (22)  $\omega L$  is abbreviated  $X_L$ , which is called the **inductive reactance**. From (21)  $X_L$  may be defined as the **factor** which is multiplied into the current to give  $e'_2$  the component of the impressed E.M.F.,  $e$ , which overcomes the counter E.M.F.,  $e_2$ , of self-induction.

**Ex. 2.** Describe and illustrate the graphs of  $e$ ,  $e_1$ ,  $e_2$ ,  $e'$ , and

**Ex. 3.** Construct (11) and (16) on semilog paper.

5. The inductance of a long solenoid is expressed in (23) in which  $L$ =inductance in henrys,  $N$ =number of turns,  $\mu$ =permeability of core,  $A$ =section area of the core in square centimeters,  $l$ =length of core in centimeters,  $r$ =mean radius of the coil in centimeters:

$$(23) \quad L = \frac{1.26N^2\mu A}{10^9 l}.$$

Show that (23) reduces to (24) for a circular section:

$$(24) \quad L = \frac{4\pi^2 N^2 r^2 \mu}{10^9 l}.$$

**Ex. 4.** Determine the inductance of the primary coil of a transformer having 500 turns. The iron core is 70 cm. in length and has a cross-section area of 325 sq.cm. Assume  $\mu$  constantly equal to 1400. Draw the curves of induced voltage for both 25 and 60 cycles when the effective primary current is 10 amps.

6. The inductance in henrys of a length  $l$  of straight cylindric wire of radius  $r$  and permeability  $\mu$  surrounded by an outside medium of permeability one is expressed by the approximate formula (25):

$$(25) \quad L = \frac{2l}{10^9} \left[ \log_e \frac{2l}{r} - 1 + \frac{\mu}{4} \right].$$

The inductance in henrys of a return circuit of two parallel wires each of length  $l$  is expressed in the approximate formula (26), wherein  $d$  is the distance between wires:

$$(26) \quad L = \frac{4l}{10^9} \left[ \log_e \frac{d}{r} + \frac{\mu}{4} - \frac{d}{l} \right].$$

**7. Mutual Inductance.** A current flowing in one conductor will induce an E.M.F. and a corresponding current in a neighboring conductor which is not electrically connected with it. The phenomenon of mutual inductance is similar to the phenomenon of self-inductance. Mutual inductance  $M$  is measured in henrys.

(27) expresses mutual inductance of two coils or solenoids wound on the same core and of approximately equal diameters.  $N_1$  and  $N_2$  are the respective number of turns on the solenoids,  $l$  the total or combined length of the solenoids in centimeters,  $\mu$  = the permeability of the core, and  $r$  the radius of the core in centimeters.

$$(27) \quad M = \frac{4\pi^2 N_1 N_2 r^2 \mu}{10^9 l}.$$

Comparing (27) with (24), we observe that the only difference is the substitution of  $N_1 N_2$  for  $N^2$ .

**Ex. 5.** From (24) and (27) show that  $M = \sqrt{L_1 L_2}$ .

**Ex. 6.** Determine the mutual inductance of two coils of 110 and 1100 turns respectively wound on the core of Ex. 2. In (27)  $\pi r^2$  may be replaced by the sectional area  $A$  of the core.

The mutual inductance of two parallel wires of length  $l$ , radius  $r$ , and at a distance  $d$  apart is expressed in the approximate formula (28).

$$(28) \quad M = \frac{2l}{10^9} \left[ \log_e \frac{2l}{d} - 1 + \frac{d}{l} \right].$$

**Ex. 7.** Transform (25) and (28) so as to express  $L$  and  $M$  with common logarithms.

**Ex. 8.** Solve (25) for  $r$ .

$$(29) \quad \frac{10^9 L}{2l} + 1 - \frac{\mu}{4} = \log_e \frac{2l}{r} \quad \text{Mult., sub. in (25)}$$

$$(30) \quad \frac{2l}{r} = \log_e^{-1} \left[ \frac{10^9 L}{2l} + 1 - \frac{\mu}{4} \right] \quad \text{log}^{-1} \text{ of (29)}$$

$$(31) \quad r = \frac{2l}{\log_e^{-1} \left[ \frac{10^9 L}{2l} + 1 - \frac{\mu}{4} \right]} \quad \text{Inversion of (30)}$$

$$(32) \quad \frac{2l}{r} = e^{\frac{10^9 L}{2l} + 1 - \frac{\mu}{4}} \quad \text{Def. log (29)}$$

$$(33) \quad r = 2l e^{-\left[ \frac{10^9 L}{2l} + 1 - \frac{\mu}{4} \right]} \quad \text{Inversion of (32)}$$

The values of  $r$  are expressed in (31) and (33), the former being the antilogarithmic form and the latter the exponential form. The shape of the graphs of (33) is illustrated in Figs. 144, 171, and 191.

**8. Practical Values of Inductances.** A pair of line wires strung on a telephone pole will have two to four milhenrys per mile, according to the displacement of the wires. The resistance of induction coils varies from 6000 to 80,000 ohms and have a corresponding inductance of 50 to 2000 henrys.

*Observation. Inductance is a property of an electric circuit which manifests itself in a counter E.M.F. whenever a change is made in the strength of the current flowing through a conductor or when a change is made in the strength of the magnetic field surrounding a conductor. Inductance may be distributed along a circuit or it may be concentrated in a coil called a reactor.*

**Ex. 9.** The inductance of a field spool of a generator is expressed in (34) wherein  $\phi$  is the total flux from one pole,  $n$  the number of turns per spool,  $I_f$  the field current of the machine.

$$(34) \quad L = \frac{\phi n}{10^8 I_f}$$

What is the inductance of a bi-polar generator having 3000 turns per spool and field current of 2.5 amps. with a flux of 2 megamaxwells?

**Ex. 10.** What is the inductance of an anchoring of cast steel wound with 200 turns of wire carrying 5 amps. The mean diameter is 5 ins. and the radius of cross-section 1 in.

**Ex. 11.** Determine the self-induction of a solenoid consisting of ten layers of No. 16 double cotton covered wire wound upon a cylindrical core 1 in. diameter. Compute for wood as well as for a soft iron core.

**9. Capacity.** Two neighboring conductors or a conductor and the earth constitute an electric pair when they are not in electric connection. Such a pair is surrounded and separated by an electrostatic field and any change in

the electric condition of either member of the pair produces a corresponding change in the electrostatic field. The latter in turn reacts upon the other member of the pair to change its electric condition. The intensity with which one member of a pair influences the potential and the charge upon the other member is measurable and this property of a circuit is called its **capacity**. The electrostatic field behaves like an elastic medium and therefore the term tension is applied to it. It may be likened to the pistons which separate the fluids in the pipe system, shown in Fig. 210A. It requires a definite quantity of electricity to charge an electric circuit, i.e., to create a potential difference between an electric pair and therefore a definite

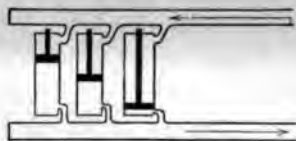


FIG. 210A.—A Mechanical Condenser.

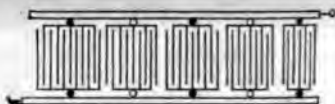


FIG. 210B.—Diagrammatic Representation of a Condenser.

expenditure of energy in order to increase the tension in the electrostatic field surrounding a conductor.

Capacity may be distributed throughout a circuit or it may be concentrated or condensed in a special device called a **condenser** or **storage pocket**, shown diagrammatically in Fig. 210B. A condenser is formed by spreading the exposed surfaces of two insulated conductors and at the same time reducing the distance between them. Two long and broad sheets of metal would make a cumbersome device, but this apparent disadvantage is overcome by winding the insulated strips into a bundle like a bolt of cloth. Another method is to divide the long strips into leaves which are interlaced so that the two sets of alternate leaves are connected to the two respective circuits. As a consequence of its charge and

manifests itself as a counter E.M.F. Its ability to take a charge of electricity is called its **capacity**.

The capacity of a condenser is a numeric quantity and equals the magnitude of the charge which produces a unit difference of potential between its pair of elements. A condenser whose difference of potential is raised one volt,  $E$ , by a charge of one coulomb,  $Q$ , has one **farad**,  $F$ , i.e., one unit capacity. A farad is  $10^{-9}$  times the absolute unit. A subsidiary unit which is more convenient in practice is one microfarad which is one millionth of a farad.

$$(35) \quad C \text{ (farads)} = \frac{Q \text{ (coulombs)}}{E \text{ (volts)}}.$$

**Ex. 12.** Determine  $C$ ,  $Q$ , and  $E$  for the following given values: (a)  $C = .00015$ ,  $E = 2000$ ; (b)  $C = .0001$ ,  $E = 110$ ; (c)  $Q = 30$ ,  $E = 550$ ; (d)  $C = .0002$ ,  $Q = 250$ .

10. The capacity  $C$  of a condenser varies directly with the total area  $A$  of the exposed leaves, and with the dielectric constant  $K$ , i.e., the specific nature of the insulating material and it varies inversely with the thickness  $d$  of the dielectric.

These relations are expressed in (36) in which  $\frac{2248}{10^7}$  is a proportionality factor:

$$(36) \quad C \propto \frac{KA}{d}.$$

$$C = \frac{2248KA}{10^7 d}.$$

The area  $A$  is in square inches and may be expressed as the product of the square inch area of one leaf times the number of dielectric separations.  $d$  expresses the dielectric thickness in mils and has a minimum dimension depending upon the potential to which the condenser is to be subjected in use. A standard condenser is one in which dielectric. The constant  $K$  for air equals

one and is a standard for reference as indicated in Table XXXVI.

TABLE XXXVI.—DIELECTRIC CONSTANTS

Dilectric.	Constant $k$ .	Dielectric.	Constant $k$ .
Air and gases.....	1	Paraffined paper.....	3.65
Manilla paper.....	1.5	Mica.....	4.00-8.00
Paraffin.....	1.68-2.30	Porcelain.....	4.38
Beeswax.....	1.86	Quartz.....	4.55
Paraffin oil.....	1.92	Castor oil.....	4.8
Resin.....	1.77-2.55	Tourmaline.....	6.05
Ebonite.....	2.05-3.15	Crown glass, hard....	6.96
India rubber, pure....	2.22-2.50	Flint glass, light.....	6.57
Turpentine.....	2.23	Flint glass, dense....	10.10
Gutta percha.....	2.45-4.20	Alcohol.....	26
Shellac.....	2.74-3.60	Methyl alcohol.....	34
Sulphur.....	2.9-4.0	Glycerine.....	56
Sperm oil.....	3	Formic acid.....	62
Olive oil.....	3.1	Distilled water.....	76
Glass.....	3.013-3.258		

**Ex. 13.** Prepare a table for the above dielectrics which expresses the microfarad (mf) capacity per square foot of area per mil thickness of dielectric.

**Ex. 14.** (a) A condenser is built with 92 sheets of beeswaxed

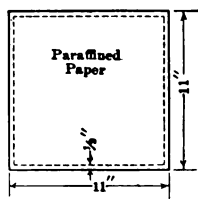


FIG. 211.—A Leaf of Tinfoil for a Condenser.

paper 7x15 ins. and 2 mils thick which separate the alternately connected sheets of tinfoil. The capacity of the condenser is 1.47 microfarads. Determine the value of  $K$ . (b) Paraffined paper 11"×11" is substituted and the tin foil 10"×10" is used as shown in Fig. (211). What is the capacity of the condenser when 100 sheets are used. (c) How many sheets of the latter must be used to give a capacity of 2 mf.?

11. The resistance of a condenser is measurable and is usually expressed in megohms per microfarad and in cable work as megohms per mile.

**Ex. 15.** In Ex. 14 the dielectric resistance is 160 megohms. Express the rating of the condenser in megohms per microfarad.

**12. Practical Values of Capacities.** The capacity of 3 miles of Atlantic cable is about 1 mf. The capacity of an ordinary overhead wire is about .03 mf. A pint size of Leyden jar has  $\frac{1}{100}$  mf. and a quart size has  $\frac{1}{300}$  mf. A 2-mf. condenser is connected with an office telephone equipment. Common potentials in wireless telegraphy are 30,000 volts and common capacities .014 mf.

From (35) we observe that the quantity of electricity which flows into a given condenser is proportional to the pressure in volts which is applied to its terminals. If the applied pressure is supplied by an A.C. circuit then the rate of change of the quantity of electricity flowing into or out of the condenser is proportional to the rate of change of the E.M.F. impressed at its terminals as expressed in (37).

$$(37) \quad \frac{dq}{dt} = C \frac{de}{dt}$$

$$(38) \quad \frac{dq}{dt} = i.$$

$$(39) \quad \therefore i = C \frac{de}{dt}.$$

(39) expresses the instantaneous value of the condenser current.

The quantity of electricity which flows into a condenser of capacity  $C$  farads in time  $t$ , when the impressed E.M.F.  $E$  is constant is expressed in (40):

$$(40) \quad q = CE \left( 1 - e^{-\frac{t}{CR}} \right).$$

$$(41) \quad q = CE e^{-\frac{t}{CR}}.$$

(41) is the equation for the discharge current of a condenser.  $CR$  is called the time constant of the condenser.

**Ex. 16.** Construct the curves of charge and discharge of a condenser of 2 microfarads capacity and one megohm dielectric resistance which is connected to a 110-volt D.C. circuit. Plot these curves on semilog paper.

**13. Formulas for Calculating Capacities.** The microfarad capacity  $C$  of a metal sheathed cable is expressed by (42) in which  $D$  and  $d$  are the respective external and internal diameters in centimeters of a dielectric whose constant is  $K$  and whose length  $l$  is expressed in centimeters.

$$(42) \quad C = \frac{2.413Kl}{10^7 \log_{10} \frac{D}{d}}.$$

The microfarad capacity  $C$  of an aerial line of twin wires is expressed by (43), in which  $l$  is the length of one wire in centimeters,  $r$  is the common radius of the conductors in centimeters, and  $\Delta$  is the spacial distance in centimeters between the conductors:

$$(43) \quad C = \frac{1.25l}{10^9 \log_{10} \frac{\Delta}{r}}.$$

**Ex. 17.** Transform (42) and (43) so that  $C$  represents the microfarad capacity per 1000 ft. and so that the other dimensions are expressed in inches.

**14. Parallel Connection of Condensers.** A variable condenser may be constructed by mounting one set of plates on a movable standard so that the exposed areas of the plates may be altered. The combined capacity of condensers, which are connected in parallel equals the sum of the individual capacities. Suppose the three condensers shown in Fig. 212 have capacities  $C_1$ ,  $C_2$ , and  $C_3$

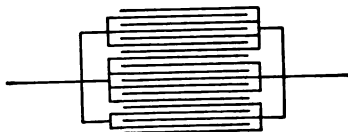


FIG. 212.—Condensers Connected in Parallel.

respectively, and corresponding charges  $Q_1$ ,  $Q_2$ , and  $Q_3$ . The potential difference  $E$  is uniform for each condenser

and is also the potential difference of the combined condensers whose total capacity is designated by  $C$  and whose total charge is designated by  $Q$ . Therefore

$$(44) \quad Q_1 = C_1 E, \quad (45) \quad Q_2 = C_2 E, \quad (46) \quad Q_3 = C_3 E,$$

$$(47) \quad Q = CE;$$

$$(48) \quad Q = Q_1 + Q_2 + Q_3.$$

$$(49) \quad \therefore CE = C_1 E + C_2 E + C_3 E.$$

$$(50) \quad \therefore C = C_1 + C_2 + C_3.$$

(44) . . . (47) are derived from (35). (48) states that the total quantity of electricity passing into the condensers equals the sum of the quantities passing into the individual condensers. The interpretation of (50) gives the law for connecting condensers in parallel; the total capacity equals the sum of the individual capacities.

**15. Series Connection of Condensers.** If three condensers  $C_1$ ,  $C_2$ , and  $C_3$  are connected in series they will all receive an equal charge  $Q$  of electricity, but their potential

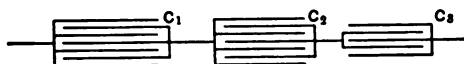


FIG. 213.—Condensers Connected in Series.

differences will be unequal and may be represented by  $E_1$ ,  $E_2$ , and  $E_3$  respectively.  $C$  is the total capacity and  $E$  the total potential difference, therefore

$$(51) \quad Q = C_1 E_1, \quad (52) \quad Q = C_2 E_2, \quad (53) \quad Q = C_3 E_3,$$

$$(54) \quad Q = CE;$$

$$(55) \quad E = E_1 + E_2 + E_3.$$

$$(56) \quad \therefore \frac{Q}{C} = \frac{Q}{C_1} + \frac{Q}{C_2} + \frac{Q}{C_3}.$$

$$(57) \quad \therefore \frac{1}{C} = \frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3}.$$

(55) states that the total difference of potential equals the sum of the differences of potential across the individual condensers. The interpretation of (57) gives the law for connecting condensers in series: the reciprocal of the total capacity equals the sum of the reciprocals of the individual capacities.

**16. Condensers Connected in Series-Parallel Arrangement.** The arrangement of the condensers, shown in Fig. 214, consists of a paralleled pair of condensers  $C_1$  and

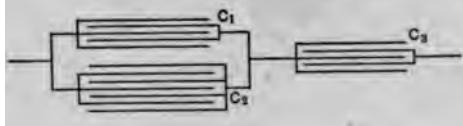


FIG. 214.—Parallel-Series Connections of Condensers.

$C_2$  which are joined in series with a third condenser  $C_3$ . The total capacity is expressed by (58):

$$(58) \quad \frac{1}{C} = \frac{1}{C_1 + C_2} + \frac{1}{C_3}.$$

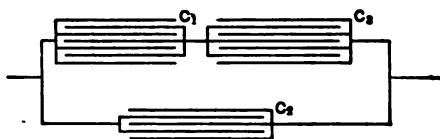


FIG. 215.—Series-Parallel Connections of Condensers.

In Fig. 215  $C_1$  and  $C_2$  are connected in series and then placed in parallel with  $C_3$ . The total capacity is expressed by (59):

$$(59) \quad C = \frac{1}{\frac{1}{C_1} + \frac{1}{C_2}} + C_3.$$

**Ex. 18.** The diagram (a) . . . (q) in Figs 216A and 216B show all the possible arrangements of three condensers used singly, doubly, or in combination. Determine all possible values for condensers of 1, 2, and 3 microfarads respectively.

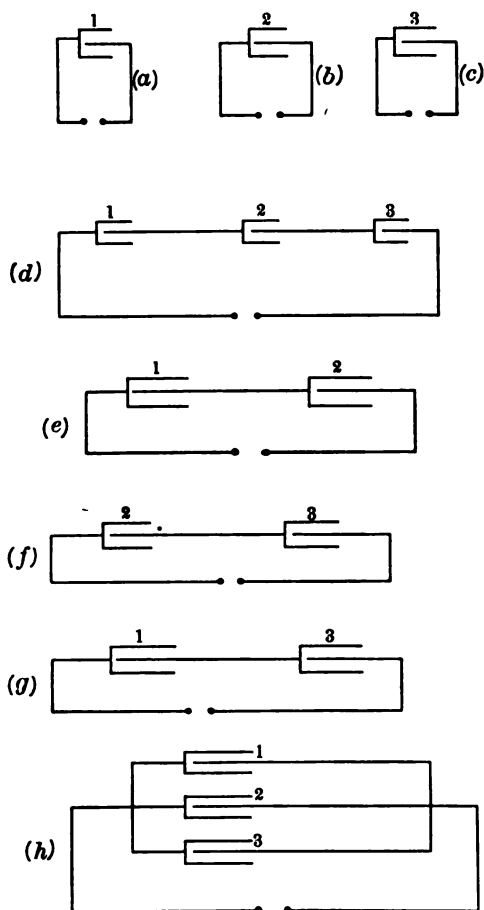


FIG. 216A.—The Possible Arrangements of Three Condensers.

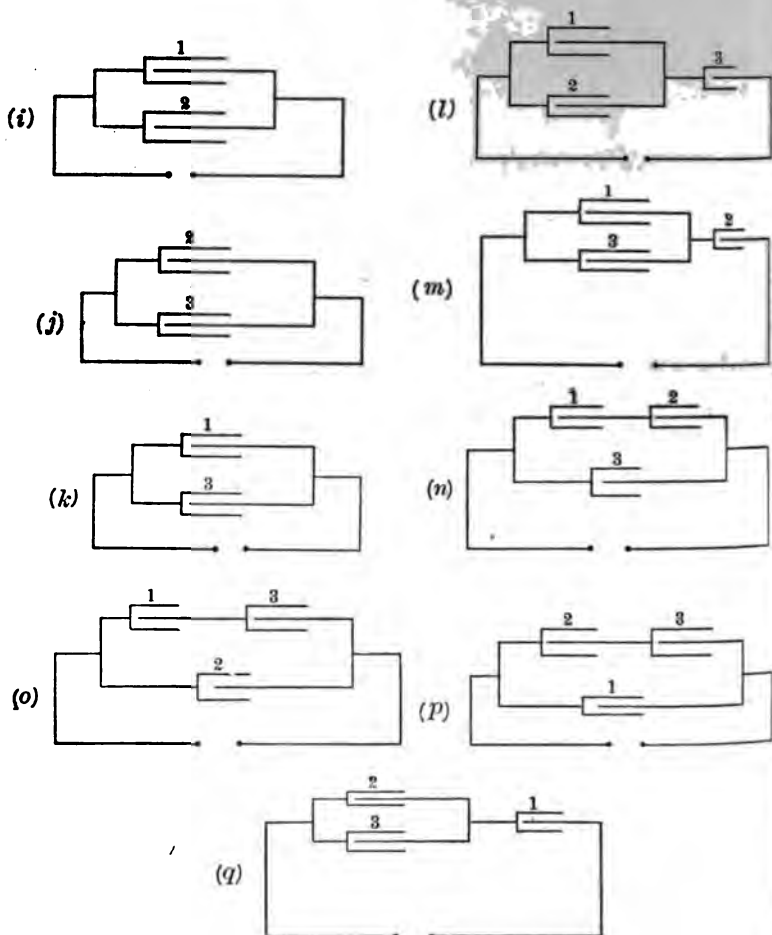


FIG. 216B.—The Possible Arrangements of Three Condensers.

**Ex. 19.** Determine all possible capacities which may be obtained from three condensers of 50, 100, and 150 microfarads respectively.

**Ex. 20.** Determine all the possible values of capacity which may be obtained by plugging the portable adjustable microfarad condenser, shown in Fig. 217.

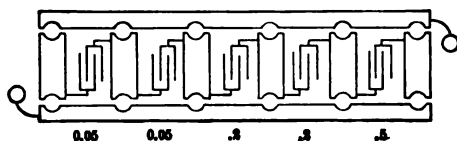


FIG. 217.—A Portable Adjustable Condenser.

**Ex. 21.** Calculate the capacity of a condenser which is built with sheets of tinfoil 4 x 5 ins. and mica 1 mm. thick. There are 200 mica leaves intervening between 201 sheets of tinfoil.

**17. Energy Equations of an A.C. Circuit.** The energy  $W$  delivered by a generator to an A.C. circuit is partly **useful power** represented by copper losses and partly **wattless power** represented by the energy stored in the magnetic and electrostatic fields. The useful power is energy which is spent in the heating of the conductors while the wattless energy is alternating stored and recovered from the field. The useful power  $W_u$  is represented by the  $I^2R$  loss in (60). The energy  $W_L$  stored in the magnetic field is expressed in (61) and the energy  $W_e$  stored in the electrostatic field is expressed in (62):

$$(60) \quad W_u = I^2R. \quad (61) \quad W_L = \frac{LI^2}{2}. \quad (62) \quad W_e = \frac{CE_e^2}{2}.$$

$$(63) \quad W = I^2R + \frac{LI^2}{2} + \frac{CE_e^2}{2}.$$

Interpret (63), (60), (61) and (62) in which  $W$  is the total power delivered to the circuit by the generator.

**18. Reactance.** The three elements  $E$ ,  $I$ , and  $R$  enter in the discussion of a D.C. circuit in their familiar relation

which is known as Ohm's Law. In such a circuit the only continued opposition or obstruction offered to the passage of a current is the resistance of the circuit. In an A.C. circuit the opposition or obstruction is called an impedance ( $Z$ ) which is expressed in (64).  $I$  and  $E$  are effective values.

$$(64) \quad I = \frac{E}{Z}.$$

Impedance consists of three distinct elements, viz.: resistance  $R$ , inductive reactance  $X_L$  due to inductance, and capacity reactance  $X_c$  due to capacity, as expressed in (65). Resistances, reactances and impedances are measured in ohms.

$$(65) \quad Z = \sqrt{R^2 + (X_L - X_c)^2}.$$

As in a D.C. circuit the resistance of an A.C. circuit depends upon length, cross-sectional area, and specific resistance of the conductor.

Inductive reactance varies directly as the inductance  $L$  of the circuit and the frequency  $f$  of the current, as expressed in (66). The proportionality factor is  $2\pi$ .

$$(66) \quad X_L = 2\pi fL = \omega L.$$

Capacity reactance varies reciprocally as the product of the frequency  $f$  and the capacity of the circuit  $C$ , as expressed in (67). The proportionality factor is  $\frac{1}{2\pi}$ :

$$(67) \quad X_c = \frac{1}{2\pi fC} = \frac{1}{\omega C}.$$

$$(68) \quad Z = \sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}.$$

$$(69) \quad Z = \sqrt{R^2 + \left(2\pi fL - \frac{1}{2\pi fC}\right)^2}.$$

(68) and (69) result from the substitution of (66), (67) and (65). (70) results from the substitution of (68) in (64).

$$(70) \quad I = \frac{E}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}}.$$

*Observation.* In general, an A.C. circuit is subject to change due to resistance and reactances, the latter caused by inductance and capacity. Inductive reactance has a positive sense, whereas capacity reactance has a negative sense. The total reactance of a circuit is the excess of the one reactance over the other and bears the same sign as the greater magnitude.

The total reactance of a circuit is designated by  $X$ , as expressed in (71), (72) and (73):

$$(71) \quad X = X_L - X_C.$$

$$(72) \quad I = \frac{E}{\sqrt{R^2 + X^2}}.$$

$$(73) \quad I^2 R^2 + I^2 X^2 = I^2 Z^2 = E^2.$$

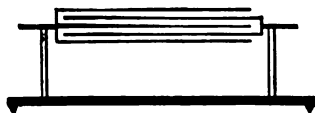


FIG. 217A.

## CHAPTER XXVI

### ELEMENTARY OPERATIONS OF VECTOR ANALYSIS

1. **Vectors.** The **magnitude** of a quantity may be represented graphically by a line, an area, or a volume. For convenience in measurement, magnitudes are indicated usually by the lengths of straight lines.

When a suitable length is chosen for a **unit**, each magnitude may be expressed either as a multiple or submultiple of the unit.

With the aid of graphic methods it becomes possible to compare and measure areas, masses, densities, temperatures, energy, quantity of heat, electric charge, potential, rainfall, moonlight, etc. When the elements under consideration are represented by any arbitrarily scaled lines they are called **scalar quantities**. A number of quantities of the same kind may be united or combined by observing the fundamental operations of algebra.

There are many other elements in applied science which involve **direction** as well as **magnitude**. Thus if we wish to represent motion graphically, it is not sufficient to consider the magnitude of the displacement, but the direction of the motion must be indicated.

Physical quantities that involve **magnitude and direction** are called **vector quantities** or simply **vectors**. Common illustrations of vectors are velocity, accelerations, forces, electric currents, magnetic fluxes, lines of force, stress and strains in structures, flow of heat, flow of fluids, tides, and currents, etc. Why are these vector quantities?

In specifying a force it is necessary to know its magnitude, its direction, and its sense, which involves the point of application. The points of the compass serve our convenience in describing direction. The bearing of a line is its angle of deviation from a north and south reference line.

In Fig. 218 the line  $c$  bears from the north by the angle 2 to the east. The horizontal distance of the extremity of  $c$  from the north and south line is called its **departure**. The vertical distance of the extremity of  $c$  from the east and

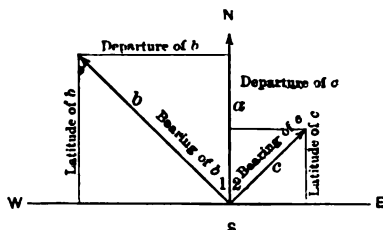


FIG. 218.—The Bearing, Departure, and Latitude of a Directed Line.

west line is called its **latitude**. When the bearing and length of a line are known its departure and latitude may be read directly from a traverse table or they may be computed by means of the respective cosine and sine of the bearing of the line.

**2. Vector Representation.** A force acting vertically is not fully described, since it may act upward or downward. This deficiency in orientation is obviated by marking arrow-heads on the vectors to indicate their sense as shown by  $g$  and  $h$  in Fig. 219. A **vector** is a **directed line**. Its length represents the **magnitude** of a quantity and its **inclination** or **bearing** to an arbitrary line of reference together with its **arrow-head** shows its **direction** and **sense of action**.

Fig. 219 represents a number of vectors of different magnitudes, directions, and senses. A vector diagram may contain vectors which represent different physical quantities and therefore there arises a necessity for distinguishing

them. As an illustration such a diagram may contain vectors which represent E.M.Fs, magnetic fluxes, and currents. Vectors may be distinguished by modifying the arrow-head and the feather, as shown in *e*, *f*, *a*, *t*, *v*, *r*, *u*. They may be distinguished by drawing them with light, medium, heavy, dotted, and double lines, as shown in *m*, *h*, *n*, *b*, and *c*. They may be distinguished by marking the lines with characters, as indicated by *d*, *q*, *o*, *s*, *p*.

Vectors are designated by one or two letters placed at or between their extremities.

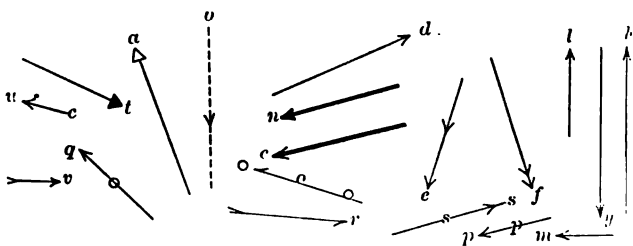


FIG. 219.—Vectors.

**3. Equal Vectors.** A vector may be represented arbitrarily on any part of the plane of our paper or blackboard. We assign to it a definite length for its magnitude according to a convenient scale and choose an arbitrary line of reference in order to specify its direction. A vector may be represented in any position of the paper by moving it parallel to itself and without altering its magnitude. Therefore **vectors are equal** when they have **equal magnitudes** and **like directions** and the **same sense**. Vectors are unequal when one of these conditions is altered. Two vectors of equal magnitude, which are respectively positive and negative, are parallel and of equal length, but their arrow-heads will point in opposite directions.

**4. The Addition of Vectors.** The two extremities of a vector may be distinguished by calling them the **head** and **nock** or **beginning** and **end**.

Two or more vectors are added graphically by placing them consecutively so that the head of one vector is in coincidence with the neck of the next adjacent vector. A vector which joins the neck of the last vector with the head of the first vector represents their sum. The vector sum is more commonly called the **resultant vector** or **resultant**.

The resultant of two or more vectors is a vector whose sense opposes that of the continuous sense of the separate vectors which have been joined.

In the left of Fig. 220 are two vectors  $a$  and  $b$  which are to be added. In the right  $a'$  and  $b'$  represent  $a$  and  $b$

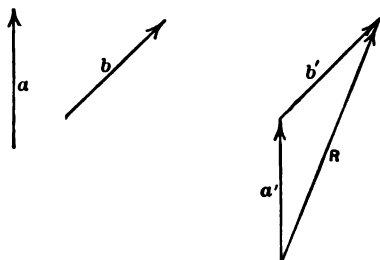


FIG. 220.—The Addition of Vectors.

respectively, in a new position with the head of  $a'$  coincident with the neck of  $b'$ .  $R$  is the resultant, i.e., the sum of  $a$  and  $b$  and its arrow-head opposes the continuous sense of  $a'$  and  $b'$ . What authority enables us to place  $a$  and  $b$  in their new positions  $a'$  and  $b'$  respectively?

In the left of Fig. 221 are two vectors  $a$  and  $b$  which are to be added. The addition of two vectors may be accomplished irrespective of the order in which they are placed in succession. In the third figure  $a''$  and  $b''$  represent  $a$  and  $b$  respectively, and in the fourth figure  $a'''$  and  $b'''$  represent  $a$  and  $b$  respectively. In both cases the resultant is  $R$ , showing that the two vectors may be added in either order. In the second figure the resultant  $R$  is drawn as the diagonal of a parallelogram which has been constructed

upon  $a'$  and  $b'$  as adjacent sides. Since the opposite sides of a parallelogram are equal it will be seen that the left and right halves of the parallelogram are identified with the third and fourth figures respectively.

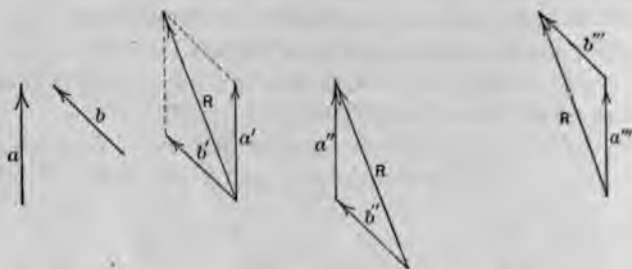


FIG. 221.—Different Arrangements for Producing an Equal Resultant.

Fig. 222 shows five different methods for obtaining the resultant of three vectors  $a$ ,  $b$ , and  $c$ . The three vectors may be added in any order as follows:

In (a) the sum of  $a$  and  $b$  is represented by  $R_1$  and the latter is added to  $c$  producing the resultant  $R$ .

In (b) the sum of  $a$  and  $c$  is represented by  $R_1$  and the latter is added to  $b$  producing the resultant  $R$ .

In (c) the sum of  $b$  and  $c$  is represented by  $R_1$  and the latter is added to  $a$  producing the resultant  $R$ .

In (d) the sum is obtained in one operation by laying off  $a$ ,  $b$ , and  $c$  consecutively. The resultant  $R$  closes the figure and completes the polygon.

In (e) the three vectors  $a$ ,  $b$ , and  $c$  are drawn from a common point  $G$ . The vectors are resolved into horizontal and vertical components by projecting them on the lines  $AK$  and  $GE$  respectively, i.e., a departure and latitude is determined for each vector.  $+KG$  and  $-AG$  are the respective horizontal projections of  $c$  and  $b$ , and  $+FG$  and  $+HG$  are the respective vertical projections of  $b$  and  $a$  is vertical and has only a vertical projection. T

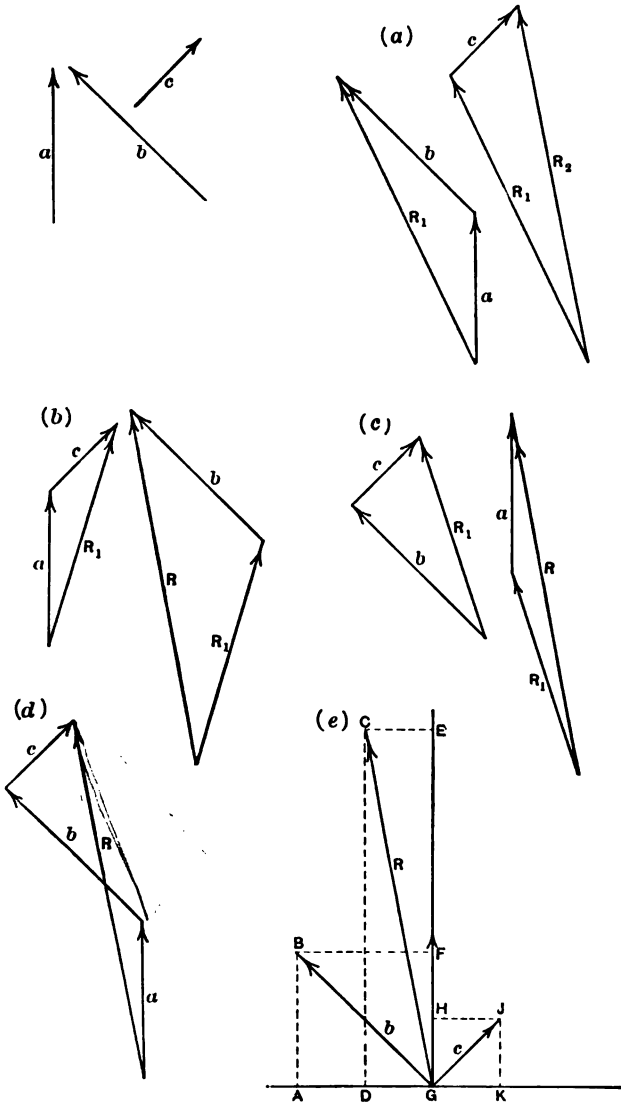


FIG. 222.—The Resultant of Three Vectors.

braic signs of the projections follow from the convention of signs in trigonometry. The sum of the vertical projections equals  $EG$  and is the vertical projection of the resultant  $R$ . The sum of the horizontal projections equals  $DG$  and is the horizontal projection of the resultant  $R$ . Therefore  $R$  may be constructed as the diagonal of a parallelogram whose two adjacent sides equal  $EG$  and  $DG$  respectively.

At the top of Fig. 223 are five vectors  $A, B, C, D$ , and  $E$  whose resultant  $R$  is obtained in (a), (b), (c), (d) and (e) by constructing polygons of vectors in five distinct ways. In (f) the two vectors  $A$  and  $B$  were added and their resultant  $R_1$  was added to  $C$ , producing  $R_2$ , the latter was added to  $D$ , producing  $R_3$  and  $R_3$  in turn was added to  $E$ , producing the final resultant  $R$ . (f) is the method of partial or continued resultants. In (g) the final resultant  $R$  is obtained by the method of projections.

*Observation.* In every case of vector addition irrespective of the method or order of the vectors the final resultant will be the same.

**Ex. 1.** By the polygon method, continued resultant and projection method, construct the resultant of the following groups of vectors illustrated in Fig. 223 : (a)  $A, B, C$ ; (b)  $A, B, D$ ; (c)  $A, B, E$ ; (d)  $B, C, D$ ; (e)  $D, C, E$ ; (f)  $C, D, E$ ; (g)  $A, C, D$ ; (h)  $A, C, E$ ; (i)  $B, D, E$ . Measure and compute the length and bearing of the resultant.

**Ex. 2.** Construct the resultant of  $A, B, C, D$ , and  $E$  in Fig. 223 by five polygons, the order of whose sides is not like those shown in Fig. 223. Obtain the resultant by the continued resultant method, changing the order shown in Fig. 223. Measure the length and bearing of the resultant.

*Observation.* Two vectors which are added constitute two sides of a triangle and the resultant the third side.

Three or more vectors may be added by forming a polygon of vectors which is completed or closed by the resultant.

The resultant may also be obtained by first adding two vectors and then adding another vector to their resultant. This

# ELEMENTARY OPERATIONS OF VECTOR ANALYSIS 531

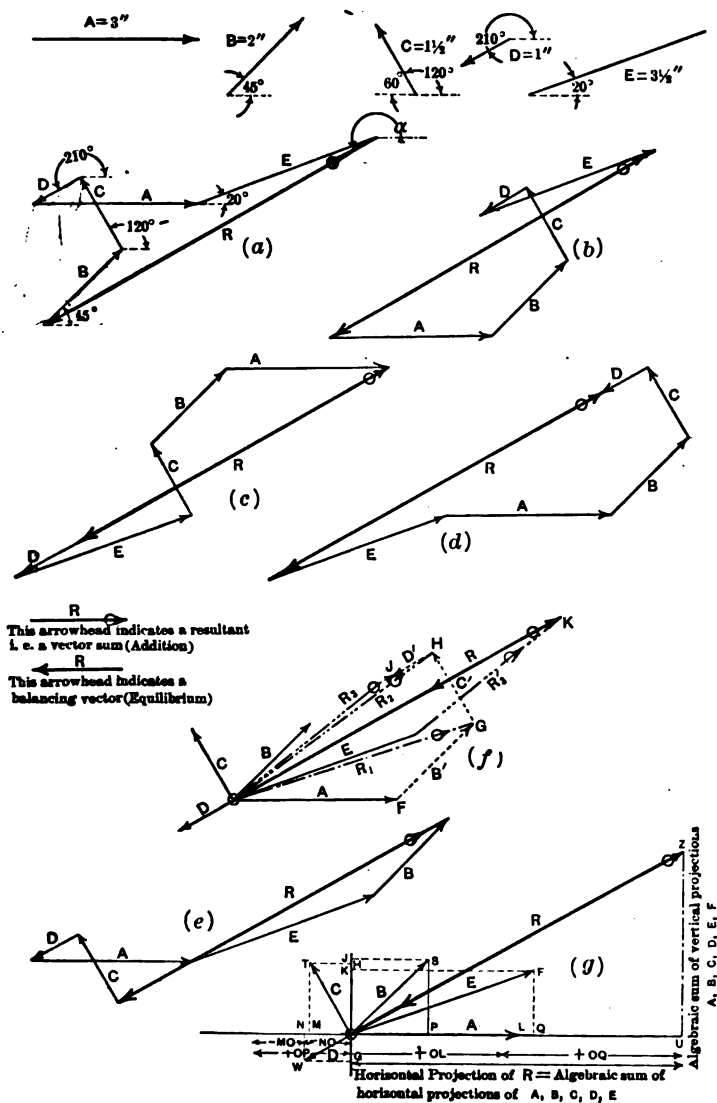


FIG. 223.—The Resultant of Five Vectors.

process is continued until all the vectors have been united. A vector which places several vectors in equilibrium is the negative of their resultant.

Another method for combining vectors is to represent them at a common point of action which shall represent also the origin of two rectangular axes. The vectors are then resolved into horizontal and vertical components by orthogonal projection. The algebraic sum of all the horizontal components is the horizontal component of the resultant. The algebraic sum of all the vertical components is the vertical component of the resultant. The resultant may then be constructed as the sum of the component vectors.

The subtraction of vectors is identical with that of addition. The vector which is to be subtracted has its sense reversed and then as in algebra it becomes a case of addition.

**Ex. 3.** In Fig. 224  $W_1$ ,  $W_2$ , and  $W_3$  are weights suspended by strings passing over pulleys whose friction is negligible. Draw

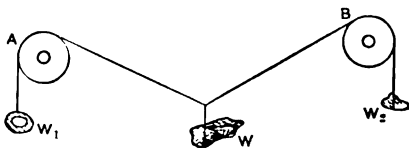


FIG. 224.—Forces in Equilibrium.

the vector diagram for the forces showing them in equilibrium: (a)  $W_1 = 5$ ,  $W_2 = 7$ ; (b)  $W_1 = 3$ ,  $W_2 = 5$ ,  $W_3 = 7$ ; (c)  $W_1 = 10$ ,  $W_2 = 15$ ,  $W_3 = 20$ .

**5. Concurrent Forces.** Lines of vectors having a common point are concurrent.

If a number of concurrent forces are in equilibrium they form a closed vector polygon.

A closed vector polygon indicates that the forces acting at a point are in equilibrium.

**Ex. 4.** Forces of 5 and 7.5 lbs. respectively act at a point. What is the resultant when the angle between them is (a)  $30^\circ$ ; (b)  $60^\circ$ ; (c)  $90^\circ$ ; (d)  $120^\circ$ ; (e)  $160^\circ$ ; (f)  $180^\circ$ ?

**Ex. 5.** The forces acting on a body at a point and their respective bearings are:  $a \equiv 20$  lbs. N.;  $b \equiv 4$  lbs. N.E.;  $c \equiv 13$  lbs.  $30^\circ$  N. of E.;  $d \equiv 17$  lbs. W.;  $e \equiv 25$  lbs. S.E. What is their resultant and what force acting with them will maintain equilibrium?

**Ex. 6.** A string is fixed to a point  $A$  and passes over a pulley  $B$  2 ft. distant and supports a weight of 10 lbs. A ring slides along the string between  $A$  and  $B$  and carries a weight of 15 lbs. Determine the tension in the string and the position of the weight.

**Ex. 7.** A street-car with its load weighs 10 tons. The average grade of the street is 5 per cent. Allowing an efficiency for the motors and gearing of 75 per cent and a horizontal speed of 10 miles an hour, what power must be applied to the motor? See Fig. 225.

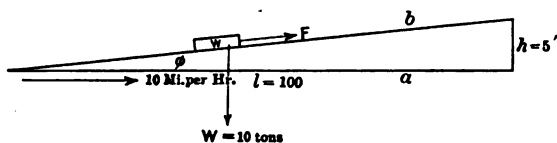


FIG. 225.—Forces Acting on an Inclined Plane.

## 6. The Vectorial Representation of Currents and E.M.Fs.

A sine curve which has been plotted to rectangular coordinates may be replotted upon polar coordinate paper. A sine curve which has been plotted to rectangular coordinates becomes a circle passing through the pole in the polar representation, as shown in Fig. 226.

In Fig. 226 the pole is located at  $O$ . The circles whose diameters are  $OA$ ,  $OE$ ,  $OD$ , and  $OB$  correspond to Eqs. (1), (2), (3), and (4) respectively, and also to (5), (6), (7), and (8) respectively.

- |                             |                             |
|-----------------------------|-----------------------------|
| (1) $e_1 = E \sin \theta.$  | (5) $i_1 = I \sin \theta.$  |
| (2) $e_2 = E \cos \theta.$  | (6) $i_2 = I \cos \theta.$  |
| (3) $e_3 = -E \sin \theta.$ | (7) $i_3 = -I \sin \theta.$ |
| (4) $e_4 = -E \cos \theta.$ | (8) $i_4 = -I \cos \theta.$ |

The maximum values of (1) . . . (8) are represented by the respective diameters  $OA$ ,  $OE$ ,  $OD$ , and  $OB$ . The instantaneous values of  $e_1$  . . .  $e_4$ ,  $i_1$  . . .  $i_4$  are the chords intercepted in the respective circles. The instantaneous values of  $e_1$  in (1) are measured by chords  $OE$ ,  $OF$ , and  $OG$  when the respective angles are  $19^\circ$ ,  $47^\circ$ , and  $71^\circ$ . The

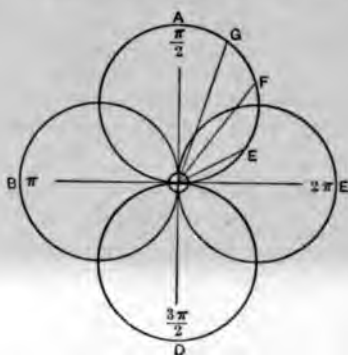


FIG. 226.—The Polar Representation of Sine Curves.

angle is measured between the chord and the horizontal reference line:

when  $\theta = 19^\circ$ , then  $e_1 = OE = E \sin 19^\circ$ ;

when  $\theta = 47^\circ$ , then  $e_1 = OF = E \sin 47^\circ$ ;

when  $\theta = 71^\circ$ , then  $e_1 = OG = E \sin 71^\circ$ .

Each chord represents an instantaneous value of the E.M.F. in (1) and is a vector, since it has both magnitude and direction.

*Observation.* The ordinates of a sine curve may be represented by the chords of a polar circle and therefore E.M.F.s, currents and magnetic fluxes may be represented vectorily.

$OA$ ,  $OE$ ,  $OD$ , and  $OB$  are the maximum values of four sine curves whose relative displacement is  $90^\circ$ . If the four diameters are held rigidly together and rotated

their respective projections are the simultaneous values of the ordinates of the four corresponding sine curves. The circles may be dispensed with if we represent the maximum values only of each sine curve by a vector.

The angular displacement of the vectors must correspond to the phase displacement of their sine curves. The effective values of E.M.F. and current are more frequently used in calculation than their maximum values. For the purpose of graphic calculation in A.C. work each E.M.F. and current is represented by a vector whose length equals the magnitude of the effective values of the respective E.M.F. or current. A diagram containing the vector representation of E.M.Fs. and currents is named a clock diagram.

The drop across a resistor is measured by the product of its resistance times the current flowing through it and is represented by a vector.

The drop across a reactor is measured by the product of its reactance times the current flowing through it and is represented by a vector.

Since reactance drop is  $90^\circ$  out of phase with resistance drop the former may be represented by a vector which is drawn at right angles to the vector which represents the latter as illustrated in Figs. 231 and 232.

## CHAPTER XXVII

### VECTOR ALGEBRA

**1. Symbolic Representation of a Vector.** The graphic representation of alternating quantities is in very extensive practical use. Many problems may be solved directly by the measurement of the lines of a vector diagram. There are cases, however, where the measurements of lines of great disparity in magnitude or of very small deviation in direction would cause errors not permitted in technical application. In such cases the numeric values of the vectors are used and from these a practical result is obtained in numeric

form. We shall proceed to examine some of the notations used in applying algebra to vector diagrams.

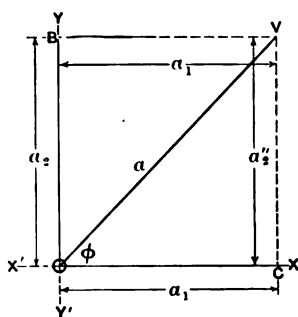


FIG. 227.—Vector Components.

In Chapter XXVI it was shown how vectors may be resolved into components which are the respective projections measured along and perpendicular to any fixed axis. Thus in Fig. 227 the vector  $OV$  has a magnitude  $a$  and it is resolved into the

horizontal component  $a_1$  and the vertical component  $a_2$ . (1) indicates that  $a$  is the vector sum of the two component vectors  $a_1$  and  $a_2$ . The symbol  $\oplus$  means vector sum.

$$(1) \quad a = a_1 \oplus a_2.$$

2. It is not only convenient but it is also necessary for calculations that we should be able to distinguish the horizontal from the vertical component. In order to provide for this necessity Steinmetz suggested the prefixing of the letter  $j$  to the term representing the vertical component, as shown in (2):

Then

$$(2) \quad V = a_1 + ja_2,$$

and since  $a_1 = a \cos \phi$  and  $a_2 = a \sin \phi$ , then (1) and (2) transform into (3) and (4) respectively:

$$(3) \quad a = a \cos \phi \oplus a \sin \phi.$$

$$(4) \quad a = a \cos \phi + ja \sin \phi.$$

Expressions (3) and (4) indicate that the vector  $a$  has a component  $a \cos \phi$  along the principal axis of reference and a component  $a \sin \phi$  measured at right angles to the latter.

Another notation which has been suggested by Cramp and Smith is to distinguish the horizontal component from the vertical component by accents. The former receives a prime or single accent (') and the latter a double prime or second accent ("). Under this notation (1) and (3) may be rewritten as (5) and (6) respectively:

$$(5) \quad a = a' + a''.$$

$$(6) \quad a = (a \cos \phi)' + (a \sin \phi)''.$$

The use of different kinds of letters for the two components has also been suggested and in fact a number of other schemes have been proposed, but instead of clarifying the work it means a burdening of the mind and a difficult task for the printer.

**3. The Magnitude and Inclination of a Vector.** The vector  $OV$ , represented in Fig. 227, has a magnitude  $a$  and an inclination  $\phi$  with the axis of reference, and since

$a_1$  and  $a_2$  are the lengths of the component vectors, then the length  $a$  of  $OV$  is expressed in (7):

$$(7) \quad a = \sqrt{a_1^2 + a_2^2}.$$

The inclination  $\phi$  is determined from (8), (9), or (10):

$$(8) \quad \tan \phi = \frac{a_2}{a_1}.$$

$$(9) \quad \sin \phi = \frac{a_2}{a}.$$

$$(10) \quad \cos \phi = \frac{a_1}{a}.$$

The magnitude of the resultant  $c$  of two vectors of magnitude  $a$  and  $b$  respectively, may be expressed by the law of the oblique triangle in (11) and (12), where  $\phi$  is the angle between the vectors:

$$(11) \quad c^2 = a^2 + b^2 + 2ab \cos \phi.$$

$$(12) \quad c = \sqrt{a^2 + b^2 + 2ab \cos \phi}.$$

**Ex. 1.** Express the value of the impressed E.M.F.  $E$  whose horizontal component is  $E_r$  and whose vertical component is  $E_v$ .

**Ex. 2.** Express the value of the current  $I$  in a circuit whose horizontal component is  $I_r$  and whose vertical component is  $I_v$ .

**Ex. 3.** Express the value of the resultant  $E$  of two vectors  $E_1$  and  $E_2$  whose phase difference is  $60^\circ$ .

**Ex. 4.** Express the value of the resultant  $E$  of two vectors  $E_1$  and  $E_2$  whose phase difference is  $\phi$ . Project  $E_1$  on  $E_2$  and also express the value of the phase angle of  $E$  measured from  $E_1$ .

## CHAPTER XXVIII

### THE GRAPHIC SOLUTION OF A.C. PROBLEMS

1. For convenience in study it is most satisfactory to separate A.C. problems into three groups: series circuits, parallel circuits, and series-parallel circuits.

In general A.C. circuits are subject to change due to resistance and reactance. They are therefore influenced by inductance, capacity, and frequency.

2. **Circuits having Resistance and Inductance.** A counter E.M.F.  $E_s'$  due to inductive reactance  $X_L$  may be expressed in (1):

$$(1) \quad E_s' = -X_L I = -2\pi f L I.$$

$$2\pi f = 377 \text{ at } 60 \text{ cycles}$$

The impressed E.M.F.  $E$  must possess a vertical component  $E_s$  which balances or overcomes  $E_s'$  as expressed in (2):

$$(2) \quad E_s = -E_s' = X_L I = 2\pi f L I.$$

The impressed E.M.F.  $E$  must possess a horizontal component  $E_r$  which balances or overcomes the drop across a resistance  $R$ . In this sense the resistance acts as a counter E.M.F.  $E_r'$ , as expressed in (3):

$$(3) \quad E_r = -E_r'.$$

These facts are illustrated in Fig. 228. According to Kirchhoff's Law: The resultant of all the E.M.Fs. is zero in a closed circuit if the counter E.M.Fs. of resistance and reactance are included.

In Fig. 228  $OB$  is the inductive reactance component of  $OC$  and  $OA$  is the corresponding resistance component.  $BC$  and  $AC$  are constructed parallel to  $OA$  and  $OB$  respectively:

$$(4) \quad E = \sqrt{E_r^2 + E_s^2}.$$

The broken lines are vectors which represent counter E.M.F.s.  $OB'$  is the counter E.M.F. set up by the induct-

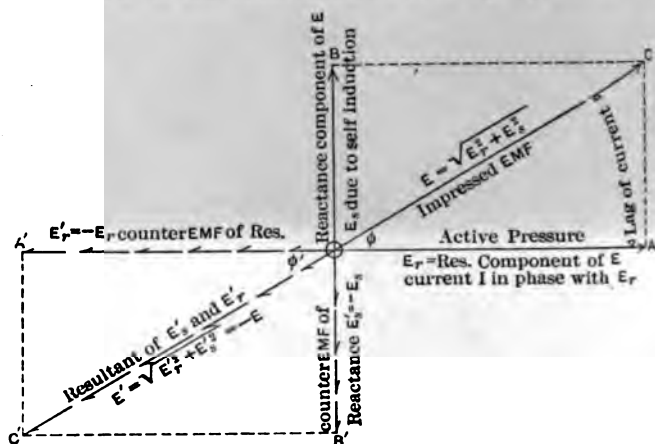


FIG. 228.—The Balanced E.M.F. Vectors for a Circuit having Resistance and Inductive Reactance.

ance and  $OA'$  is the counter E.M.F. due to resistance.  $OC'$  is the resultant of  $OA'$  and  $OB'$ .

$OB$  balances  $OB'$ ,  $OA$  balances  $OA'$ , and  $OC$  balances  $OC'$ .

$E_r$  is constructed in phase with the current. Since the counter E.M.F. of inductance  $E_s'$  lags  $90^\circ$  behind the current, then for counter clockwise rotation  $E_s'$  will be constructed vertically below  $O$ . Therefore the reactance component  $E_s$  of the impressed E.M.F is constructed vertically above  $O$ .

Fig. 228 may be separated into two parts; one of them is shown in Fig. 229. The impressed E.M.F. may

lated from  $E_r$  and  $E_s$  or from  $E_r'$  and  $E_s'$ . For purposes of simplicity we shall choose the former and accordingly construct Fig. 229 from which the following observations are noted.

The impressed E.M.F.  $E$  is resolvable into two components: one of these represents a drop of potential across the resistance, and the other the drop across the reactance.

To overcome the resistance  $R$  an E.M.F.  $E_r = IR$  is required in phase with the current represented by the vector  $OA$ .

To overcome the counter E.M.F. of self-induction an E.M.F.  $E_s = IX_L$  is required  $90^\circ$  ahead of the current represented by vector  $OB$ .

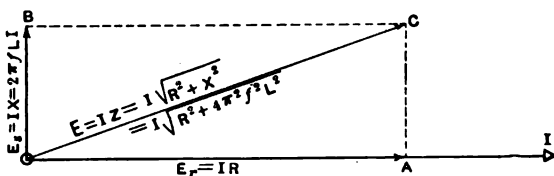


FIG. 229.—The Components of an Impressed E.M.F. in an Inductive-Resistive Circuit.

The resistance consumes E.M.F. in phase and inductive reactance an E.M.F.  $90^\circ$  ahead of the current.

The current  $I$  being in phase with  $IR$  may be represented upon the same line by choosing a suitable unit for  $R$ . This is shown by  $OI$  with a solid head.

The current  $I$  lags behind the impressed E.M.F.  $E$  by the angle  $\phi$ .

3. The useful power  $P$  which is consumed in a circuit is due to the heating of the resistance:

$$(5) \quad P = I^2 R = \frac{E_r^2 R}{R^2} = \frac{E_r^2}{R} = E_r I = EI \cos \phi.$$

The interpretation of (5) states that the active power in an A.C. circuit may be computed by multiplying the

effective values of the E.M.F. and current by the cosine of their phase difference. The factor  $\cos \phi$  is called the **power factor** of the circuit.

**4. Circuits having Resistance and Capacity.** A counter E.M.F.  $E_c'$  due to capacity leads the current by  $90^\circ$  and therefore an impressed E.M.F. must furnish a component  $E_c$  which balances  $E_c'$ .

$$(6) \quad E_c = -E_c'.$$

$$(7) \quad E_c' = IX_c = \frac{I}{\omega C} = \frac{I}{2\pi fC}.$$

$$(8) \quad \therefore E_c = -\frac{I}{2\pi fC} = -\frac{I}{\omega C} = -IX_c.$$

In Fig. 230  $OB$  is the capacity reactance component of  $OC$  and  $OA$  is the corresponding resistance component.  $BC$  and  $AC$  are constructed parallel to  $OA$  and  $OB$  respectively.

$$(9) \quad E = \sqrt{E_r^2 + E_c^2}.$$

The broken lines are vectors which represent counter E.M.F.s.  $OB'$  is the counter E.M.F. set up by the capacity and  $OA'$  is the counter E.M.F. due to resistance.  $OC'$  is the resultant of  $OA'$  and  $OB'$ .

$OB$  balances  $OB'$ ,  $OA$  balances  $OA'$ , and  $OC$  balances  $OC'$ .

$E_r$  is constructed in phase with the current. Since  $E_c'$  leads the current by  $90^\circ$ , then for counter clockwise rotation  $E_c'$  will be constructed vertically above  $O$ . Therefore the reactance component  $E_c$  of the impressed E.M.F. is constructed vertically below  $O$ .

It is customary to consider the impressed E.M.F. and its components only and therefore the lower right-hand part of the vector diagram is used for simplicity.

*Observation.* E.M.F.s. of inductance and capacity are opposite in phase, i.e.,  $180^\circ$  apart. In any circuit in which they are both present their effect is one of the complete or

*partial neutralization. Both inductance and capacity contribute to make the total reactance and therefore we may distinguish their respective reactances by subscripts.*

$E_r = E_L =$  component of  $E$  which overcomes inductive  
reactance voltage  $= IX_L = 2\pi fLI$

$E_c =$  component of  $E$  which overcomes capacity  
reactance voltage  $= -IX_c = -\frac{I}{2\pi fC}$

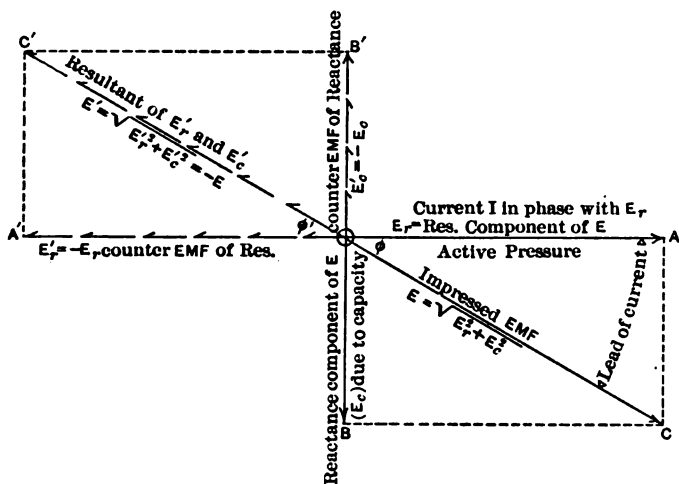


FIG. 230.—The Balanced E.M.F. Vectors for a Circuit having Resistance and Capacity Reactance.

The total reactance component of  $E$  is the algebraic sum of the inductance reactance voltage and the capacity reactance voltage.

### 5. Circuits having Inductance, Capacity, and Resistance.

The vector diagram for circuits having inductance, capacity, and resistance is a combination of the right-hand halves of Figs. 228 and 230, as shown separately in Fig. 231. Therefore the vector diagram for every simple circuit will contain  $ng$  vectors,  $E_r$ ,  $E_L$ ,  $E_c$ , and their resultant  $E$ .

In A.C. vector diagrams an open arrow-head indicates an E.M.F. and a closed arrow-head indicates a current.

The left-hand diagram of Fig. 232 shows an excess of inductive reactance and a corresponding lagging current. The right-hand diagram of Fig. 232 shows an excess of capacity reactance and a corresponding leading current.

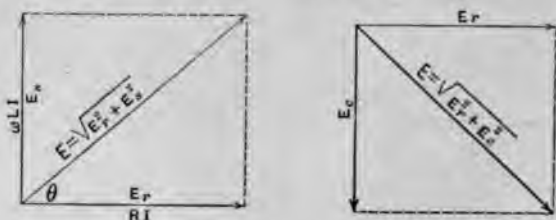


FIG. 231.—The Component and Resultant E.M.F. Vectors of a Simple Circuit.

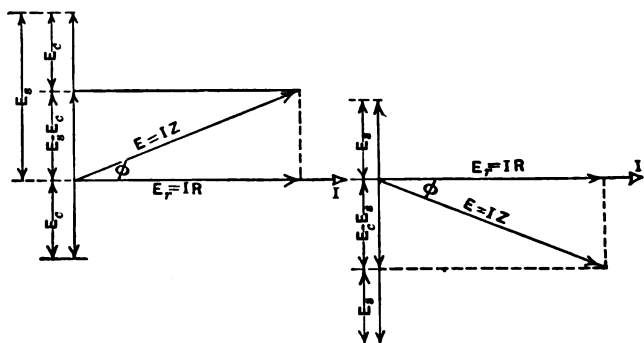


FIG. 232.—Vector Diagrams for Circuits having Resistance, Inductance and Capacity.

## 6. General Expression for an A. C. Circuit.

(10) The current  $I = \frac{\text{impressed E.M.F. } E}{\text{impedance } Z}.$

(11) 
$$I = \frac{E}{\sqrt{R^2 + X^2}}$$

$$(12) \quad I = \frac{E}{\sqrt{R^2 + \left[2\pi fL - \frac{1}{2\pi fC}\right]^2}}$$

$$(13) \quad I = \frac{E}{\sqrt{R^2 + [X_L + X_C]^2}}$$

$$(14) \quad I = \frac{E}{Z} = EY.$$

The interpretation of (14) states that  $I$  equals the impressed E.M.F. times a factor  $Y$  which is named admittance and is defined as the reciprocal of impedance.

$$(15) \quad \phi = \tan^{-1} \left\{ \frac{2\pi fL - \frac{1}{2\pi fC}}{R} \right\} = \tan^{-1} \left\{ \frac{X_L - X_C}{R} \right\}.$$

**Ex. 1.** What are the magnitude relations for the inductive and capacity reactances which make  $\phi$  successively positive, zero, and negative?

**Ex. 2.** Construct a right triangle showing the relation between resistance, reactance, and impedance, and also  $\phi$ .

**Ex. 3.** Construct a right triangle similar to the one in Ex. 2, substituting admittance for impedance, conductance for resistance, and susceptance for reactance.

Designate conductance by  $g$  and susceptance by  $b$ .

**Ex. 4.** Express  $\cos \phi$  and  $\sin \phi$  in terms of  $R$ ,  $X$ , and  $Z$  from Ex. 2.

**Ex. 5.** Express  $g$  in terms of  $\phi$  and  $Z$  and substitute for  $Z \cos \phi$  from Ex. 4.

**Ex. 6.** Express  $b$  in terms of  $\phi$  and  $Z$  and substitute for  $Z \sin \phi$  from Ex. 4.

**Ex. 7.** Express admittance in terms of  $g$  and  $b$  from the relations in Ex. 3.

**Ex. 8.** From the similar triangles in Exs. 2 and 3 prove the following:

$$(16) \quad g = \frac{R}{Z^2} = \frac{R}{R^2 + X^2} \quad (17) \quad b = \frac{X}{Z^2} = \frac{X}{R^2 + X^2}$$

$$(18) \quad R = \frac{g}{y^2} = \frac{g}{g^2 + b^2} \quad (19) \quad X = \frac{b}{y^2} = \frac{b}{g^2 + b^2}$$

**Ex. 9.** Explain why diagrams constructed of  $Z$ ,  $R$ , and  $X$  are not vector diagrams.

## CHAPTER XXIX

### SERIES CIRCUITS

**1. The Symbols for  $R$ ,  $X$ ,  $Z$ .** Every electrical circuit possesses resistance, inductance, and capacity and therefore reactance and impedance. If the reactance of the circuit is negligible, the circuit is represented with resistance only which is signified by using the symbol  $R$  in Fig. 233. If the circuit has negligible resistance and capacity

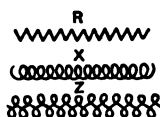


FIG. 233.—Symbols  
Used in an A.C.  
Circuit.

reactance this fact is indicated by placing the symbol  $X$ , Fig. 233, in the circuit. The symbol  $X$  is also used when only the capacity reactance is negligible. If the circuit has negligible resistance and inductive reactance this fact is indicated by placing the symbol  $C$  for capacity in the circuit, as shown in Figs. 233–235. When neither

resistance, inductive, or capacity reactance are negligible the three symbols for  $R$ ,  $X$ , and  $C$  may be used, but they are merged more conveniently into the impedance symbol  $Z$  Fig. 233. The use of these symbols is intended to show that the distributed resistances and reactances of a circuit may be considered as concentrated or condensed to facilitate the physical interpretation of the problem without causing appreciable error in the results. The power factor of the circuit is abbreviated P.F. or  $\cos \phi$ , wherein  $\phi$  is the angle of lead or lag.

**2. Standard Frequencies.** The selection of a suitable frequency in A.C. work depends largely upon the kind of

load to which the power is furnished. As will be indicated later every circuit has a **definite critical** frequency at which it will operate most efficiently. It is customary to design A.C. apparatus to meet the general demands of practice for which 60 and 25 cycles have been adopted. The symbol for cycles is written ( $\sim$ ).

**Ex. 1.** Determine the numeric values for  $\omega$  and  $\frac{1}{\omega}$  for 60  $\sim$  and 25  $\sim$ .

**Ex. 2.** Determine the value of  $X_L$  for a 60  $\sim$  and also for a 25  $\sim$  circuit, Fig. 234, when  $L$  equals: (a) .001h; (b) .0015h; (c) .01h; (d) .015h; (e) .1h; (f) .15h.

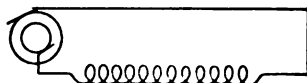


FIG. 234.—An Inductive Circuit.

**Ex. 3.** Determine the value of  $X_c$  for both 60  $\sim$  and 25  $\sim$  circuits when  $C$  equals: (a) 100 mf.; (b) 50 mf.; (c) 150 mf.; (d) 200 mf.; (e) 250 mf.; (f) 300 mf. The value of  $X_c$  is computed from (1) in which  $C$  is expressed in microfarads:

$$(1) \quad X_c = \frac{10^6}{\omega C}$$

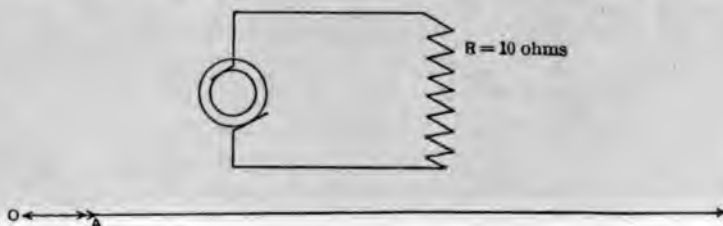
**Ex. 4.** What is the value of  $\frac{10^6}{\omega}$  for a 60  $\sim$  and 25  $\sim$  respectively?

**Ex. 5.** Construct a chart of **ohmic factors** whose ordinates represent  $\omega = 2\pi f$  and whose abscissas represent  $\frac{1}{\omega} = \frac{1}{2\pi f}$ . The values of  $f$  are to extend from 15  $\sim$  to 150  $\sim$ .  $\omega$  is named the **ohmic factor** and  $\frac{10^6}{\omega}$  is named the **reciprocal ohmic factor**.

Solve the following problems both graphically and numerically for 60  $\sim$  and also 25  $\sim$ . Determine (a) the reactances and impedances of the entire circuit and also for its parts; (b) the phase angles and power factors for the entire circuit and also for its parts indicating lag and lead by + and - angles respectively; (c) the current which flows through the circuit when 110 volts

impressed upon it; (d) the required impressed E.M.F. to pass 10 amps. through the circuit. Arrange the diagram of the circuit at the top of the page, showing the elements of the circuit and their numeric values, as given in the data. Show the graphic and numeric solutions for 60~ and 25~ under two vertical parallel columns.

**Ex. 6.** The circuit consists of a non-inductive resistance of  $10\ \Omega$ . The vector diagram consists of a current line  $OA$  and an E.M.F. line in phase with  $OA$ . Since this circuit consists of resist-



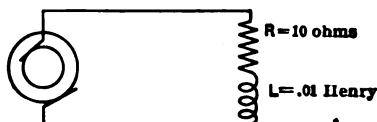
ance only there will be no phase difference between current and E.M.F. Inductive reactance and capacity reactance are the only factors which tend to cause a lag or lead in the current:

(a)  $Z = R = \underline{10\Omega}$ . (b)  $\Phi = \underline{0}$  for both 60~ and 25~.

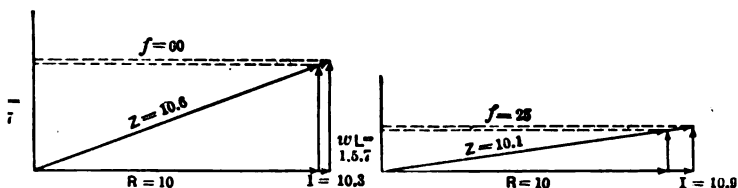
(c)  $I = \frac{E}{Z} = \frac{110}{10} = \underline{11}$  amps. when 110 volts are impressed.

(d)  $E = IZ = 10 \times 10 = \underline{100}$  volts required to produce 10 amps.

**Ex. 7.** The circuit consists of an inductive resistance of 100 and .01 henry. Complete the omitted calculations and check all



results by graphic determination. The upper diagrams are called impedance diagrams and show the relations between  $R$ ,  $X$ , and  $Z$ .



$$R = 10\Omega.$$

$$X_L = \omega L = 377 \times .01 = 3.77\Omega.$$

$$Z = \sqrt{R^2 + X^2} = \sqrt{100 + 14.21}.$$

$$Z = \underline{10.68\Omega}.$$

$$\cos \phi = \frac{R}{Z} = \frac{10}{10.68} = \underline{\hspace{1cm}}$$

$$\phi = \underline{\hspace{1cm}} \text{ lag.}$$

$$I = \frac{E}{Z} = \frac{110}{10.68} = \underline{10.3 \text{ amps.}}$$

$$E = IZ = 10 \times 10.68 = \underline{106.8 \text{ volts.}}$$

$$(1) \quad R = 10\Omega.$$

$$(2) \quad X_L = \omega L = 157 \times .01 = 1.57\Omega.$$

$$(3) \quad Z = \sqrt{100 + 2.47}.$$

$$(a) \quad Z = \underline{10.12\Omega}.$$

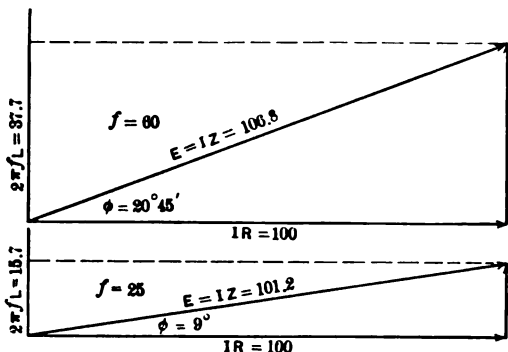
$$(b) \quad \cos \phi = \frac{10}{10.12} = \underline{\hspace{1cm}}$$

$$(4) \quad \phi = \underline{\hspace{1cm}} \text{ lag}$$

$$(c) \quad I = \frac{110}{10.12} = \underline{10.9 \text{ amps.}}$$

$$(d) \quad E = 10 \times 10.12 = \underline{101.2 \text{ volts.}}$$

**Ex. 8.** The circuit consists of a non-inductive coil of  $5\Omega$  in series with an inductive coil of  $5\Omega$  and  $.01 \text{ h.}$  Show that the diagrams for Ex. 8 are identical with those of Ex. 7, excepting that

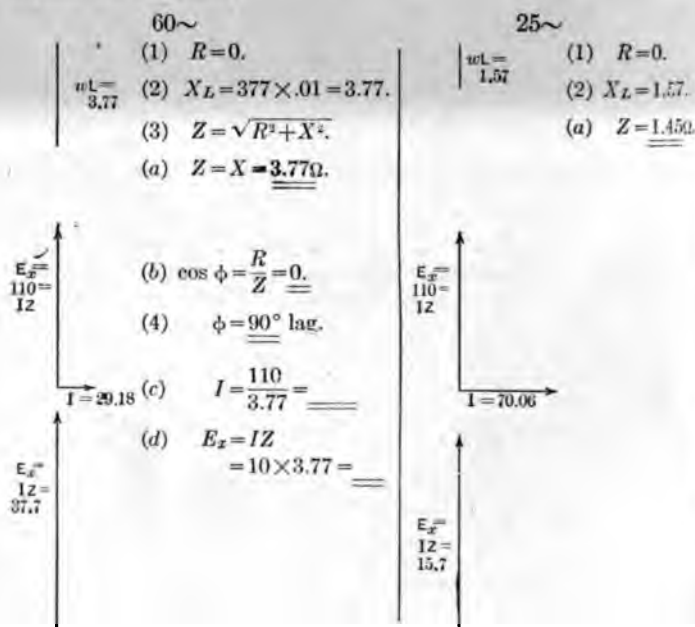
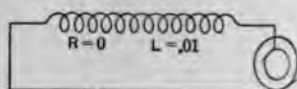


the resistance drop consists of two equal parts. How does this diagram enable us to show that the pressure impressed upon the

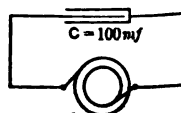
circuit is not equal to the algebraic sum of the drops between the terminals of the parts of the circuit but is equal to the vector sum of the separate pressures.

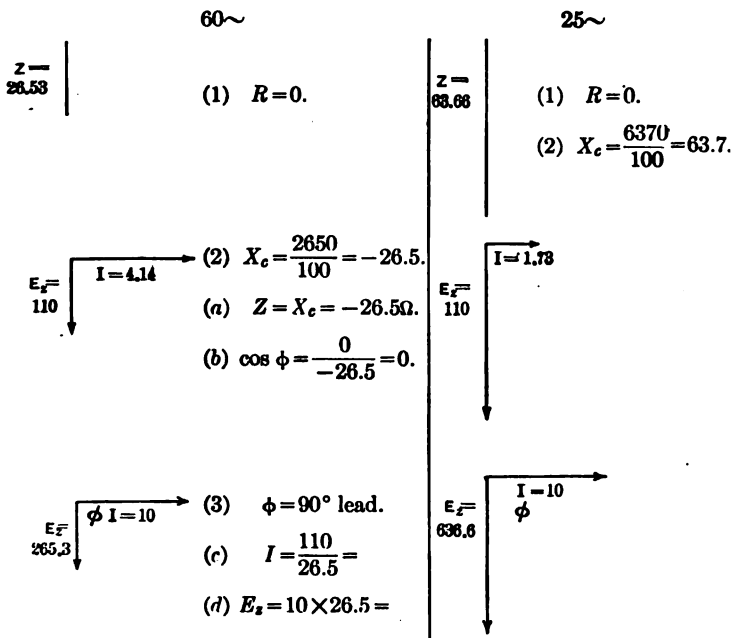
*Observation.* For every series circuit there is an impedance diagram whose three sides  $R$ ,  $X$ , and  $Z$  are in the same ratio as the three vectors  $E_R$ ,  $E_x$ , and  $E$  of the vector diagrams for the same circuit.

**Ex. 9.** The circuit consists of a coil which has .01 h. inductance but negligible resistance. Complete the calculations for both cycles and explain the diagrams.

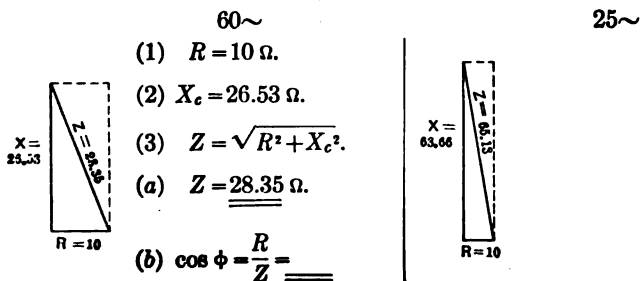
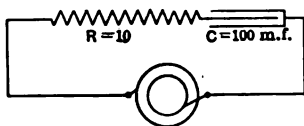


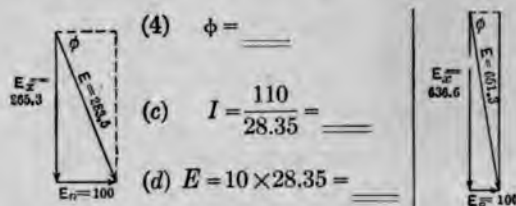
**Ex. 10.** The circuit contains a condenser of 100 mf. capacity. Complete the calculations and explain the diagrams.



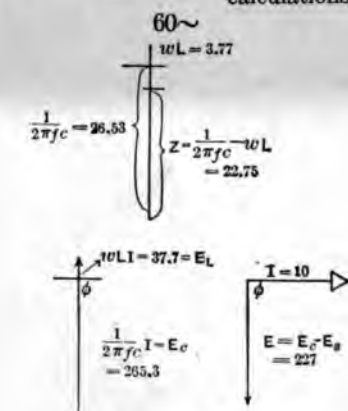


**Ex. 11.** The circuit consists of a resistance of  $10\ \Omega$  in series with a condenser of  $100\ \text{mf.}$  capacity. Complete the calculations and explain the diagrams. Check the numeric with the graphic solution. The upper diagrams are called impedance diagrams and indicate the magnitude relations between  $R$ ,  $X$ , and  $Z$ .





**Ex. 12.** The circuit consists of a condenser of 100 mf. capacity in series with an inductance of .01 h. The upper diagrams show the method of obtaining the excess of reactances which is a negative or capacity excess for both cycles. The solid arrow head indicates current. Complete the calculations.



(1)  $X_L = 377 \times .01 = 3.77 \Omega.$

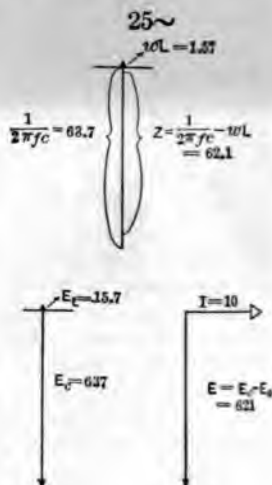
(2)  $X_C = \frac{2653}{100} = -26.53 \Omega.$

(3)  $X = X_L - X_C = -22.75.$

(a)  $Z = X = \underline{\underline{22.75.}}$

(b)  $P.F. = \frac{R}{Z} = \frac{0}{X_L - X_C} = 0.$

(4)  $\phi = \underline{\underline{90^\circ \text{ lead.}}}$

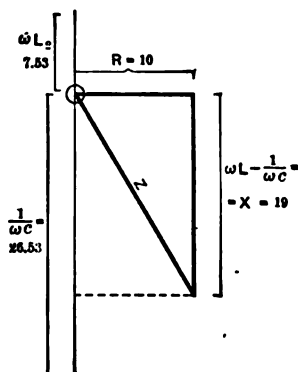


**Ex. 13.** Perform the work for the graphic and numeric solution of the following A.C. circuits: (a)  $R=10\ \Omega$ ,  $L=.015\ \text{h.}$ ; (b)  $R=15\ \Omega$ ,  $L=.015\ \text{h.}$ ; (c)  $R=20\ \Omega$ ,  $L=.03\ \text{h.}$ ; (d)  $R=30\ \Omega$ ,  $L=.03\ \text{h.}$ ; (e)  $R=10\ \Omega$ ,  $C=100\ \text{mf.}$ ; (f)  $R=15\ \Omega$ ,  $C=150\ \text{mf.}$ ; (g)  $R=20\ \Omega$ ,  $C=200\ \text{mf.}$ ; (h)  $R=30\ \Omega$ ,  $C=300\ \text{mf.}$ ; (i)  $R=5\ \Omega$ ,  $C=50\ \text{mf.}$ ; (j)  $C=100\ \text{mf.}$ ,  $L=.01\ \text{h.}$ ; (k)  $C=200\ \text{mf.}$ ,  $L=.02\ \text{h.}$ ; (l)  $C=50\ \text{mf.}$ ,  $L=.005\ \text{h.}$

**Ex. 14.** Use the values given in Ex. 13 for a frequency of 100.

**Ex. 15.** Use the values given in Ex. 13, using an impressed voltage of 120 volts.

**Ex. 16.** The circuit consists of  $10\ \Omega$  resistance in series  $100\ \text{mf.}$  capacity and  $.02\ \text{h.}$  inductance. Construct the impedance diagram shown in shaded lines. Its vertical side represents the magnitude  $X$ , the excess of reactance, its horizontal side represents the magnitude of  $R$  and its diagonal side represents the magnitude of the impedance of the circuit. The current and impressed E.M.F. will be directionally coincident with  $R$  and  $Z$  respectively, and their phase angle will be numerically the same as the angle between  $R$  and  $Z$ .



**Ex. 17.** The circuit consists of the following elements: (a)  $R=10\ \Omega$ ,  $C=150\ \text{mf.}$ ,  $L=.015\ \text{h.}$ ; (b)  $R=10\ \Omega$ ,  $C=100\ \text{mf.}$ ,  $L=.015\ \text{h.}$ ; (c)  $R=10\ \Omega$ ,  $C=100\ \text{mf.}$ ,  $L=.03\ \text{h.}$ ; (d)  $R=20\ \Omega$ ,  $C=150\ \text{mf.}$ ,  $L=.015\ \text{h.}$ ; (e)  $R=20\ \Omega$ ,  $C=100\ \text{mf.}$ ,  $L=.015\ \text{h.}$ ; (f)  $R=20\ \Omega$ ,  $C=100\ \text{mf.}$ ,  $L=.03\ \text{h.}$ ; (g)  $R=20\ \Omega$ ,  $C=300\ \text{mf.}$ ,  $L=.03\ \text{h.}$ ; (h)  $R=10\ \Omega$ ,  $C=150\ \text{mf.}$ ,  $L=.15\ \text{h.}$ ; (i)  $R=10\ \Omega$ ,  $C=150\ \text{mf.}$ ,  $L=.3\ \text{h.}$

**3. Resonance.** A series circuit is said to be resonant or in resonance when the current and E.M.F. in the supply circuit are in phase. This means that the current attains its greatest value and the impedance formula (2) reduces to (3). It also implies that the total reactance is neutralized and therefore equals zero and accordingly (4) follows when  $X=0$ .

$$(2) \quad I = \frac{E}{\sqrt{R^2 + X^2}}$$

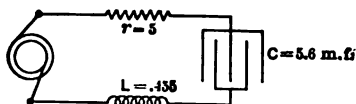
$$(3) \quad I = \frac{E}{R}$$

$$(4) \quad 2\pi fL - \frac{1}{2\pi fC} = 0.$$

Solve (4) for  $f$ . The resulting equation gives the **critical frequency**, i.e., the frequency which makes the circuit resonant. The **natural period** of a circuit is the reciprocal of its critical frequency.

**Ex. 18.** Determine the critical frequency and the natural period of the circuits specified in Ex. 17.

**Ex. 19.** Construct a resonance curve, shown in Fig. 235, plotting amperes vertically and frequency horizontally for a circuit having 5 ohms resistance, an inductance of .455 henry and a capacity of 5.6 microfarads. An E.M.F. of 100 volts is impressed



upon this circuit. What frequency is required for resonance? At this frequency what is the potential difference between the terminals of the condenser and that across the inductance coils?

**Ex. 20.** Construct resonance curves for the circuits described in Ex. 17.

**4. Problems Solved by Adding Impedances.** A circuit may have any number of pieces of apparatus in series each of which may or may not possess resistance, inductance, and capacity. The fundamental principle guiding us in the investigation of such a circuit is that when an E.M.F. is impressed there is but one current which has the same value throughout the circuit. The pressure at the terminals of each piece of apparatus has a magnitude and phase dependent upon the respective values of  $R_s$ ,  $L_s$ , and  $C_s$ . We have seen that in order to determine the pressure nec-

to send a definite A.C. through such a series we must add vectorially the pressures required to send the current through the separate pieces of the circuit. In the vector diagram the quantity  $I$  appears as a common factor and therefore the problem resolves itself into the addition of impedances.

**Ex. 21.** The circuit consists of four pieces of apparatus,  $R_1 = 80$  ohms,  $L_1 = .2$  henry,  $C_1 = 20$  microfarads;  $R_2 = 50$  ohms:

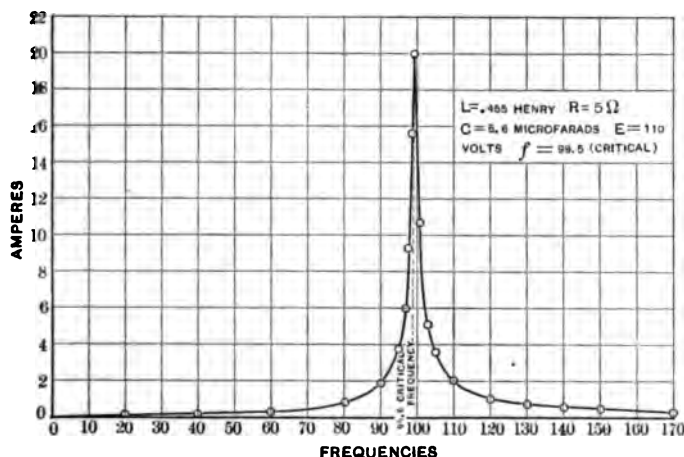


FIG. 235.—A Resonance Curve.

$L_2 = .25$  henry;  $C_2 = 50$  microfarads;  $R_3 = 75$  ohms. Determine the impedance for the four respective pieces of apparatus, designating them by  $Z_1, Z_2, Z_3, Z_4$ . Add the four  $Z$ s vectorially. Designate the terminal pressure of the four parts of the circuit by  $E_1, E_2, E_3$ , and  $E_4$  respectively, and the current by  $I$ . The drop across any impedance is expressed in the general formula (5) from which (6), (7), (8), and (9) follow:

$$(5) \quad E = I \times \text{impedance.}$$

$$(6) \quad E_1 = I \sqrt{R_1^2 + \left[ \omega L_1 - \frac{1}{\omega C_1} \right]^2} = I Z_1.$$

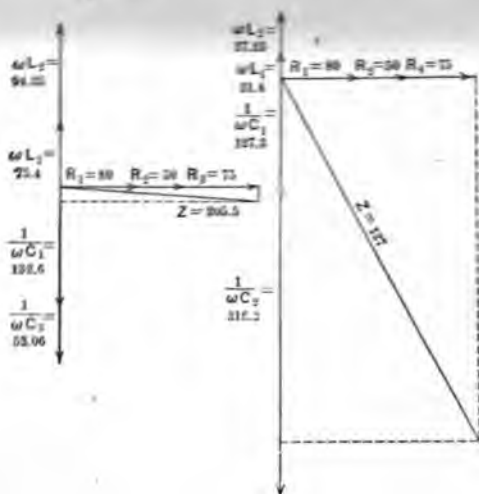
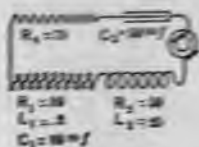
$$(7) \quad E_c = I \sqrt{R_c^2 + (-X_c)^2} = IZ_c^* \quad \text{---}$$

$$(8) \quad E_c = I \sqrt{\left(\frac{-1}{\omega C_c}\right)^2} = IZ_c \quad \text{---}$$

$$(9) \quad E_c = I \sqrt{(R_c)^2} = IZ_c \quad \text{---}$$

$$(10) \quad \therefore E = IZ = IZ_1 \oplus IZ_2 \oplus IZ_3 \oplus IZ_4 \quad \text{---}$$

$$(11) \quad \therefore E = I(Z_1 \oplus Z_2 \oplus Z_3 \oplus Z_4) \quad \text{---}$$



The interpretation of (11) states that the current times vector sum of the impedance equals the impressed E.M.F.

The combined impedance is the vector sum of the separate impedances and may be obtained by the method of vertical horizontal components, i.e., by  $R_s$  and  $X_s$  respectively, as explained in (12) and (13):

$$(12) \quad E = IZ = I\sqrt{(\sum \text{res})^2 + \left[\sum \frac{\text{inductive}}{\text{reactances}} - \sum \frac{\text{capacity}}{\text{reactances}}\right]^2}.$$

$$(13) \quad E = I\sqrt{(R_1 + R_2 + R_4)^2 + \left[\omega(L_1 + L_2) - \frac{1}{\omega}\left(\frac{1}{C_1} + \frac{1}{C_2}\right)\right]^2}.$$

Substitute the data from Ex. 21 and compute  $E$ ,  $E_1$ ,  $E_2$ ,  $E_3$ , and  $E_4$ . State the interpretation of (14):

$$(14) \quad E_1 + E_2 + E_3 + E_4 \neq E.$$

**Ex. 22.** Complete the following statements from observations upon series circuits:

I. Non-inductive circuits in series.

How is the total impressed pressure determined?

How is the total resistance determined?

II. Inductive circuits of equal time constants connected in series.

How is the total impressed pressure determined?

How is the total impedance determined?

III. Inductive and non-inductive circuits in series or inductive circuits of unequal time constant in series.

How is the total impressed pressure determined?

How is the total impedance determined?

What relation exists between total and individual pressures, also between the total and individual impedances?

IV. Condensers connected in series by conductors of negligible resistance.

How is the total impressed pressure determined?

How is the total impedance determined?

V. Condensers in series with non-inductive resistances.

How is the total impressed pressure determined?

How is the total impedance determined?

What magnitude relations exist for total and individual pressures, also the relation of impedances?

VI. State the relations for condensers in series with inductances and resistances.

**Ex. 23.** Make complete calculations for the following circuits made up with the following parts of Ex. 17: (a) and (b); a) and (c); (a) and (d); (a) and (e); (a) and (f); (a) and (g); a) and (h); (a) and (j); (a) and (i); (b) and (c); (b) and (d); b) and (e); (b) and (f); (b) and (g); (b) and (h); (b) and (i); c) and (d); (c) and (e); (c) and (f); (c) and (g); (c) and (h); c) and (i); (d) and (e); (d) and (f); (d) and (g); (d) and (h);

(d) and (i); (e) and (f); (e) and (g); (e) and (h); (e) and (i);  
 (f) and (g); (f) and (h); (f) and (i); (g) and (h); (g) and (i);  
 (h) and (i).

*Observation.* In a series circuit there is one current line which is constructed horizontal and becomes a line of reference from which to measure the phase angles of the respective E.M.F.s. The impedance diagram is a triangle similar to the triangle whose sides are  $E_r$ ,  $E_x$ , and  $E$ .  $R$ 's,  $X$ 's, and  $Z$ 's, although not vectors, are added vectorially. By placing arrow-heads on an impedance diagram the  $R$ ,  $X$ , and  $Z$  lines become respectively  $E_r$ ,  $E_x$ , and  $E$ . The  $E_r$  vector represents the drop across resistance  $R$  and is always in phase with the current  $I$ .

## CHAPTER XXX

### PARALLEL CIRCUITS

1. The graphic treatment of problems relating to parallel circuits is analogous to the treatment for series circuits. In the preceding chapter series circuits were solved by adding E.M.F.'s vectorially and in such cases the current line was laid off as the horizontal reference line. In parallel circuits the E.M.F. vector is the horizontal line of reference and the circuits are solved by adding currents vectorially. For series circuits an impedance diagram is constructed, whereas for parallel circuits an admittance diagram is constructed.

2. An **Equivalent Impedance** is a single impedance which may replace several impedances in a circuit. The equivalent impedance  $Z$  of a group of parallel circuits whose individual impedances are expressed by  $Z_1$ ,  $Z_2$ ,  $Z_3$ , and  $Z_4$  is determined indirectly by considering their respective admittances. Admittance has been defined as the factor which when multiplied into the terminal voltage of a circuit or of a piece of apparatus determines the current passing through it. The common terminal voltage of four branches of a parallel circuit is  $E$  and the respective currents through each circuit is expressed in (1), (2), (3), and (4):

$$(1) \quad I_1 = \frac{E}{Z_1} = EY_1.$$

$$(2) \quad I_2 = \frac{E}{Z_2} = EY_2.$$

$$(3) \quad I_3 = \frac{E}{Z_3} = EY_3.$$

$$(4) \quad I_4 = \frac{E}{Z_4} = EY_4.$$

The total current  $I$  flowing through the mains is the vector sum of the currents supplied to the branches, as expressed in (5):

$$(5) \quad I = I_1 \oplus I_2 \oplus I_3 \oplus I_4.$$

$$(6) \quad I = E(Y_1 \oplus Y_2 \oplus Y_3 \oplus Y_4).$$

$$(7) \quad Y = Y_1 \oplus Y_2 \oplus Y_3 \oplus Y_4.$$

$$(8) \quad \therefore I = EY = \frac{E}{Z}.$$

(6) is obtained by substituting (1), (2), (3), and (4) in (5). The interpretation of (6) states that the total current  $I$  may be obtained by multiplying the common terminal voltage by  $Y$  the vector sum of the admittances of the branches. The total current or the current in any branch circuit is the product of the voltage in that circuit times its admittance as expressed in (8).

3. If we consider two branch circuits, then from (1) and (2), we have (9), which states that the total current  $I$  will divide vectorially so that the currents  $I_1$  and  $I_2$  will be directly proportional to the admittances of the respective circuits or inversely proportional to their impedances:

$$(9) \quad \frac{I_1}{I_2} = \frac{Y_1}{Y_2} = \frac{Z_2}{Z_1}.$$

The phase angle of each circuit may be obtained from the power-factor, i.e.,  $\cos \phi = \frac{R}{Z}$ . The phase angles are measured from the horizontal, E.M.F. line. In Fig. 236,

$I_1 = EY_1$  and  $I_2 = EY_2$  are two currents whose vector sum is  $I = EY$ . A parallelogram having its two sides equal to  $I_1$  and  $I_2$  respectively has a diagonal equal to  $I$ .

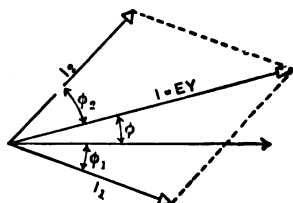


FIG. 236.—The Vector Diagram for a Divided Circuit.

Fig. 237 shows the current  $I$  resolved into a component  $I_g$  in phase with the E.M.F.  $E$  and another component  $I_b$  perpendicular to the E.M.F. The component  $I_g$  may be obtained by multiplying  $E$  by a factor  $g$ , which is designated the **conductance** factor, and  $I_b$  may be obtained by multiplying  $E$  by a factor  $b$ , which is designated the **susceptance** factor.

Each line of the left-hand triangle of Fig. 237 has a common factor  $E$  and therefore is similar to the admittance

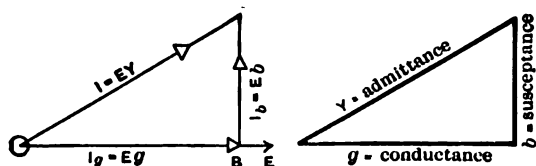


FIG. 237.—A Current Resolved into Two Orthogonal Components.

triangle which is the heavily shaded figure on the right. Therefore (10), (11), (12), and (13) follow and by using subscripts these equations apply to any circuit or branch:

$$10) \quad \cos \phi = \frac{g}{Y}, \quad (11) \quad \sin \phi = \frac{b}{Y},$$

$$12) \quad \tan \phi = \frac{b}{g}, \quad (13) \quad Y = \sqrt{g^2 + b^2}.$$

In (10), (11), and (12)  $\phi$  is the phase angle between current and E.M.F. If  $E$  be resolved into a component  $E_r = IR$  in phase with  $I$  and a component  $E_x = IX$  perpendicular to  $E_r$ , then the three vectors will constitute a right triangle which will be similar to an impedance triangle, containing  $R$ ,  $X$ , and  $Z$  and  $\phi$ . The impedance triangle and the admittance triangle are similar and therefore their homologous sides are proportional, as expressed in (14) and (15), and therefore (16), (17), (18), and (19) follow:

$$(14) \quad \frac{Y}{Z} = \frac{g}{R} \quad (15) \quad \frac{Y}{Z} = \frac{b}{X}$$

$$(16) \quad \therefore g = \frac{RY}{Z} = \frac{R}{Z^2} = \frac{R}{R^2 + X^2}$$

$$(17) \quad \therefore b = \frac{XY}{Z} = \frac{X}{Z^2} = \frac{X}{R^2 + X^2}$$

$$(18) \quad \therefore R = \frac{gZ}{Y} = \frac{g}{Y^2} = \frac{g}{g^2 + b^2}$$

$$(19) \quad \therefore X = \frac{bZ}{Y} = \frac{b}{Y^2} = \frac{b}{g^2 + b^2}$$

$$(20) \quad g = Y \cos \phi \quad (21) \quad b = Y \sin \phi$$

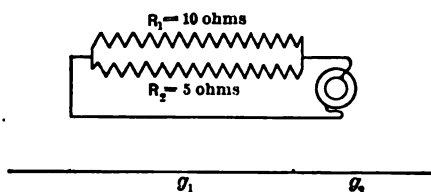
$$(22) \quad Y = \sqrt{g^2 + b^2} = \frac{1}{Z} = \frac{1}{\sqrt{R^2 + X^2}}$$

By means of (16) and (17)  $g$  and  $b$  are expressed in terms of resistance, reactance, and impedance. By means of (18) and (19)  $R$  and  $X$  are expressed in terms of conductance, susceptance, and admittance.

In the following problems it is desired to have the following quantities determined for each branch, as well as for the total circuit at frequencies of 60 ~ and 25 ~ : (a) The impedance and admittance; (b) the current which flows through the circuit when 120 volts is applied to the circuit; (c) the impressed E.M.F. required to overcome the drop across the mains; (d) the phase

angle and power factors for the circuits. The circuits are distinguished by assigning a like subscript to everything connected in the same branch.

**Ex. 1.** A circuit contains two non-inductive resistances  $R_1 = 5 \Omega$  and  $R_2 = 10 \Omega$  in parallel. The admittance diagram consists of two  $g$  lines which represent the two conductances .2 and .1 respectively. Their sum equals .3, and therefore the two



circuits may be replaced by a single circuit with an equivalent impedance, resistance, and reactance:

$$(1) \quad g_1 = \frac{1}{5} = .2, \quad b_1 = 0.$$

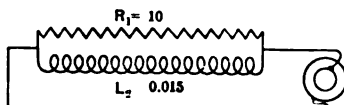
$$(2) \quad g_2 = \frac{1}{10} = .1, \quad b_2 = 0.$$

$$(3) \quad g = g_1 + g_2 = .3.$$

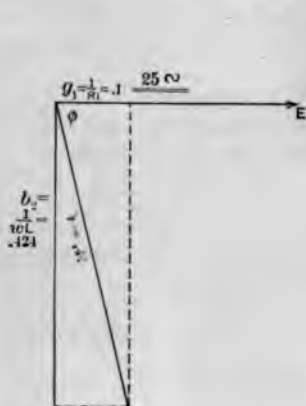
$$(4) \quad Y = \sqrt{g^2 + b^2} = g = .3.$$

$$(5) \quad Z = \frac{1}{Y} = \frac{1}{.3} = 3.33.$$

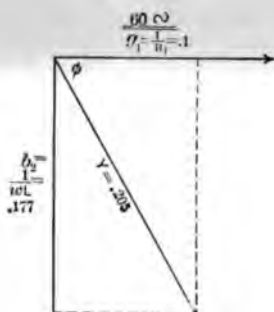
**Ex. 2.** A circuit consists of a non-inductive branch of  $10 \Omega$  resistance and another inductive branch of .015 h. inductance.



Complete the calculations and explain the admittance diagrams. Construct the vector diagrams.

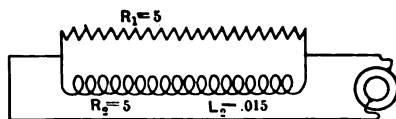


- 25 ~
- (1)  $g_1 = \frac{1}{R_1} = .1.$
  - (2)  $b_2 = \frac{1}{X_2} = .424.$
  - (3)  $Y = \sqrt{g_1^2 + b_2^2} = .43.$
  - (4)  $Z = \frac{1}{Y} = 2.32.$
  - (5)  $I = EY = 51.6.$
  - (6)  $E = IZ = 232.$
  - (7)  $\cos \phi = \frac{g}{Y} =$
  - (8)  $\phi =$
- 60 ~

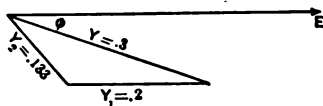
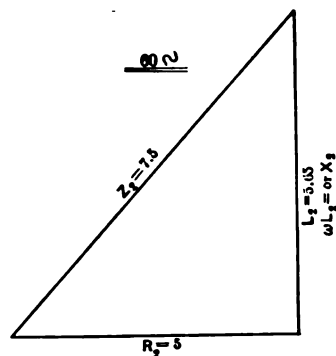
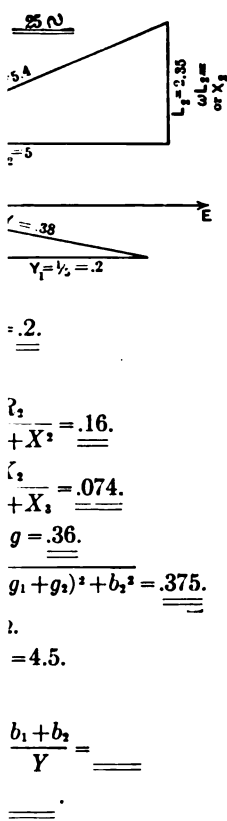


- (1)  $g_1 =$
- (2)  $b_2 =$
- (3)  $Y = \sqrt{g_1^2 + b_2^2} =$

**Ex. 3.** The circuit consists of a non-inductive branch of resistance, and a second branch of  $5\Omega$  resistance and  $.015$  h. in. reactance. The impedance and angle of lag may be determined

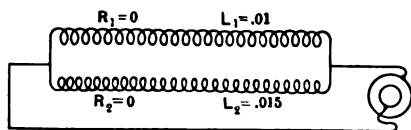


one circuit by means of the impedance diagram. The lower diagram shows the vector sum  $Y$  of the admittances  $Y_1$  and  $Y_2$ . The angle between  $E$  and  $Y$ , equals the angle between  $Z$  and  $R$ . Construct the vector diagrams and explain. Complete the calculations.



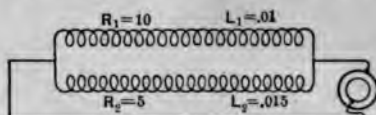
- (1)  $g_1 = \frac{1}{R_1} =$
- (2)  $b_1 = 0.$
- (3)  $g_2 = \frac{R_2}{R_2^2 + X_2^2} =$

The circuit consists of two inductive branches of .01 h. inductance respectively. The admittance diagram



st of a b line only which would also represent the y  
ruct the vector diagram.

**Ex. 5.** The circuit consists of two branches,  $R_1 = 10\Omega$ ,  $L_1 = .01$  h., and  $R_2 = 5\Omega$ ,  $L_2 = .015$  h. respectively. Determine the impedance of each branch separately and combine the corresponding recip-

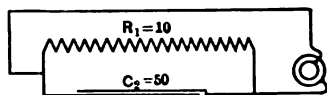


rocals. Calculate  $b$  and  $g$  for each branch and substitute in the formula which expresses  $Y$  in terms of the vertical and horizontal components of  $Y_1$  and  $Y_2$ :

$$Y = \sqrt{(g_1 + g_2)^2 + (b_1 + b_2)^2}.$$

**Ex. 6.** When two or more condensers are joined in parallel by wires whose resistances are negligible their combined effect is equivalent to a single condenser whose capacity is the sum of the capacities. Illustrate by an admittance diagram for condensers of .01 and .015 microfarad respectively.

**Ex. 7.** The circuit consists of a non-inductive branch  $R_1 = 10$  ohms and a capacity branch  $C_2 = 50$  mf.



25 ~

- (1)  $g_1 = \underline{.1.}$
- (2)  $b_1 = 0.$
- (3)  $g_2 = 0.$
- (4)  $b_2 = \underline{.00787.}$
- (5)  $Y = \sqrt{g_1^2 + b_2^2} = \underline{.10039.}$
- (6)  $Z = \underline{9.96\Omega.}$
- (7)  $E = IZ = 10 \times 9.96 = \underline{9.96 \text{ volts.}}$
- (8)  $I = EY = \underline{12.05 \text{ amps.}}$
- (9)  $\cos \phi = \underline{\hspace{1cm}}$
- (10)  $\phi = \underline{\hspace{1cm}}.$

60 ~

- (1)  $g_1 = \underline{.1.}$
- (2)  $b_1 = 0.$
- (3)  $g_2 = 0.$
- (4)  $b_2 = \underline{.0188.}$

**Ex. 8.** The circuit consists of two branches,  $R_1 = 5 \Omega$ ,  $R_2 = 5 \Omega$ , and  $C_2 = 50 \text{ mf.}$

**Ex. 9.** The circuit consists of two reactive branches  $R_1 = 10 \Omega$ ,  $C_1 = 50 \text{ mf.}$ , and  $R_2 = 15 \Omega$ ,  $C_2 = 75 \text{ mf.}$

**Ex. 10.** The following branches are to be grouped as designated by the subscripts: (a) -1, 2; (b) -1, 3; (c) -1, 4; (d) -1, 5; (e) -1, 6; (f) -1, 7; (g) -1, 8; (h) -1, 9; (i) -1, 10; (j) -1, 11; (k) -1, 12; (l) -3, 4; (m) -5, 6; (n) -7, 8; (o) -9, 10; (p) -11, 12; (q) -2, 3; (r) -2, 4; (s) -2, 5; (t) -2, 6; (u) -2, 7; (v) -2, 8; (w) -2, 9; (x) -2, 10; (y) -2, 11; (z) -2, 12.

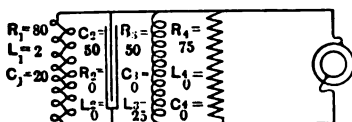
- |   |   |
|---|---|
| (1) $R_1 = 5 \Omega$ , $L_1 = .01 \text{ h.};$    | (7) $R_7 = 10 \Omega$ , $L_7 = .005 \text{ h.};$        |
| (2) $R_2 = 10 \Omega$ , $C_2 = 50 \text{ mf.};$   | (8) $R_8 = 0 \Omega$ , $C_8 = 100 \text{ mf.};$         |
| (3) $R_3 = 5 \Omega$ , $L_3 = .005 \text{ h.};$   | (9) $R_9 = 0 \Omega$ , $L_9 = .015 \text{ h.};$         |
| (4) $R_4 = 15 \Omega$ , $C_4 = 150 \text{ mf.};$  | (10) $R_{10} = 10 \Omega$ , $C_{10} = 100 \text{ mf.};$ |
| (5) $R_5 = 10 \Omega$ , $L_5 = .0105 \text{ h.};$ | (11) $R_{11} = 10 \Omega$ , $L_{11} = .015 \text{ h.};$ |
| (6) $R_6 = 10 \Omega$ , $C_6 = 100 \text{ mf.};$  | (12) $R_{12} = 0 \Omega$ , $C_{12} = 150 \text{ mf.}$   |

If a number of circuits are placed in parallel their combined current is the vector sum of the currents in each branch. The total admittance is the vector sum of the admittances of the branches. The total admittance is expressed in (23) and is obtained by adding the square of the sum of the conductances, i.e., the  $g$ 's of the branches to the square of the sum of the susceptances, i.e., the  $b$ 's of the branches.

$$(23) \quad Y = \sqrt{(g_1 + g_2 + g_3 + \dots)^2 + (b_1 + b_2 + b_3 + \dots)^2}.$$

$$(24) \quad I = E \sqrt{(g_1 + g_2 + g_3 + g_4)^2 + (b_1 + b_2 + b_3 + b_4)^2}.$$

**Ex. 11.** A circuit has the following four branches. Determine the current in each branch and determine the total current by substituting in (24):



$$R_1 = 80 \Omega, L_1 = .2 \text{ h.}, C_1 = 20 \text{ mf.};$$

$$R_2 = 0, L_2 = 0, C_2 = 50 \text{ mf.};$$

$$R_3 = 50 \Omega, L_3 = .25 \text{ h.}, C_3 = 0;$$

$$R_4 = 75 \Omega, L_4 = 0, C_4 = 0.$$

**Ex. 12.** Combine the following circuits which are described in Ex. 9: (a) -1, 2, 3, 4; (b) -5, 6, 7, 8; (c) -9, 10, 11, 12; (d) -1, 2, 3, 5; (e) -1, 2, 3, 6; (f) -1, 2, 3, 7; (g) -1, 2, 3, 8; (h) -1, 2, 3, 9; (i) -1, 2, 3, 10; (j) -1, 2, 3, 11; (k) -1, 2, 3, 12; (l) -1, 2, 4, 5; (m) -1, 2, 4, 6; (n) -1, 2, 4, 7; (o) -1, 2, 4, 8; (p) -1, 2, 4, 9; (q) -1, 2, 4, 10; (r) -1, 2, 4, 11; (s) -1, 2, 4, 12; (t) -1, 2, 5, 6; (u) -1, 2, 5, 7; (v) -1, 2, 5, 8; (w) -1, 2, 5, 9; (x) -1, 2, 5, 10; (y) -1, 2, 5, 11; (z) -1, 2, 5, 12.

**Ex. 13.** From observations upon parallel circuits, state the conclusions for determining the total current and admittances for the following conditions:

- I. Non-inductive resistances in parallel.
- II. Inductive circuits of equal time constants.
- III. Non-inductive and inductive resistances.
- IV. Condensers connected with negligible resistance.
- V. When condensers are connected in parallel with non-inductive resistances.
- VI. When condensers are in parallel with inductive circuits.

**4. The Circle Diagram.** From (12), (14), and (15) we derive (24):

$$(25) \quad \tan \phi = \frac{b}{g} = \frac{X}{R}.$$

Formula (25) enables us to compute the problems for parallel circuits by using E.M.F.s and impedances. It depends upon the geometric principle that an angle inscribed in a semicircle is a right angle. Two mutually orthogonal sides of a right angle are used to determine the active and wattless components of the total current, as shown in Fig. 238. For the purpose of the discussion we shall use the following data:  $R_1 = 10\Omega$ ,  $L_1 = .01$  h.,  $R_3 = 5\Omega$ ,  $L_3 = .015$  h.

In Fig. 238  $OX$  represents the impressed E.M.F. = 120 volts and is used for the diameter of the semicircle  $OCDAX$ .

From  $O$  lay off  $\phi_1 = \tan^{-1} \frac{b_1}{g_1} = \tan^{-1} \frac{X_1}{R_1} = 20^\circ 39'$ . Then  $OA = I_1 R_1$  and  $XA = \omega L_1 I_1$ . The current in branch 1 is in phase with  $OA$  and is represented by  $I_1 = \frac{OA}{R_1}$ . From  $O$

lay off  $\phi_2 = \tan^{-1} \frac{b_2}{g_2} = \tan^{-1} \frac{X_2}{R_2} = 48^\circ 31'$ . Then  $OC = I_3 R_3$  and  $XC = \omega L_3 I_3$ . The current in branch 3 is in phase with  $OC$  and is represented by  $OH = I_3 = \frac{OC}{R_3}$ . The vector sum of  $I_1$  and  $I_3$  is represented by  $I_2$  whose phase angle  $\phi_2 = 36^\circ 12'$ . The equivalent resistance drop and equivalent

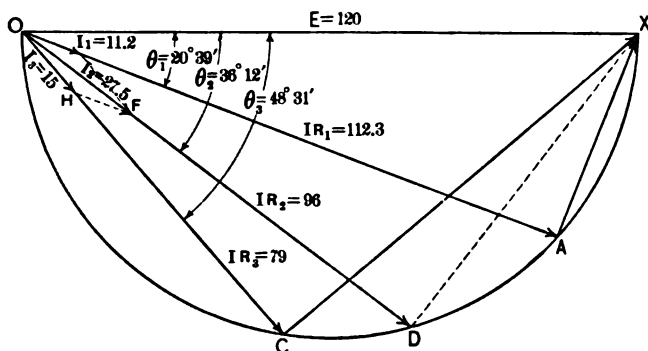


FIG. 238.—The Impedance Solution of Parallel Circuits.

reactance drop for the combination are represented by  $OD$  and  $DX$  respectively. The equivalent resistance  $R = \frac{OD}{I_2}$  and the equivalent reactance  $X = \frac{DX}{I_2}$ .

**Ex. 14.** Apply the principle of the circle diagram to the solution of Exs. 8 and 9.

## CHAPTER XXXI

### SERIES PARALLEL CIRCUITS

1. A series circuit may contain a number of groups of impedances in which each group consists of a number of impedances in parallel. Such a circuit is known as a **series parallel circuit**. The equivalent impedance of each group is determined by the method for solving a parallel circuit and then the entire circuit with its equivalent impedances is treated as a series circuit. When several impedances of a group are replaced by an equivalent impedance  $Z_n$  the latter has an equivalent resistance  $R_n$  and an equivalent reactance  $X_n$ , which are obtained by multiplying the impedance by the respective cos and sin of the phase angle  $\phi_n$  of the group:

$$(1) \quad R_n = Z_n \cos \phi_n.$$

$$(2) \quad X_n = Z_n \sin \phi_n.$$

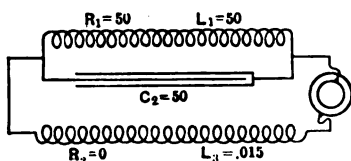
Determine all the data and properties of the branches and groups as well as for the entire circuit in the following examples. Use 60~ and 25~ and calculate the results:

**Ex. 1.** A circuit consists of an impedance  $B$  in series with a group  $A$ . The group  $A$  consists of two branch circuits  $R_1 = 50\Omega$ ,  $L_1 = 50$  milhenrys, and  $C_2 = 50$  mf. The impedance  $B$  consists of an inductance  $L_3 = .015$  h. The following equations suggest the method of procedure. Check by vector diagram:

$$(1) \quad g_1 = \frac{R_1}{R_1^2 + X_1^2} = \frac{50}{50^2 + (18.85)^2} = \quad (3) \quad b_2 = \frac{1}{-X_2} = \frac{50}{2650} =$$

$$(2) \quad b_1 = \frac{X_1}{R_1^2 + X_1^2} = \frac{18.85}{50^2 + (18.85)^2} \quad (4) \quad Y_A = \sqrt{g_1^2 + (b_1 - b_2)^2}$$

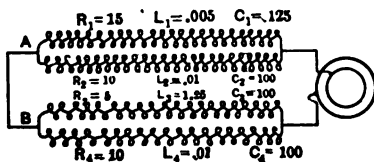
- (5)  $Z_A = \frac{1}{Y_A}$ . (12)  $E_A = I_1 Z_A$ .
- (6)  $\phi_A = \cos^{-1} \frac{g_1}{y_A}$ . (13)  $E_B = I_1 Z_B$ .
- (7)  $R_A = Z_A \cos \phi_A$ . (14)  $I_1 = \frac{E_A}{Z_1}$  lagging.
- (8)  $X_A = Z_A \sin \phi_A$ . (15)  $I_2 = \frac{E_A}{Z_2}$  leading.
- (9)  $Z_{\text{total}} = \sqrt{(R_A + R_B)^2 + (X_A + X_B)^2}$ . (16)  $\phi_1 = \cos^{-1} \frac{R_1}{Z_1}$  lag.
- (10)  $\phi_{\text{total}} = \cos^{-1} \frac{R_A + R_B}{Z_{\text{total}}}$ . (17)  $\phi_2 = 90^\circ$  lead.
- (11)  $I_{\text{total}} = \frac{E}{Z_{\text{total}}}$ . (18)  $\phi_3 = 90^\circ$  lag.



**Ex. 2.** The circuit consists of *A* and *B* in series. *A* consists of two branches  $R_1 = 10\Omega$ ,  $L_1 = .01$  h., and  $R_2 = 5\Omega$ ,  $L_2 = .015$  h. *B* consists of  $R_3 = 10\Omega$ ,  $L_3 = .005$  h.

**Ex. 3.** The circuit consists of *A* and *B* in series. *A* consists of two branches  $R_1 = 15\Omega$ ,  $L_1 = .005$  h.,  $C_1 = 125$  mf., and  $R_2 = 10\Omega$ ,  $L_2 = .01$  h.,  $C_2 = 100$  mf. *B* consists of  $R_3 = 5\Omega$ ,  $L_3 = .125$  h.,  $C_3 = 100$  mf.

**Ex. 4.** The circuit consists of *A* and *B* in series, as shown in the figure.



## CHAPTER XXXII

### ALTERNATING CURRENT PROBLEMS

**Ex. 1.** Fig. 239 represents the cross-section of a 20-pole revolving field of an alternator. How many revolutions will it require to give frequencies of 25~ and 60~ respectively?

**Ex. 2.** Ascertain from trade catalogues and other sources the number of poles adapted for a line of commercial alternators. From the information compute the number of revolutions which will produce 25 and 60 cycles per second respectively. Tabulate the data.

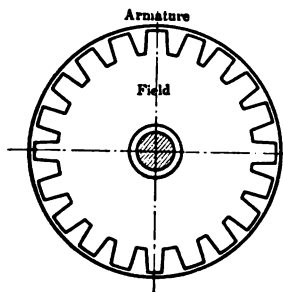


FIG. 239.—Revolving Field of Alternator.

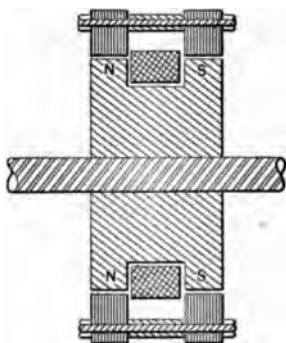


FIG. 240.—Inductor Type Alternator.

**Ex. 3.** A machine of the inductor type, Fig. 240, is run at 1500 revolutions per minute and has 200 polar projections on each end. What is its frequency?

In the following problems make a sketch of each circuit showing the symbols for the different apparatus. Indicate the numeric values of the supply voltage and other data. Number

each formula and equation and arrange the work neatly and methodically according to the standard set for this department.

**Ex. 4.** Determine the power factor, P.F., for each of the following coils which are connected with an ammeter to 110-volt A.C. mains: (a)  $3\Omega$  resistance takes 15 amps; (b)  $2.5\Omega$  takes 12 amps; (c)  $10\Omega$  takes 10 amps; (d)  $110\Omega$  takes .2 amp.; (e)  $100\Omega$  takes 1 amp.; (f)  $50\Omega$  takes 2 amps.; (g)  $25.3\Omega$  takes 3.95 amps.

**Ex. 5.** Determine the resistance of the following coils which are connected to 110-volt A.C. mains and have the following properties: (a) a P.F. = .9 and takes 25 amps.; (b) a P.F. = .2 takes 10 amps.; (c) a P.F. = .85 and takes 1 amp.; (d) a P.F. = .9 and takes 22.5 amps.

**Ex. 6.** What current will pass through the following coils connected to 110-volt A.C. mains: (a) a P.F. = 1 and resistance =  $10\Omega$ ; (b) a P.F. = .9 and resistance  $50\Omega$ ; (c) a P.F. = .85 and resistance =  $25\Omega$ .

**Ex. 7.** An ammeter in series with an arc lamp on an A.C. circuit indicates 6.3 amperes. A voltmeter indicates 80 volts across the arc. A wattmeter connected to the lamp circuit shows 450 watts delivered to the arc. What is the power factor of the lamp?

**Ex. 8.** Two coils Fig. 241 are joined in series across a 110

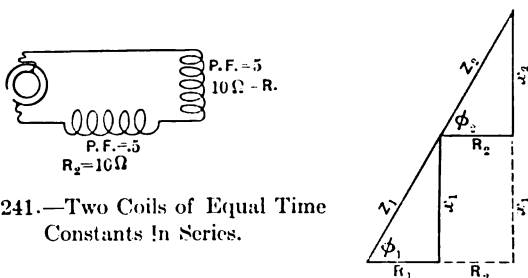


FIG. 241.—Two Coils of Equal Time Constants in Series.

A.C. main. Each coil has a power factor of 0.5 and a resistance of 10 ohms. What current flows through the coils?

**Ex. 9.** Determine the current flowing through the circuit and its power factor when the following coils described in Ex. 5 are connected in series: (A)-(a) (b); (B)-(a) (c); (C)-(a) (d); (D)-(b) (c); (E)-(b) (d); (F)-(c) (d).

**Ex. 10.** Determine the total current flowing into the circuit and its power factor when the following coils described in Ex. 4 are connected in parallel: (A)-(a) (b); (B)-(a) (c); (C)-(a) (d);

(D)-(a) (e); (E)-(a) (f); (F)-(a) (g); (G)-(b) (c); (H)-(b) (d); (I)-(b) (e); (J)-(b) (f); (K)-(b) (g); (L)-(c) (d); (M)-(c) (e); (N)-(c) (f); (O)-(c) (g); (P)-(d) (e); (Q)-(d) (f); (R)-(d) (g); (S)-(e) (f); (T)-(e) (g); (U)-(f) (g).

**Ex. 11.** The current coil (C) of a wattmeter Fig. 242 has a resistance =  $4.5\Omega$  and the series coil (S), has a resistance =  $1050\Omega$ . L is a bank of lamps each of which takes .5 amp. when the supply

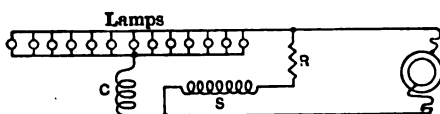


FIG. 242.—Wattmeter Connection.

voltage = 115. (a) How many watts are indicated by the instrument when 1 lamp is burning; (b) when two lamps are burning?

**Ex. 12.** The wattmeter described in Ex. 11 is reconnected as shown in Fig. 243. Determine the readings of the instrument

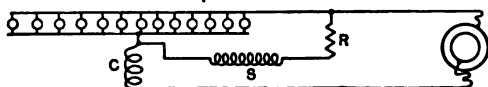


FIG. 243.—Wattmeter Connection.

when (a) one lamp is connected; (b) when two lamps are connected.

**Ex. 13.** Two alternators A and B are connected in series. Each has an E.M.F. of 1000 volts. The two machines are  $90^\circ$  out of phase. The alternators give a current of 120 amperes

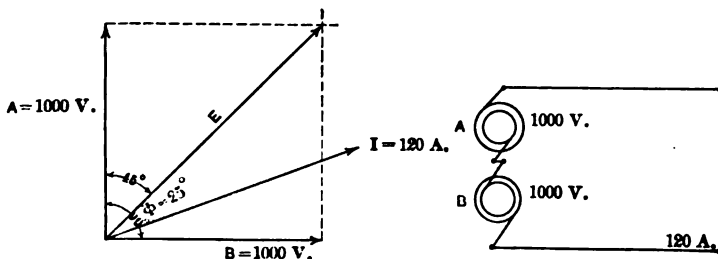


FIG. 244.—Alternators in Series.

which lags  $25^\circ$  in phase behind the resultant E.M.F. Determine the resultant voltage and the output of each generator. Construct A vertically and B horizontally as the adjacent sides of a



**Ex. 17.** Two coils  $E$  and  $F$  are connected across a 110-volt A. C. main. The voltage across  $E$  is 75 volts and the voltage across  $F$  is 81 volts. What is the phase difference in their E.M.F.s?

**Ex. 18.** Two coils  $G$  and  $H$  are connected across a 110-volt A.C. main. The voltage across  $G$  is 90 volts. Their phase angle is  $85^\circ$ . What should be the voltage across  $H$ ?

**Ex. 19.** An alternator supplies 200 amperes to a lamp circuit and delivers 80 amperes to start an induction motor whose power factor is .3 at starting. What is the total power delivered? The lamps and motor are in parallel, as shown in Fig. 240, and therefore

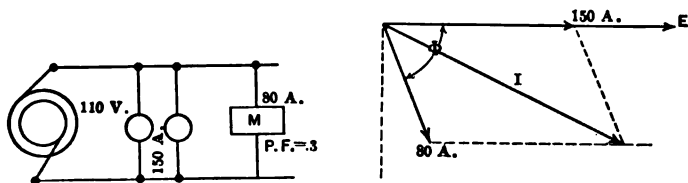


FIG. 247.

the supply voltage  $E = 110$  is laid off horizontally. The current  $I_L$  in the lamps = 150 amps. and is in phase with  $E$ , whereas the the current  $I_m$  in the motor = 80 amps. and lags behind  $E$  by the angle  $\phi = \cos^{-1}(.3)$ . The total current  $I$  is the vector sum of  $I_L$  and  $I_m$ :

$$(1) \quad I = I_L \oplus I_m.$$

Resolve  $I_m$  into a horizontal component  $I_{mg}$  and a vertical component  $I_{mb}$  and determine  $I$  from the sum of the squares of the horizontal and vertical components:

$$(2) \quad I = \sqrt{(150 + 80 \cos \phi)^2 + (80 \sin \phi)^2} =$$

$$(3) \quad \phi_{\text{total}} = \cos^{-1} \frac{150 + 80 \cos \phi}{I} =$$

$$(4) \quad P_{\text{total}} = EI \cos \phi_{\text{total}} =$$

**Ex. 20.** Determine the facts described in Ex. 19, making the following changes: (a)  $I_L = 200$  amps.; (b)  $I_L = 100$  amps.; (c)  $I_L = 80$  amps.; (d)  $I_m = 75$  amps.; (e)  $I_m = 85$  amps.; (f) P.F. at starting = .28; (g) P.F. at starting = .32; (h) P.F. at starting = .25.

**Ex. 21.** Prepare a table for a 10-amp. A.C., giving (a) the change of current at the instant it passes through its zero value which the current would attain at the end

of a twelfth of its cycle if its zero rate continued uniformly; (c) its actual value at one-twelfth of the cycle. Calculate for gradations of 5 from 15~ to 60~ and in gradations of 10 from 60~ to 150~.

**Ex. 22.** Prepare a table for currents in an A.C. circuit supplied with 110 volts. The circuit contains a constant resistance of 100 and a variable inductance ranging from .01 h. to .1 h. in gradations of .01 h. (a) Use 60~. (b) Use 25~. (c) Use 15~. (d) Use 100~.

The tables should be filled out under the following headings, current, inductance, reactance, power factor, phase angle. A note of the constants of the circuit should be entered above the table.

**Ex. 23.** A circuit has a power factor = .9 and is supplied with 200 amps. (effective) from 110-volt (effective) A.C. mains. Determine (a) the maximum E.M.F.  $E_m$ , (b) the maximum current  $I_m$ , (c) the maximum value of the power  $P_{+m}$ , and (d) the minimum value of the power  $P_{-m}$ :

$$(1) \quad E_m = \frac{E}{.707} = \quad (2) \quad I_m = \frac{I}{.707} =$$

In Fig. 248 the vectors represent effective values, whereas in Fig. 249 the vectors represent maximum values. The vertical

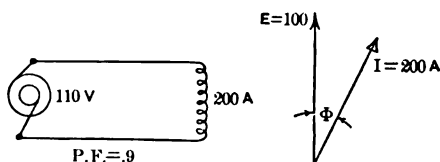


FIG. 248.—The Usual Representation with Effective  $E$  and  $I$ .

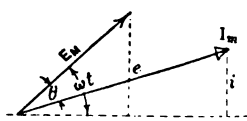


FIG. 249.—Maximum E.M.F. and Current.

dotted lines in Fig. 249 are the instantaneous values  $e$  and  $i$  of E.M.F. and current respectively:

$$(1) \quad P_{+m} = E_m I_m \cos^2 \frac{\phi}{2} = EI(1 + \text{P.F.}) =$$

$$(2) \quad P_{-m} = -E_m I_m \sin^2 \frac{\phi}{2} = -EI(1 - \text{P.F.}) =$$

**Ex. 24.** Determine the maximum and minimum power of the following alternators: (a)  $E = 220$ ,  $I = 500$ ,  $\text{P.F.} = .8$ ; (b)  $E = 110$ ,  $I = 500$ ,  $\text{P.F.} = .8$ ; (c)  $E = 500$ ,  $I = 500$ ,  $\text{P.F.} = .8$ ; (d)  $E = 220$ ,  $I = 500$ ,  $\text{P.F.} = .75$ ; (e)  $E = 1000$ ,  $I = 500$ ,  $\text{P.F.} = .8$ ; (f)  $E = 110$ ,  $I = 50$ ,  $\text{P.F.} = .95$ .

**Ex. 25.** Determine the power factor of the circuits which are supplied by alternators as follows: (a)  $P_{+m} = 50$  K.W.,  $P_{-m} = -5$  K.W.,  $E_m = 120$  volts; (b)  $P_{+m} = 55$  K.W.,  $P_{-m} = -7.5$  K.W.,  $E = 220$  volts; (c)  $P_{+m} = 50$  K.W.,  $P_{-m} = -5$  K.W.,  $I_m = 350$  amps; (d)  $P_{+m} = 55$  K.W.,  $P_{-m} = -7.5$  K.W.,  $I = 225$  amps.

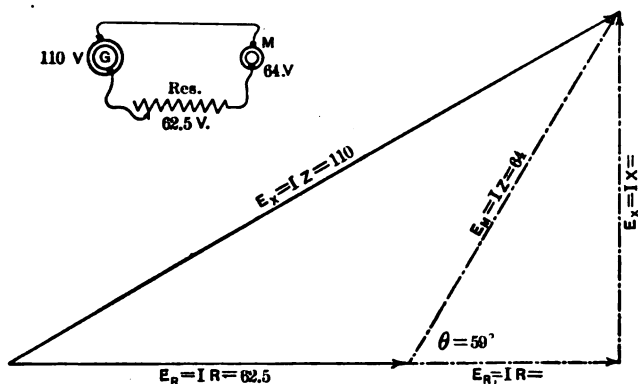
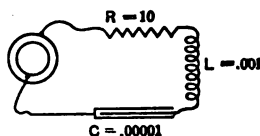


FIG. 250.

**Ex. 26.** Determine the active and wattless component of the power supplied to a fan motor which is connected in series with a non-inductive rheostat across 110-volt mains. When the fan motor takes 1 amp. the drop across it is 64 volts and the drop across the rheostat is 62.5 volts. Explain the vector diagram in Fig. 250.

**Ex. 27.** An A.C. has a maximum value = 135.5 amps. and is connected to a series circuit having a resistance  $R$ , an inductance  $L$ , and a capacity  $C$ . Draw the impedance and vector diagrams for 60~ and 25~ and determine the phase and effective values of the E.M.F. across  $R$ ,  $X_L$ , and  $X_C$  for the following: (a)  $R = 10\Omega$ ,  $L = .001$  h.,  $C = .00001$  F; (b)  $R = 5\Omega$ ,  $L = .001$  h.,  $C = 10$  mf.; (c)  $R = 25\Omega$ ,  $L = .001$  h.,  $C = 10$  mf.; (d)  $R = 2.5\Omega$ ,  $L = .01$  h.,  $C = 100$  mf.; (e)  $R = 10\Omega$ ,  $L = .025$  h.,  $C = 10$  mf.; (f)  $R = 10\Omega$ ,  $L = .02$  h.,  $C = 10$  mf.; (g)  $R = 10\Omega$ ,  $L = .005$  h.,  $C = 0$  mf. Determine the critical frequency for each circuit and also determine the



required reactance which should be added to each circuit to make it resonant at 25~.

**Ex. 28A.** An A.C. circuit is supplied with 10 amps. at 110 volts. The current leads the E.M.F. by 25°. Determine the resistance, reactance, and impedance of the circuit graphically, and numerically:

$$(1) \quad Z = \frac{E}{I} =$$

$$(2) \quad R = Z \cos \phi = Z \cos 25^\circ =$$

$$(3) \quad X = Z \sin \phi = Z \cos 25^\circ = R \tan 25^\circ =$$

In Fig. 251 lay off  $R$  horizontally and  $Z$  25° behind  $R$ , then construct  $X_c \perp R$ . The scale  $R$ ,  $X$ ,  $Z$  is determined from (1), (2), or (3).

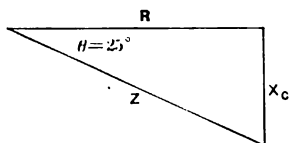


FIG. 251.

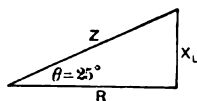


FIG. 252.

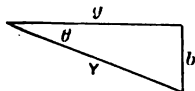
**Ex. 28B.** Explain why Fig. 252 represents the impedance diagram for the circuit described in Ex. 25, when the current lags 25° behind the E.M.F. How can the impedance diagram be supplemented in order to become the vector diagram of the circuit?

**Ex. 29.** Determine the resistance, reactance, and impedance of A.C. circuits which are supplied with 15 amperes at 110 volts when the phase angles are as follows: (a) 30°; (b) 35°; (c) 45°; (d) 55°; (e) 75°; (f) -15°; (g) -5°; (h) -35°; (i) -45°; (j) -55°; (k) -75°; (l) 15°; (m) 5°; (n) 90°; (o) 0°.

Determine the capacity of the above circuits when (I) the inductance is .01h. and the frequency = 60~; (II) when the capacity is 100 mf. and the frequency = 25~.

**Ex. 30.** Determine the resistance, reactance, and impedance of A.C. circuits which are supplied with 10 amps. at 110 volts when the power factors are as follows: (a) .7; (b) .85; (c) .8; (d) .9; (e) .95; (f) .75; (g) .55; (h) .96; (i) .97; (j) .98; (k) .985; (m) .99; (n) .995; (o) 1. Determine the capacity of the above circuits when the inductance = .02h. and the frequency = 25~.

**Ex. 31A.** A lamp circuit Fig. 253 receives 100 amperes from an A.C. main. An inductive circuit of negligible resistance takes 15 amperes when connected across the mains. What is the total current delivered to the mains?  $I_2$  is in phase with the E.M.F. whereas  $I_1$  is normal to the E.M.F. and therefore  $I$  is expressed by (1). Determine the power factor of the circuit.



$$(1) \quad I = \sqrt{I_1^2 + I_2^2} = \sqrt{100^2 + 15^2} =$$

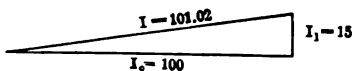
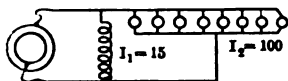


FIG. 253.

**Ex. 31B.** Determine the power factor and the total current supplied to the circuit shown in Fig. 246 when the lamp current  $I_2$  has the following values: (a) 50 amps.; (b) 15 amps.; (c) 75 amps.; (d) 60 amps. Calculate the admittance, susceptance, and conductance of the circuit. Why is  $Y$  shown behind  $g$  in Fig. 253?

**Ex. 31C.** Determine the power factor and the total current supplied to the circuit shown in Fig. 253 when the inductive current  $I_1$  has the following values: (a) 10 amps.; (b) 20 amps.; (c) 25 amps.; (d) 5 amps. Calculate the admittance, susceptance, and conductance of the circuit.

**Ex. 32A.** A capacity of 60 mf. and an inductance of .025h. are placed in series with a resistance of  $10\Omega$ . (a) Determine the frequency which will make the circuit resonant. (b) At the critical frequency what is the drop across the condenser when a 110-volt A.C. is impressed upon the circuit?

**Ex. 32B.** If the capacity is halved and the inductance doubled in Ex. 31A what is the effect upon the calculated values (a) and (b)?

**Ex. 33.** Determine the admittance, conductance, and suscep-

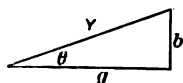


FIG. 254.

tance of a circuit, one branch of which contains .25 h. and  $5\Omega$  whereas the other branch contains 50 mf. and  $1\Omega$ . Fig. 254 represents the admittance diagram when the frequency = 25~. When the capacity reactance equals

the inductive reactance of the branches the parallel circuit is said to be resonant.

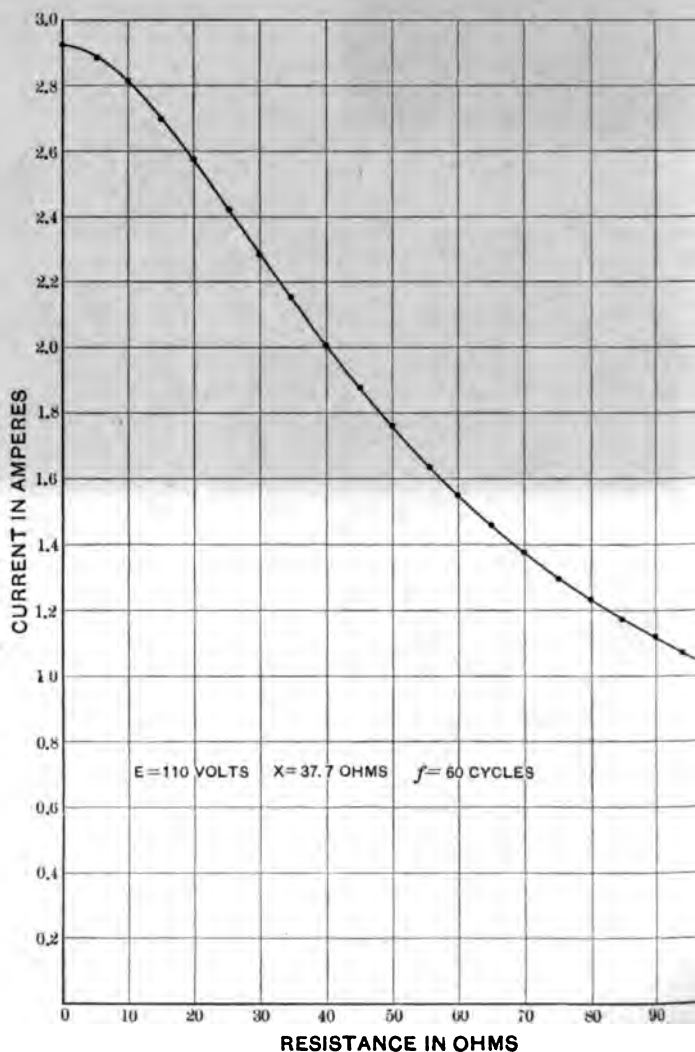


FIG. 255.—An Inductive Circuit with Variable Resistance.

**Ex. 34.** Make the graphic and numeric calculations for the circuit described in Ex. 33 when (I) the following values are substituted for the inductance: (a) .2 h.; (b) .3 h.; (II) when the following values are substituted for the capacity: (c) 25 mf.; (d) 100 mf.

**Ex. 35.** Calculate, tabulate, and plot a curve as shown in Fig. 255 between current and resistance in an inductive circuit. The circuit is supplied from a 60-cycle 110-volt main. The resistance ranges from 0 to 100 $\Omega$ . and the inductance has the following constant values: (a)  $L = .1$  h.; (b)  $L = .3$  h.; (c)  $L = .5$  h.; (d)  $L = .75$  h. (e)  $L = 1$  h.; (f) use the value of  $L$  given in (a) but plot  $R$  between 0 and 1000 $\Omega$ .

**Ex. 36.** A transmission line delivers 100 amperes to a non-inductive circuit. The E.M.F. across the terminals of the latter is 10,000 volts. The transmission line has a resistance = 5 $\Omega$  and a reactance = 2.5 $\Omega$ . (a) What is the generator voltage? (b) What is the phase difference between generator E.M.F. and receiver voltage?

**Ex. 37.** Make the following substitutions in Ex. 36 and determine (a) and (b) when: (I) line resistance = 4 $\Omega$ ; (II) when line resistance = 3.5 $\Omega$ ; (III) when line reactance = 2 $\Omega$ ; (IV) when line reactance = 2.25 $\Omega$ ; (V) when line reactance = 1 $\Omega$ ; (VI) when line reactance = 5 $\Omega$ .

**Ex. 38.** A transmission line with a negligible inductance has a resistance = 5 $\Omega$  and delivers 100 amps. to a receiver circuit. The latter is entirely inductive and has 10,000 volts at its terminals. In Fig. 256  $E_c = BC$  is the receiver voltage,  $E_L = AB$  is the line drop, and  $E_g = AC$  is the generator voltage and therefore (1) follows:

$$(1) \quad E_g = \sqrt{E_c^2 + E_L^2} =$$

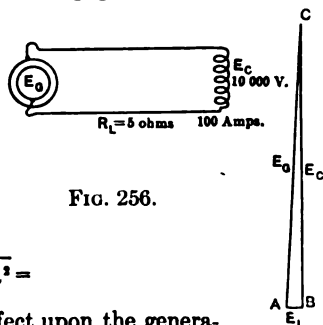


FIG. 256.

**Ex. 39.** What would be the effect upon the generator voltage if the inductive resistance of the receiving circuit in Ex. 38 were replaced by an equal capacity reactance.

**Ex. 40.** A transmission line, Fig. 257, with negligible inductance has a resistance of 4.5 $\Omega$  and delivers 100 amps. at 10,000 volts to a receiving circuit whose power factor = .7. What is the value of the generator voltage. In Fig. 250  $\phi = \cos^{-1} (.7)$  and  $E_c = CB$  is the voltage at the receiving circuit,  $E_L = AC$  is the drop in the

line, and  $E_G = AB$  is the generator voltage and therefore (1) follows:

$$(1) \quad E_G = \sqrt{(E_L + E_c \cos \phi)^2 + (E_c \sin \phi)^2} =$$

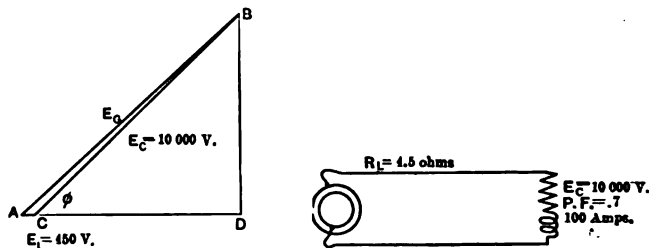


FIG. 257.—A Transmission Line.

**Ex. 41.** Determine the generator voltage for the transmission line described in Ex. 40 when the receiving circuit has the following values for its power factor: (a) .9; (b) .95; (c) .85; (d) .8; (e) .75; (f) .92; (g) .93; (h) .94; (j) .96; (k) .97; (l) .975; (m) .91; (n) .945.

**Ex. 42.** Determine the drop across two condensers of negligible resistance, which are connected in series to 110-volt A.C. mains: (a)  $C_1 = .5$  mf.,  $C_2 = .25$  mf.; (b)  $C_1 = .5$  mf.,  $C_2 = .01$  mf.; (c)  $C_1 = .05$  mf.,  $C_2 = .5$  mf.; (d) what is the effect upon the drop if the voltage is increased to 1100 volts; (e) what is the effect upon the respective drops if both condenser capacities are doubled.

**Ex. 43.** An electrostatic voltmeter has a capacity of .0006 n.f. when it deflects with 75 volts at its terminals. An auxiliary condenser of .001 mf. is connected in series with it. What voltage across the combination will produce a like deflection?

**Ex. 44.** An electrodynamic meter has a resistance of 1650 ohms and an inductance of .0205 henry gives the same deflection for a 60-cycle E.M.F. of unknown value as it does for a D.C. voltage of 125 volts. What is the effective value of the 60-cycle E.M.F.?

**Ex. 45.** An alternator delivers current to two receiving circuits A and B in parallel. A has a power factor of .8 and takes 25 amps. B has a power factor of .7 and takes 20 amps. What is the power factor of the combination? Construct the vector diagram by laying off  $E$  the E.M.F. as a reference line, and then the currents in A and B will lag behind  $E$ . Their resultant will be the total current. The total phase angle is expressed in (1):

$$(1) \quad \cos \phi_{\text{total}} = \frac{I_B \cos \phi_B + I_A \cos \phi_A}{I_{\text{total}}}.$$

**Ex. 46.** The receiving circuit is the same as described in Ex. 45, excepting: (a)  $A$  takes 20 amps.; (b)  $A$  takes 30 amps.; (c)  $B$  takes 25 amps.; (d)  $B$  takes 15 amps.; (e)  $B$  has a P.F. = .9; (f)  $B$  has a P.F. = .6; (g)  $B$  has a P.F. = .95; (h)  $A$  has a P.F. = .9; (i)  $A$  has a P.F. = .7; (j)  $A$  has a P.F. = .95; (k)  $A$  has a P.F. = 1.

**Ex. 47.** An alternator delivers a 60-cycle, 1100-volt, 200-amp. current to a receiving circuit. What capacity would be required to compensate for the lagging current when the power factor is: (a) .9; (b) .8; (c) .95; (d) 1; (e) when the P.F. = .9 and the frequency = 25 ~?

**Ex. 48.** The power  $P$  in an A.C. circuit may be measured directly with a wattmeter. It may be measured also by connecting a resistance  $R$  in series with the circuit whose impedance is represented by  $D$ , as shown in Fig. 258. Three voltmeters whose readings are  $E$ ,  $E_1$ , and  $E_2$  are connected between the points  $C$  and  $A$ ,  $B$  and  $A$ , and  $A$ ,  $C$  and  $B$ , respectively. The working equation is given in (1). Determine the power in the circuit when  $E = 110$ ,  $E_1 = 80$ ,  $E_2 = 40$ , and  $R = 10\Omega$ :

$$(1) \quad P = \frac{1}{2R}(E^2 - E_1^2 - E_2^2).$$

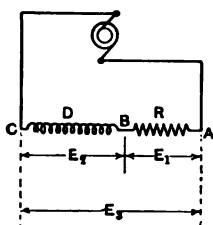


FIG. 258.—Three-Voltmeter Method.

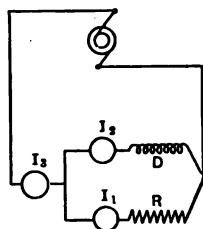


FIG. 259.—Three-Ammeter Method.

**Ex. 49.** Another method of measuring the power in an A.C. circuit is by connecting a resistance  $R$  in shunt with the circuit whose impedance is represented by  $D$ , as shown in Fig. 259. Three ammeters whose readings are  $I$ ,  $I_1$ , and  $I_2$  are connected in the main and in the branches  $D$  and  $R$  respectively. The working equation is given in (1). Determine the power in the circuit when  $I = 10.2$ ,  $I_1 = 8.1$ ,  $I_2 = 5.5$ , and  $R = 20\Omega$ :

$$(1) \quad P = \frac{R}{2}(I^2 - I_1^2 - I_2^2).$$

**Ex. 50.** A balanced two-phase system has a common return wire. Determine the current in the return wire when each phase has a current of: (a) 150 amps.; (b) 200 amps.; (c) 225 amps. What is the current in each phase when the current in the return wire is: (d) 175 amps.; (e) 150 amps.; (f) 200 amps.

- (1)  $I_A = I_B = 200 \text{ amps.}$   
 (2)  $I = \sqrt{I_A^2 + I_B^2} = I_A \sqrt{2} =$

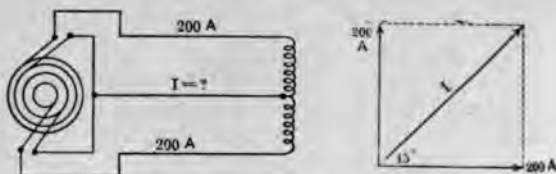


FIG. 260.—Balanced Two-Phase System with Common Return.

**Ex. 51.** A balanced two-phase system, Fig. 261, with a common return has an E.M.F. of 110 volts in each phase. (a) W

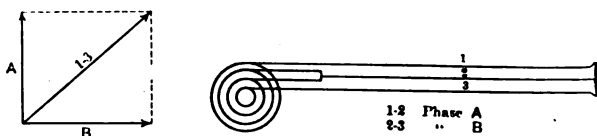


FIG. 261.—Balanced Two-Phase System with Common Return.

is the E.M.F.,  $E$ , between outside wires. If the two windings of the machine are connected in series what E.M.F. will the generator develop?

- (1)  $E_A = E_B = 110 \text{ volts.}$   
 (2)  $E = \sqrt{E_A^2 + E_B^2} = E_A \sqrt{2} =$

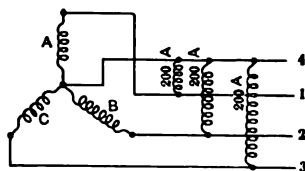


FIG. 262.—Balanced Three-Phase System.

**Ex. 52.** Fig. 262 represents a three-phase alternator with four collecting rings connected to the four mains 1, 2, 3, 4, respectively.

three receiving circuits, *a*, *b*, *c*, each take 200 amps. (a) What is current in main 4? (b) If the connections for coil *A* are reversed so that the vector diagram changes from Fig. 263 to Fig. 264. What is the value of the current in main 4? (c) If the connections of coils *A* and *B* are both reversed show that the vector diagram changes from Fig. 263 to Fig. 265. What is the value of the current in main 4? (d) If the three windings of the generator are connected in series, what E.M.F. will the machine develop?

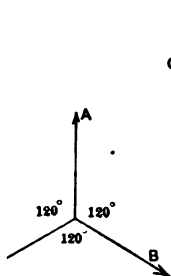


FIG. 263.

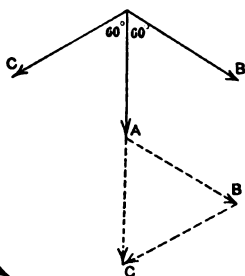


FIG. 264.

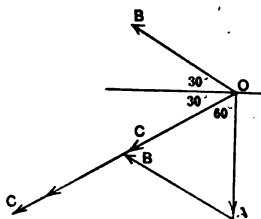
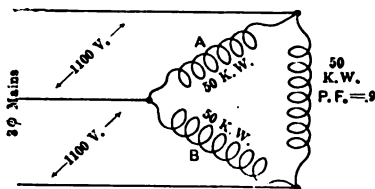


FIG. 265.

**Ex. 53.** Fig. 259 represents three similar receiving circuits which are  $\Delta$ -connected to three phases between each pair of which there is an E.M.F. = 1100 volts. In a balanced system the total power equals the power in each phase multiplied by the number of phases. The total power = 150 K.W., what is the value of the


 FIG. 266.— $\Delta$ -Connection.

current in each main and the current in each receiving circuit when the power factor of each receiving circuit equals: (a) .96; (b) .95; (c) .94; (d) .93; (e) .92; (f) .91; (g) .9; (h) what is the corresponding effect upon the current values when the total power is increased to 100 K.W.?

**Ex. 54.** In Ex. 53 substitute a Y-connection for the three receiving circuits and determine the corresponding currents in the mains and in each receiving circuit when the power factor of

each of the latter equals: (a) .97; (b) .98; (c) .96; (d) .95; (e) .94; (f) .93; (g) .92; (h) .91; (i) .9.

**Ex. 55.** Fig. 267 represents a three-phase  $\Delta$ -connected generator supplying three similar Y-connected receiving circuits. Determine (I) the current in each receiving circuit, (II) the current in each main, (III) the current in each armature winding, (IV) the voltage in each receiving circuit, (V) the voltage between mains. Each armature winding develops an E.M.F. = 110 volts. The receiving circuits have the following respective resistances and reactances: (a)  $R = 5\Omega$ ,  $X = 2.5\Omega$ ; (b)  $R = 4\Omega$ ,  $X = 2.5\Omega$ ; (c)  $R = 3\Omega$ ,  $X = 2.5\Omega$ ; (d)  $R = 2.5\Omega$ ,  $X = 2.5\Omega$ ; (e)  $R = 5\Omega$ ,  $X = 3\Omega$ ; (f)  $R = 5\Omega$ ,  $X = 4\Omega$ ; (g)  $R = 5\Omega$ ,  $X = 5\Omega$ .

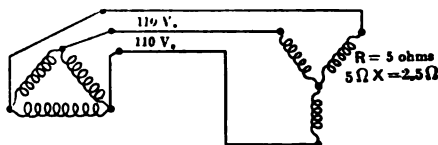


FIG. 267.—Three-Phase  $\Delta$ -Connected Generator Supplying Y-Connected Receiving Circuit.

**Ex. 56.** A three-phase Y-connected generator supplies three similar  $\Delta$ -connected receiving circuits. Determine (I) the current in each receiving circuit; (II) the current in each main; (III) the voltage between mains; (IV) the current in each armature winding; (V) the voltage in each receiving circuit. Each armature winding develops an E.M.F. = 110 volts. The receiving circuits have the following respective resistances and reactances: (a)  $R = 5\Omega$ ,  $X = 2.5\Omega$ ; (b)  $R = 4\Omega$ ,  $X = 2.5\Omega$ ; (c)  $R = 3.5\Omega$ ,  $X = 2\Omega$ ; (d)  $R = 2.5\Omega$ ,  $X = 2.5\Omega$ ; (e)  $R = 5\Omega$ ,  $X = 3.5\Omega$ ; (f)  $R = 5\Omega$ ,  $X = 4\Omega$ ; (g)  $R = 5\Omega$ ,  $X = 5\Omega$ .

**Ex. 57.** Make a sketch of a three-phase  $\Delta$ -connected generator with the mains supplying a  $\Delta$ -connected receiving circuit. Represent two wattmeters  $W_1$  and  $W_2$  so connected that their ammeter coils (series coils) are inserted directly in mains 1 and 3 respectively. Connect the voltmeter coils of  $W_1$  between mains 1 and 2 and the voltmeter coils of  $W_2$  between mains 2 and 3. The total power delivered to the three circuits is 25 K.W. Draw the vector diagram for current and voltages and determine the reading of each instrument when each receiving circuit has a power factor equal to: (a) .9; (b) .8; (c) .85; (d) .95; (e) 1. Describe the conditions which would result if the receiving circuits were Y-connected.

**Ex. 58.** Two alternators  $A$  and  $B$  are connected in series. Their E.M.Fs. are 1200 and 1000 volts respectively. Plot the power curves for each generator and also for the copper losses in the receiving circuit, when the resistance and reactance of the latter are: (a)  $R=1\Omega$ ,  $X=1.5\Omega$ ; (b)  $R=1\Omega$ ,  $X=1\Omega$ ; (c)  $R=1.5\Omega$ ,  $X=1.5\Omega$ ; (d)  $R=1\Omega$ ,  $X=2\Omega$ .

**Ex. 59.** An alternator is operated as a synchronous motor when connected to  $60 \sim 1100$ -volt mains and takes a leading current. Its resistance  $=1.2\Omega$  and its reactance  $=.6\Omega$ . Determine the following facts for zero load: (a) the value of the current; (b) the component of the current which is  $90^\circ$  ahead of the E.M.F. of the supply; (c) the capacity of condensers which could be substituted so as to take the same amount of leading current.

**Ex. 60.** A synchronous motor  $A$  and an induction motor  $B$  are operated in parallel from 110-volt mains. Determine the generator voltage when  $A$ 's power factor  $=.92$  and  $B$ 's power factor  $=.85$ .  $A$  takes a leading current of 50 amperes and  $B$  takes a lagging current of 85 amperes.

**Ex. 61.** A  $25 \sim 220$ -volt generator delivers 100 amperes to operate a number of induction motors whose power factor is .8. What capacity must be inserted to raise the power factor to: (a) .85; (b) .9; (c) .95?

## CHAPTER XXXIII

### THE ALGEBRA OF A TRANSFORMER

1. A **transformer** is an electrical device for raising or lowering potential and accordingly is known as a **step-up** or **step-down** transformer. It consists of a core of laminated soft iron or annealed sheet steel on which two insulated coils are wound. The coil which is connected to the supply end of the circuit is called the **primary** whereas the coil which is connected to the receiving circuit is called the **secondary**. Transformers are classified as core-types when the coils are wound around the cores and as shell-types when the iron core is built around the coils as illustrated in cross-section in Figs. 269 and 276. Transformers are represented diagrammatically in Figs. 270 to 275.

When an alternating current is supplied to the primary coil of a transformer it causes an alternating magnetic flux through the core, which in turn induces an alternating E.M.F. in the secondary, which produces the alternating current in the receiving circuit.

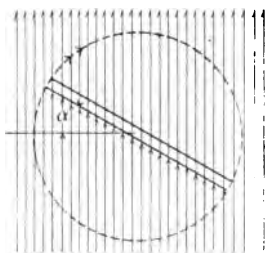


FIG. 268.

2. The stationary coils of a transformer with their corresponding periodically varying fields may be likened to a coil rotating through a uniform field as shown in Fig. 268. Therefore the formulas for the induced E.M.F. of

a generator apply to the induced E.M.Fs. of a transformer.

rapid reversals of magnetization in the iron core an E.M.F.  $e$  in each turn of both primary and secondary coils and if their respective number of turns be denoted by  $N_1$  and  $N_2$  then the corresponding total fluxes are expressed by (1) and (2).

$$E_1 = N_1 e,$$

$$E_2 = N_2 e,$$

$$\therefore \frac{E_1}{E_2} = \frac{N_1}{N_2}.$$

interpretation of (3) states that the ratio of the primary and secondary E.M.Fs. equals the ratio of their respective number of turns.

The magnetizing action of the secondary current  $I_2$  is opposed by an equal and opposite magnetizing action of the primary current  $I_1$  as expressed in (4) from which it follows that

$$N_1 I_1 = N_2 I_2,$$

$$\therefore \frac{I_2}{I_1} = \frac{N_1}{N_2},$$

$$\therefore \frac{E_1}{E_2} = \frac{I_2}{I_1}.$$

It results from equating (3) and (5) and interpreted that the currents in the primary and secondary are inversely as their respective E.M.Fs. In an ideal transformer the active power  $I_2^2 R_2$  of the secondary is equal to the active power  $I_1^2 R_1$  in the primary current as expressed in (7).

$$I_1^2 R = I_2^2 R_2,$$

$$\therefore R = \left( \frac{I_2}{I_1} \right)^2 R_2,$$

$$\therefore R = \left( \frac{N_1}{N_2} \right)^2 R_2.$$

(9) results from the substitution of (5) in (8). In (9)  $R$  is called the **equivalent resistance**, i.e., the resistance which would have to be placed in the primary to give a corresponding loss due to  $R_2$  in the secondary.

The wattless power,  $I_2^2 X_2$ , of the secondary should have an equal primary power equivalent  $I_1^2 R$  in terms of the primary current as expressed in (10).

$$(10) \quad I_1^2 R = I_2^2 X_2,$$

$$(11) \quad \therefore X = \left(\frac{I_2}{I_1}\right)^2 X_2,$$

$$(12) \quad \therefore X = \left(\frac{N_1}{N_2}\right)^2 X_2.$$

(12) results from the substitution of (5) in (11). In (12)  $X$  is called the **equivalent reactance**, i.e., the reactance which would have to be placed in the primary to require a corresponding energy due to  $X_2$  in the secondary.

**3. Transformer Losses and Efficiency.** The total losses  $W_T$  in a transformer are composed of copper losses  $W_c$  due to the resistance in the primary and secondary, hysteresis losses  $W_h$  and eddy current losses  $W_e$ .

$$(13) \quad W_c = I_1^2 R_1 + I_2^2 R_2,$$

$$(14) \quad W_h = K_1 V f B^{1.6},$$

$$(15) \quad W_e = K_2 V f^2 B^2,$$

$$(16) \quad W_T = W_c + W_h + W_e.$$

(14) and (15) are identical with (37) and (38) respectively in Chap. IX. The efficiency  $\eta$  of a transformer is expressed in (17) in which  $W$  is the input of the transformer.

$$\eta = \frac{\text{output}}{\text{input}} = \frac{W - W_T}{W}.$$

**Ex. 1.** (a) What elements affect both eddy and hysteresis losses? (b) Of these elements which produce like effects upon

both losses? (c) Of the remaining elements which have the greater effect upon these losses? (d) Are there any elements which affect one loss and not the other? (e) How could (14) and (15) be changed to express the losses in terms of weight?

**4. The Design of the Core.** The core losses in a transformer vary directly as the weight of the core and therefore the latter should be minimized.

**Ex. 2.** (a) What determines the cross-sectional area and what determines the length of the core?

(b) Increase of length has what effect upon the copper losses?

(c) The shape of the cross-section of the core has a considerable influence upon the length of the turns, therefore what is its corresponding effect upon the resistance of the coils?

(d) Compare the length of turns required on rectangular square and circular cross-sections of equal area.

(e) Under what circumstances will a circular cross-section be preferable?

(f) If the linear dimensions are the same what is the ratio of volumes of circular and square cores?

(g) Which cross-section therefore gives the greater E.M.F. induced per foot of wire? How does the difference increase?

**Ex. 3.** Make a comparative study of the different types of transformers illustrated in Fig. 269. Assume a like output, i.e.,

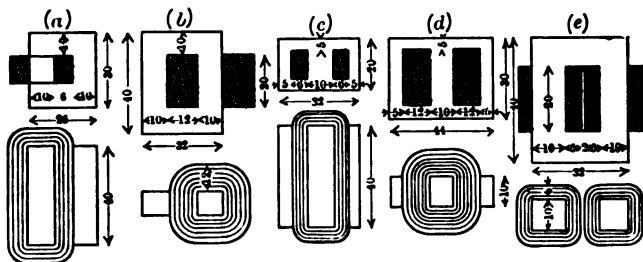


FIG. 269.—Types of Transformers.

product of current and E.M.F., for each transformer. An equal current, current density, and gauge of wires will be used for all windings. The number of turns will be directly proportional to the winding space and for the same induction the cross-section will be inversely proportional to the number of turns. The weight of iron and the length of wire may be taken as a basis to judge the design. The dimensions of the figures are given in

centimeters. Type (a) has a core whose cross-section = 400 sq.cm. and winding space = 60 sq.cm. The weight of the iron = 200 kg. The mean length of one turn =  $2 \times 40 + 2 \times 10 + 2\pi \times 3 = 118.9$  cm. 100 turns on the primary require 118.9 meters of wire.

**Ex. 4.** Calculate the eddy current and hysteresis losses in the iron of a 60-cycle transformer for which  $B_m = .15$  megamaxwells. The mean length of the magnetic circuit is 30 ins. and the cross-section 8 sq.ins.

**Ex. 5.** A transformer has a primary winding of 100 turns of No. 4 B. & S. copper wire and the secondary winding of 2000 turns of No. 16 B. & S. copper wire. The mean lengths of these respective windings are 30 ins. and 18 ins. Determine the total equivalent primary resistances.

**Ex. 6.** A 110-volt D.C. main supplies a current to a coil of 200 turns wound on an iron core, as shown in Fig. 270. The

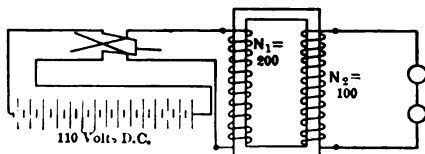


FIG. 270.

current is reversed 120 times a second. The E.M.F. in the primary is shown as curve  $E_1$  in Fig. 271. The core flux is represented by curve  $\phi$  in Fig. 271; it has a constant slope and reaches its maximum and minimum values at the instant of current reversals, i.e., when the primary E.M.F. is zero. The secondary E.M.F. is represented by  $E_2$ , and is  $180^\circ$  out of phase with  $E_1$ , i.e., when  $E_1$  is positive  $E_2$  is negative, whereas when  $E_1$  is negative  $E_2$  is positive.

A secondary coil of 100 turns is wound on the above core and supplies a current to a non-inductive receiving circuit =  $50\Omega$ . Both primary and secondary coils have negligible resistance. Determine the secondary E.M.F., secondary current, total primary current, core flux when its reluctance = .0001.

$$(1) \quad \frac{E_1}{E_2} = \frac{N_2}{N_1}$$

$$(2) \quad \therefore E_2 =$$

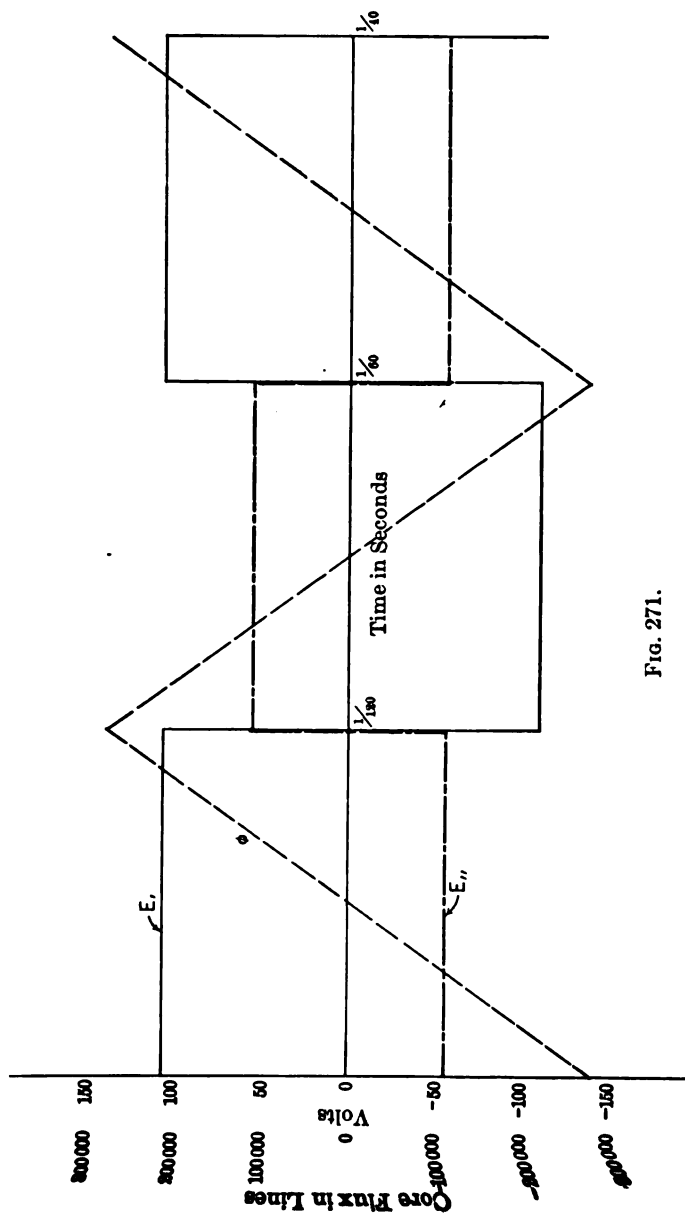


Fig. 271.

$$(3) \quad \frac{d\phi}{dt} = \frac{E_1}{N_1} = \frac{110 \times 10^8}{200} = 55 \times 10^6.$$

$$(4) \quad \phi = 55 \times 10^6 t = \frac{55 \times 10^6}{120}.$$

**Ex. 7.** A transformer has its primary and secondary coils wound with 1100 and 110 turns of wire respectively, as shown in Fig. 272. The supply current is from 1100-volt mains. What

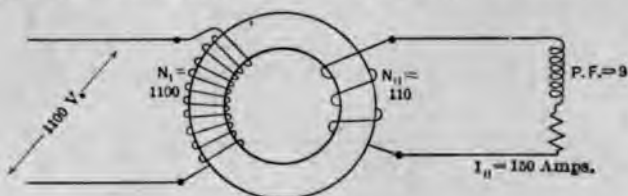


FIG. 272.

is the equivalent resistance and equivalent reactance when the secondary delivers 150 amps. to a receiving circuit having a power factor equal to: (a) .97; (b) .96; (c) .95; (d) .94; (e) .93; (f) .92; (g) .91; (h) .90; (i) .89; (j) .88; (k) .87; (l) .86; (m) .85; (n)  $C_s = 200$  amps. and P.F. = .95.

**Ex. 8.** A 10:1 transformer has its primary coil connected to a 1100-volt main, as shown in Fig. 273. The primary coil is supplied through a variable non-inductive resistance,  $R_1 = 100\Omega$ . Determine the resistance and reactance of a simple circuit which would replace the transformer and secondary.

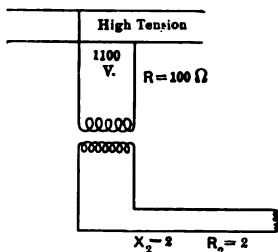


FIG. 273.

Determine the primary and secondary currents, the primary and secondary terminal voltages. The secondary delivers current to a circuit having the following resistances and reactances: (a)  $R_2 = 5\Omega$ ,  $X_2 = 2\Omega$ ; (b)  $R_2 = 3\Omega$ ,  $X_2 = 1\Omega$ ; (c)  $R_2 = 3\Omega$ ,  $X_2 = 2\Omega$ ; (d)  $R_2 = 1\Omega$ ,  $X_2 = .5\Omega$ ; (e)  $R_2 = 0\Omega$ ,  $X_2 = 3\Omega$ ; (f)  $R_2 = 3\Omega$ ,  $X_2 = 3\Omega$ ; (g)  $R_2 = 2.5\Omega$ ,  $X_2 = 3\Omega$ ; (h)  $R_2 = 3\Omega$ ,  $X_2 = 2.5\Omega$ .

**Ex. 9.** An iron core consisting of a bundle of wires of 15 sq.cm. cross-sectional area is magnetized from an A.C. 60-cycle 110-volt

main. (a) How many turns are required to give a maximum flux density = 4000 lines per square centimeter? How is the result altered when (b) the frequency = 25~; (c) when the E.M.F. of mains = 550 volts; (d) when the flux density = 3500 lines per square centimeter. See Fig. 274.

**Ex. 10.** A step-up transformer raises the voltage of an alternator from 110 volts to 15,000 volts. There are 50 turns in the primary coil and 500 cm. of cross-section are allowed for each ampere of current. Determine the number of turns in the secondary and the cross-sections of primary and secondary windings when the alternator supplies: (a) 500 amps.; (b) 750 amps.; (c) 250 amps.; (d) 200 amps.; (e) 100 amps.; (f) 150 amps., Fig. 276A.

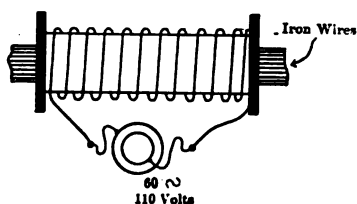


FIG. 274.

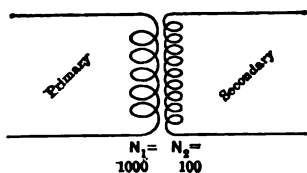


FIG. 275.

**Ex. 11.** A transformer, Fig. 275, has 1000 turns in the primary and 100 turns in the secondary. The primary wire is 40 mils in diameter and 500 cir. mils are allowed per ampere. The sectional area is 10 sq.cm. Determine the E.M.F., current, and power rating of the transformer when the frequency and flux density are: (a)  $f=60\sim$ ,  $B=4200$  lines per square centimeter; (b)  $f=25\sim$ ,  $B=4100$  lines; (c)  $f=60\sim$ ,  $B=4100$ ; (d)  $f=25\sim$ ,  $B=4000$ ; (e)  $f=60\sim$ ,  $B=4000$ ; (f)  $f=25\sim$ ,  $B=4200$ .

**Ex. 12.** Determine the efficiency of a 10 K.V.A. transformer at unity power factor whose iron loss constantly equals 160 watts and whose load  $L$  and copper losses  $P_c$  are as follows: (a)  $L=2500$ ,  $P_c=10$ ; (b)  $L=5000$ ,  $P_c=40$ ; (c)  $L=7500$ ,  $P_c=90$ ; (d)  $L=10,000$ ,  $P_c=160$ ; (e)  $L=12,500$ ,  $P_c=250$ . Construct the efficiency curve.

**Ex. 13.** A shell type transformer is shown in cross-section and in longitudinal section in Fig. 276. The mean length of both primary and secondary coils is 29.5 ins. The primary consists of 480 turns of No. 7 B. & S. copper wire. The secondary consists of 24 turns of two strands of No. 7 wires in parallel. The section of the magnetic circuit is  $10 \times 1.75$  ins. .85 of the section is iron. The volume of the iron is 24 sq.ins.  $\times 10 \times .85$  ins. Thick-

ness of laminations 15 mils. Allow 500 cir. mils per ampere and a flux density of 4500 lines per square centimeter. Determine (a) the E.M.F., current, and power ratings of the transformer at 60 cycles; (b) the copper and iron losses at full load; (c) the efficiency of the transformer at full load; (d) the all-day efficiency of the transformer when operated for 4.5 hours each day at full load and at zero load for the remainder of the day.

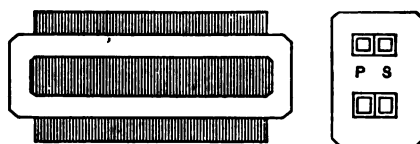


FIG. 276.—Cross-sections of a Shell Type Transformer.

**Ex. 14.** A shunt motor has an output of 5 K.W. At zero load the loss is 850 watts and the losses in the motor at full load are 1050 watts. Determine the all-day efficiency when the motor is operated at full load intermittently three minutes out of every ten minutes during ten hours of the day.

**Ex. 15.** A 10-K.W. 1100 : 110-volt transformer is connected as an autotransformer to step up the voltage from 1100 volts to 1175 volts. I. Determine (a) the power delivered at 1175 volts to non-reactive circuit without exceeding the rating of the transformer; (b) the current in each coil of the transformer; (c) the power delivered to and by the coils. II. The transformer steps down so as to deliver 1000 volts to the non-reactive circuit. III. The transformer steps up to 1200 volts. IV. The transformer steps down to 1025 volts.

**Ex. 16.** Three 1100 : 110-volt transformers have their 1100-volt coils delta connected to a three-phase 1100-volt mains. The secondaries are Y-connected to service mains, as shown in Fig. 277.

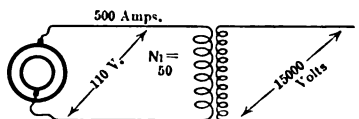


FIG. 276A.

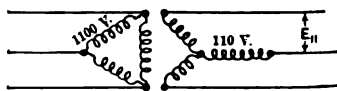


FIG. 277.

(a) Determine the E.M.Fs. between each set of mains. (b) Determine the E.M.Fs. between each set of mains when the delta and Y-connections are interchanged, as shown in Fig. 278.

**Ex. 17.** Two 100-K.W. transformers, Fig. 279, are arranged for the step-down transformation of a three-phase supply. Deter-



FIG. 278.

mine the power which can be delivered to balance the receiving circuits without exceeding the voltage and current ratings of the two transformers when the power factor of the receiving circuits is: (a) P.F. = .9; (b) .95; (c) .90; (d) .85.

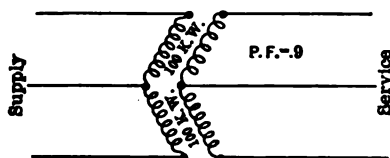


FIG. 279.

**Ex. 18.** Two transformers are connected to two-phase mains. Each phase has an E.M.F. = 1000 volts. The primary coils have each 480 turns. The secondaries are joined in series and give an E.M.F. =  $30^\circ$  ahead of one of the two E.M.F.s. Determine the proper winding for each secondary coil when its E.M.F. equals: (a) 550 volts; (b) 600 volts; (c) 500 volts.

**Ex. 19.** A Scott transformer is to transform from 1100-volt 60-cycle two-phase to a 110-volt three-phase. The cross-section of the core is 70 sq.cm. The maximum flux density = 4000 lines per square centimeter. I. Determine (a) the number of turns of wire in each primary coil; (b) the number of turns of wire in each secondary coil. II. Determine (a) and (b) when  $f = 25$ .

**Ex. 20.** A three-ring converter is to supply a D.C. to a car line at 600 volts. Three similar transformers are provided to accomplish the step-down transformation from 10,000 volts. The primaries are delta connected to the high voltage mains and the secondaries are Y-connected to the three rings of the converter. What is the ratio of transformation for each transformer when the voltage of the car line equals: (a) 600 volts; (b) 575 volts; (c) 550 volts?

**Ex. 21.** The thirty-six windings of an ordinary ring-wound D.C. armature are disconnected from the commutator and numbered consecutively from 1 . . . 36. Specify and illustrate with a diagram the manner in which the thirty-six coils should be connected to two-phase 60-cycle mains in order to produce in the ring a rotating state of magnetism of (a) 4 poles, (b) 6 poles, (c) 12 poles, (d) 18 poles, (e) state the speed of the magnetism in each in revolutions per second.

**Ex. 22.** An ideal three-phase induction motor takes 5 amps. into each phase of its delta connected primary member at 110 volts 60 cycles. The rotor runs at two-thirds synchronous speed. Determine (a) the ratio of the stator to the rotor turns; (b) the rotor terminal voltage; (c) total intake of power; (d) electrical output of power; (e) the mechanical output of power. The rotor which has a three-phase winding delta-connected to collector rings supplies 15 amps. to each of three similar circuits, each having a power factor equal to: (a) .9; (b) .85; (c) .8; (d) .75.

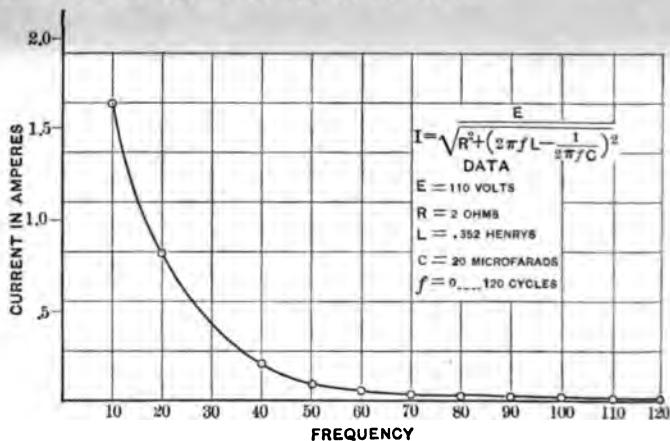


FIG. 279A.

**Ex. 23.** Make the necessary calculations and construct the curve shown in Fig. 279A. Plot between current and frequency. Use the data given for the figure.

## CHAPTER XXXIV

### THE CIRCLE DIAGRAM

**1. Variable Elements in an A.C. Circuit.** In the preceding chapters A.C. problems were solved by numeric calculation and also by vector diagrams when a sufficient and necessary amount of data was specified. When any data includes definite values of resistance, reactance, frequency and impressed E.M.F. there is a corresponding definite vector diagram and its corresponding impedance or admittance diagram. Every new set of data implies its corresponding diagrams. In this chapter a general diagram called a **circle diagram** or a **locus diagram** will be described by means of which it will be possible to study the variation of currents, E.M.Fs., power factors, phase angles, power, etc., in a circuit which is subject to variable resistance, reactance, and frequency. Fig. 280 shows the usual vector diagram for a circle supplied with constant E.M.F.  $E$  and in which the resistance and reactance have the constant values  $R$  and  $X$  respectively.  $AB$  represent the constant E.M.F.  $E$ , and  $AI$  represents the current  $I$  lagging  $\theta^\circ$  behind the E.M.F.  $AD=IR=E_R$ =the drop across the resistance  $R$  whereas  $DB=IX=E_x$ =the drop across the reactance  $X$  and  $\theta = \tan^{-1} \frac{X}{R}$ . From the property of a circle

a right angle is inscribed in a circle. A circle may be drawn upon  $AB$  as a diameter so as to pass through  $D$ .

$$(1) \quad I = \frac{E}{\sqrt{R^2 + X^2}},$$

$$(2) \quad \therefore E = I \sqrt{R^2 + X^2}.$$

(1) and (2) express the relation of four quantities. If we consider  $E$  constant then it is possible to change the current by changing either the resistance or the reactance as shown in (3) and (4) respectively, or by changing both the resistance and the reactance as shown in (5). Hereafter

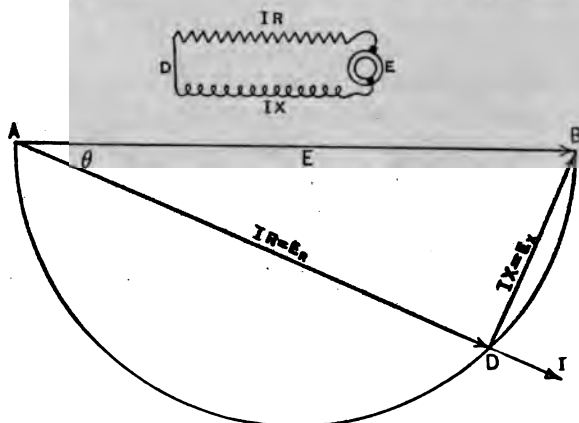


FIG. 280.—The Diagram for a Circuit having Constant Elements.

constant values are represented by capital letters and variable or instantaneous values are represented by l. c. letters.

$$(3) \quad E = i\sqrt{r^2 + X^2},$$

$$(4) \quad E = i\sqrt{R^2 + x^2},$$

$$(5) \quad E = i\sqrt{r^2 + x^2},$$

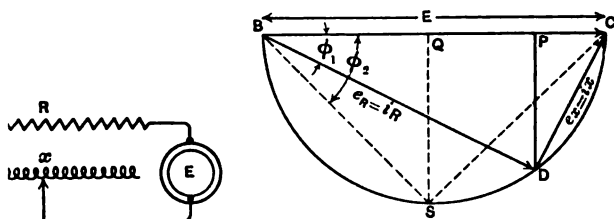
In the above cases  $E$  is always the hypotenuse of a right triangle and the mutually orthogonal components  $E_r$  and  $E_x$  are the legs. The right angle of each right triangle will lie on the circumference of a circle described on the hypotenuse as a diameter. The circle is the locus of the point  $D$  for circuits in which  $E$  is constant and  $i$ ,  $r$ , and  $x$  are variable. The lower semicircle is used when the phase angle  $\theta$  is positive, i.e., when the inductive reactance is

The upper semicircle is used when  $\theta$  is negative, when the capacity reactance is in excess.

**Fundamental Principles in an A.C. Circuit.** (I) The  $E$  which is impressed upon a circuit may be resolved into two mutually orthogonal components, the one  $E_R$  in phase with the current and the other  $E_x$  being perpendicular to the current.

(2) The current in a circuit may be resolved into two mutually orthogonal components, the one  $I_r$  being in phase with the applied voltage and the other  $I_x$  being perpendicular to the voltage.

A circuit having constant resistance  $R$  and variable reactance  $x$  with constant impressed voltage  $E$  is represented



281.—Circle Diagram for a Circuit with Variable Reactance.

281. In triangle  $BDC$ ,  $\phi_1 = \tan^{-1} \frac{x}{R}$ . The line  $DC$  will vary with both  $i$  and  $x$  whereas the line  $BD = e_R$  is proportional to current  $i$  only since  $R$  is constant.  $e_x$  is moved in a new position  $SC$  and  $e_R$  in the corresponding position  $BS$  and  $\phi_2 = \tan^{-1} \frac{e_x}{e_R} = \tan^{-1} \frac{x}{R}$ . By choosing a suitable scale the line  $e_R$  may represent the successive instantaneous values of the current  $i$  as  $x$  changes. The power  $W = i^2 R$  in the circuit is proportional to  $\overline{BD}^2$ .

$$W \propto \overline{BD}^2,$$

but

$$(6) \quad \overline{BD}^2 = \overline{BC} \times \overline{BP} = E \times \overline{BP},$$

$$(7) \quad \therefore \overline{BD}^2 \propto \overline{BP} \quad \text{and} \quad W \propto \overline{BP}.$$

In (6)  $E = BC$  is a constant. The interpretation of (7) states that the active power in the circuit is proportional to and represented by  $BP$  the projection of  $E_R$  on  $E$ . By choosing a suitable scale for  $BP$  the power in the circuit may be read directly from the circle diagram.

$$(8) \quad \cos \phi = \frac{BD}{BC} = \frac{BD}{E},$$

$$(9) \quad \therefore \cos \phi \propto BD.$$

The interpretation of (9) states that the power factor of the circuit is proportional to  $BD$ . Since the maximum value of the chord  $BD$  equals the diameter  $E$ , then the scale for reading power factors may be determined from the fact that a unity power factor is represented by a line equal to  $BC$ .

**Ex. 1.** Explain the method for determining the scale when  $BD$  in Fig. 282 is used for reading the values of the current in the

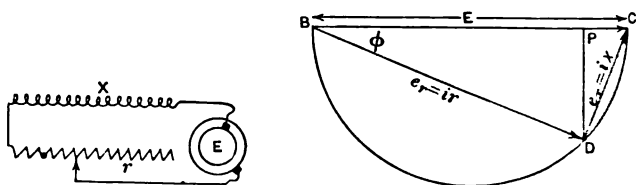


FIG. 282.

circuit. Explain why  $BC$  cannot represent the current.

**Ex. 2.** Explain the method for determining the scale when  $BP$  is used for reading the values of the active power in the circuit. What line in the figure represents wattless power and what is its scale.

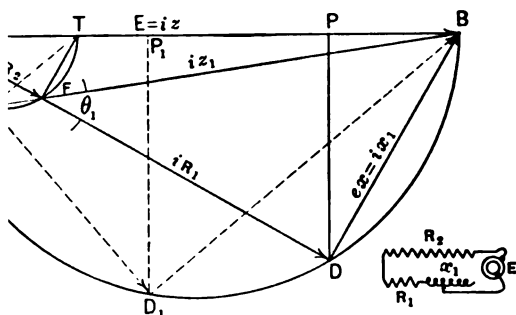
**Ex. 3.** Construct a circle diagram, Fig. 282, for a circuit having constant  $E$  and  $X$  but variable  $i$  and  $r$ . In such a diagram

lines represent current, power factor, and active  
is sufficient proof.

Construct the circle diagram for Ex. 3, when the reactance is in excess.

*mn.* In the circle diagram for constant voltage active power line is the horizontal projection of the current line when  $R$  is constant but it is the vertical projection of the current line when  $X$  is constant.

**Horizontal Auxiliary Semicircle.** It was observed in chapter that when a series circuit contains two impedances the latter could be separated by combining the individual resistances



### 283.—A Series Circuit with Variable Reactance.

group and the individual reactances into a second algebraic sum of the drops across the individual reactances equals the drop across the group of resistances, and the algebraic sum of the drops across the individual reactances equals the drop across the group of

These principles are applied to the circle a series circuit. The circuit diagram in Fig. two constant resistances  $R_1$  and  $R_2$  in series ble reactance  $x_1$ , which are supplied with current urce of constant impressed E.M.F.  $E$ . The : the current in the circuit may be written as ctive of whether  $R_1$  and  $x_1$  are considered sep-

arately or as constituting the two elements of an impedance  $z_1 = \sqrt{R_1^2 + x_1^2}$ .

$$(10) \quad i = \frac{E}{\sqrt{(R_1 + R_2)^2 + x_1^2}}$$

$$(11) \quad \tan \theta = \frac{ix_1}{iR_1 + iR_2} = \frac{x_1}{R_1 + R_2}$$

The circle diagram in Fig. 283 is constructed so that the diameter  $AB = E$  and then

$$\theta = \tan^{-1} \frac{BD}{AD} = \tan^{-1} \frac{x_1}{R_1 + R_2}$$

$BD$  represents the instantaneous drop across the variable reactance  $x_1$  and  $AD$  represents the instantaneous drop across the total resistance  $R_1 + R_2$ . Since  $R_1$  and  $R_2$  are constant their respective drops are proportional to their resistances.

$$(12) \quad \frac{AF}{FD} = \frac{iR_2}{iR_1} = \frac{R_2}{R_1}$$

It is necessary to be able to distinguish the segments of  $AD$  which represent the respective drops across  $R_1$  and  $R_2$  for each new position of  $AD$ . Draw  $FT \perp AF$ , intersecting  $AB$  at  $T$ . Construct the **horizontal auxiliary semicircle**  $AF_1FT$ , passing through  $A$ ,  $F$ , and  $T$ . The point  $F$  travels around the arc  $TFF_1A$  as arm  $AD$  rotates about  $A$ , i.e., as the point  $D$  travels around the arc  $BDD_1A$ . Therefore the points  $F$  and  $D$  will always represent the right angled vertexes of two similar right triangles. Accordingly the auxiliary semicircle  $TFF_1A$  will continually divide the rotating arm  $AD$  into proportional segments whose ratio is  $R_2 : R_1$ , as expressed in (13).

$$(13) \quad \frac{AF}{FD} = \frac{AT}{TB} = \frac{AF_1}{F_1D_1} = \frac{iR_2}{iR_1} = \frac{R_2}{R_1}$$



For the minimum value of the phase angle,

$$\tan \theta_1 = \frac{2}{15} = .133.$$

Construct  $BK \perp AB$  so that  $BK = .133 AB$ . Join  $A$  with  $K$ , then  $\tan BAK = .133 = \tan \theta_1$ . It is advantageous to construct the circle diagram on cross-section paper or polar paper so that its diameter is 10 units in length. Its scale is adjusted accordingly.

**6. The Vertical Auxiliary Circle.** The circuit diagram in Fig. 285 shows a constant resistance  $R_2$ , a constant reactance  $X_1$  and a variable resistance  $r_1$ , which are supplied with current from a source of constant E.M.F.  $E$ . The equation for the current in the circuit may be written irrespective of whether  $X_1$  and  $r_1$  are considered separately or as constituting the two elements of the impedance  $z_1 = \sqrt{r_1^2 + X_1^2}$ .

The circle diagram in Fig. 285 is constructed so that the diameter  $AB = E$  and then

$$\phi = \tan^{-1} \frac{BD}{AD} = \tan^{-1} \frac{X_1}{r_1 + R_2}.$$

$BD$  represents the instantaneous drop across the constant reactance  $X_1$  and  $AD$  represents the drop across the total resistance  $r_1 + R_2$ . The line  $AD$  is divided into two segments  $AF = iR_2$  and  $FD = ir_1$ .  $AF$  varies with  $i$  alone whereas  $FD$  varies with both  $i$  and  $r_1$ . Therefore for each new position of  $AD$  its segments will bear a different ratio, i.e.,  $AF$  must vary with  $i$  alone. Construct  $AT \perp AB$  and  $FT \perp AF$ . Through the points  $A$ ,  $F$ , and  $T$  construct the **vertical auxiliary** circle  $AFF_1T$ . The point  $F$  travels around the arc  $AFF_1T$  as the arm  $AD$  rotates about  $A$ , i.e., as the point  $D$  travels around the arc  $BDD_1A$ . Therefore the points  $F$  and  $D$  will always represent the right angled vertexes of two similar right triangles. Accordingly the segment

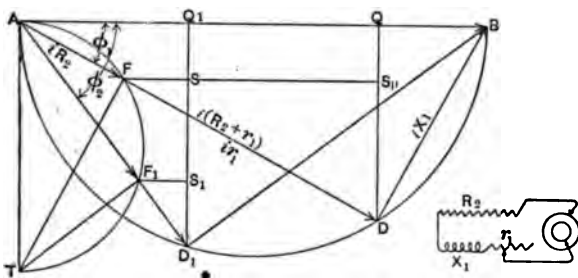
$AF$  of the rotating arm  $AD$  which is intercepted by the arc  $AFF_1T$  is proportional to the current  $i$  only as expressed in (14). The two diameters  $AB$  and  $AT$  are constant.

$$(14) \quad \frac{AF}{AT} = \frac{DB}{AB},$$

(15)  $\therefore AF \propto BD$  but  $BD \propto i$ .

$$(16) \quad \therefore AF \propto i.$$

Therefore  $AF$  varies with  $i$  alone and represents the instantaneous drop across the constant resistance  $R_2$



**FIG. 285.—A Series Circuit with Variable Resistance.**

**Ex. 7.** Which two lines of Fig. 285 may be used as the current lines of the diagram. Show which lines represent the several power factors of the circuit, and its elements, the several phase angles, and the several active powers in the circuit and its elements.

**7. The Eccentric Circle.** An alternative construction for the preceding problem is shown in Fig. 286 in which  $AB=E$  is the usual diameter and  $FB$  is the drop across the constant reactance  $X_1$  and  $AF$  is the drop across the total resistance  $r_1+R_2$ .

$AF$  is divided into two segments  $AD=ir_1$  and  $DF=iR_2$ . Therefore  $DF$  varies with current only. Construct the circular arc  $ADB$  which is known as the arc of the **eccentric circle** whose center  $H$  may be determined by the intersection of the perpendicular bisectors of  $AB$  and  $AD$ . The point

$F$  travels around the arc  $BDA$  as the arm  $AF$  rotates about  $A$ , i.e., as the point  $D$  travels around the arc  $BFF_1A$ . Therefore the points  $F$  and  $D$  are the vertexes of two inscribed angles which intercept constant arcs in their respective circles. Therefore the triangles  $DBF$  and  $D_1BF_1$  are similar.

$$(17) \quad \frac{D_1F_1}{DF} = \frac{F_1B}{FB} = \frac{i_1}{i_2}.$$

Therefore  $DF$  the segment of the chord  $AF$  which is intercepted between arcs  $BDD_1A$  and  $BFF_1A$  will vary with current only.

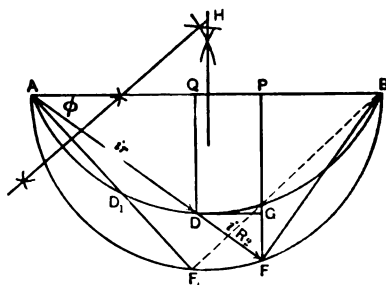


FIG. 286.—The Eccentric Auxiliary Circle.

**Ex. 8.** Which lines in Fig. 286 represent current, power factor, and power.  $DG$  is drawn parallel to  $AB$  and  $QD$  and  $PF$  are perpendicular to  $AB$ .

**Ex. 9.** Construct the circle diagram for a 25~ circuit having a constant capacity of 50 mf., a constant resistance  $R_2 = 10\Omega$ , and a variable resistance  $r_1$  which changes from (a)  $5\Omega$  to  $10\Omega$ ; (b) 0 to  $10\Omega$ ; (c) 2.5 to  $10\Omega$ ; (d) 2.5 to  $12.5\Omega$ ; (e) 2.5 to  $7.5\Omega$ . Determine the maximum and minimum power factors, currents, and power for the elements, as well as for the entire circuit.

*Observation.* In a circle diagram any line which is affected by a single variable may be used to represent the instantaneous magnitude of that variable. The chords of a circle diagram are divided into segments by a horizontal auxiliary circle when their resistance or reactance elements are constant quantities. The chords of a circle diagram are divided

into segments by a vertical auxiliary circle when their resistance or reactance elements consist of both constant and variable quantities.

**8. Multiple Auxiliary Circles.** In Fig. 287 the circuit consists of two constant and one variable resistance  $R_3$ ,  $R_2$ , and  $r_1$  respectively, and a constant reactance  $X$ , which are supplied with current from a source of constant E.M.F.  $E$ .

The resistance chord  $AF$  is divided into the segments  $AN = i(R_3 + R_2)$  and  $NF = ir_1$ . In order to separate the drops across  $R_3$  and  $R_2$ ,  $AN$  is subdivided at  $D$  by the second vertical auxiliary circle  $ADT_1$ .  $D$  is located by

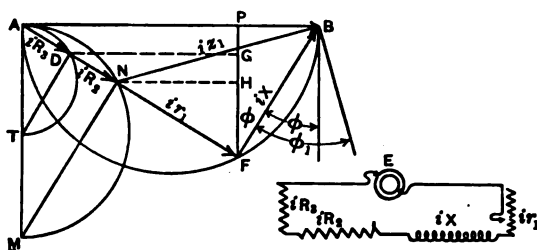


FIG. 287.

dividing  $AN$  into two segments  $AD$  and  $DN$  whose ratio equals  $R_3 : R_2$ .  $DT$  is drawn perpendicular to  $AD$  and a semicircle is drawn through  $A$ ,  $D$ , and  $T$ .

$$(18) \quad \frac{AD}{AT} = \frac{AN}{AM} = \frac{DN}{TM} = \frac{BF}{AB},$$

$$(19) \quad \therefore \frac{AD}{DN} = \frac{\text{constant } AT}{\text{constant } TM} = \text{constant}.$$

Therefore the two segments  $AD$  and  $DN$  of the entire chord  $AF$  which are intercepted by the auxiliary circles preserve a constant ratio  $R_3 : R_2$ . This method may be extended to a further subdivision of the constant part of the total resistance.  $NB$  represents the drop across the

impedance  $z_1 = \sqrt{r_1^2 + X^2}$ . The phase angle  $\phi$  for the entire circuit and the phase angle  $\phi_1$  for the impedance  $z_1$  may be measured from the perpendiculars drawn to  $AB$  and  $NB$  respectively as indicated in Fig. 287. Acute angles are equal when their respective sides are perpendicular.

**Ex. 10.** Fig. 288 represents an alternator with constant excitation and variable non-inductive load. The resistance and

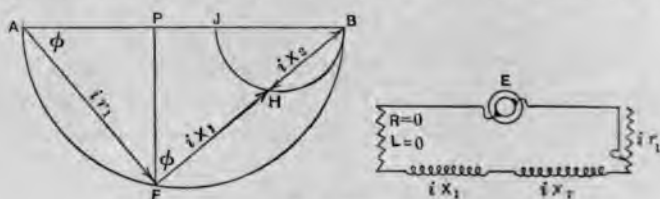


FIG. 288.

reactance of the alternator's armature are assumed constant. Show how the changes of the terminal voltage may be read from a circle diagram as the load resistance varies.

**Ex. 11.** Construct and explain the diagrams in Fig. 289, which represent a circuit with two constant reactances  $X_1$  and  $X_2$ ,

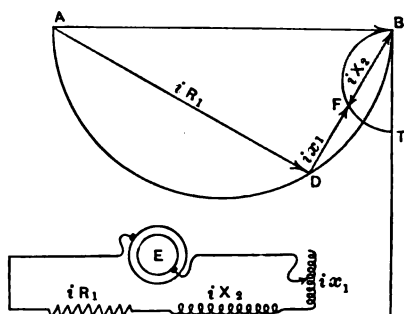


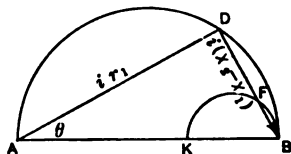
FIG. 289.

in series with a variable resistance  $r_1$ . The impressed voltage  $E$  is constant.

**Ex. 12.** Construct and explain the diagrams in Fig. 289, which represent a circuit with a constant reactance  $X_2$  in series

with a constant resistance  $R_1$  and a variable reactance  $x_1$ . The impressed voltage  $E$  is constant.

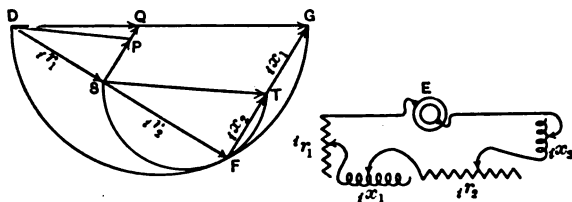
**Ex. 13.** Construct and explain the circle diagram in Fig. 290, which represents a circuit with variable resistance  $r_1$  in series



**FIG. 290.**

with a constant capacity reactance  $X_2$  and a constant inductive reactance  $X_1$  in which  $X_2 > X_1$ . The impressed voltage  $E$  is constant.

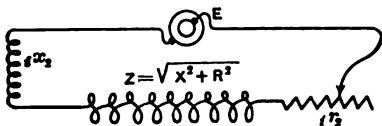
**Ex. 14.** Fig. 291 represents the diagram for a series circuit



**FIG. 291.**

consisting of two impedances  $z_1 = \sqrt{r_1^2 + x_1^2}$  and  $z_2 = \sqrt{r_2^2 + x_2^2}$ , supplied with current from a source of a constant E.M.F.  $E$ . Why does  $DF$  represent the energy component of  $E$  and why does  $FG$  represent the wattless component of  $E$ ? What significance do we attach to lines  $ST$ ,  $TF$ , and  $SF$ ? Can a single line be drawn in the figure to represent the drop across  $z_1$ ? What is the difficulty in the use of this diagram?

**Ex. 15.** Construct and explain the use of the circle diagram to correspond to the circuit shown in Fig. 292.



**FIG. 292.**



be applied to the circuit may be determined by the circuit diagram as illustrated in Figs. 294 and 295.

In both case  $AB = E$  is the diameter of a circle diagram and is the drop across the impedance.  $AF$  in Fig. 294 is an extension of  $AD$  and is the additional voltage drop across the additional

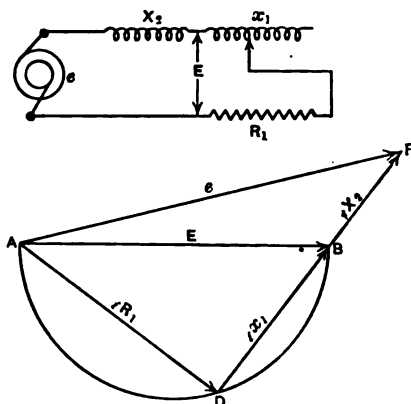


FIG. 295.

resistance  $R_2$ .  $FB = e$  is the generator voltage.  $BF$  in Fig. 287 is an extension of  $DB$  and is the additional voltage drop across the additional reactance  $X_2$ .  $AF = e$  is the generator voltage.

Fig. 296 shows a modification of Fig. 287 for the case in which the extra reactance  $x_2$  is variable. Explain Fig. 287.

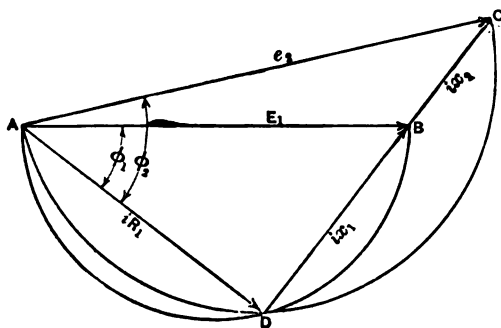


FIG. 296.

### 10. Parallel Circuits with Constant Applied Voltage.

Fig. 297 represents a circuit with two parallel branches I and II which are both supplied by the same constant voltage. Branch I contains a constant resistance  $R_1$  and a variable reactance  $x_1$ . Branch II contains a constant resistance  $R_2$  and a variable reactance  $x_2$ . Each branch has its distinct circle diagram as represented by (a) and (b) respectively in Fig. 298. The current in I is proportional to  $AF = i_1 R_1$  and the

current in II is proportional to  $DB = i_2 R_2$ . In order to obtain the resultant current for the entire circuit, the circle diagrams for I and II must be superposed and modified so as to place the vectors  $AF$  and  $DB$  adjacent and consecutive and they must both be represented to the

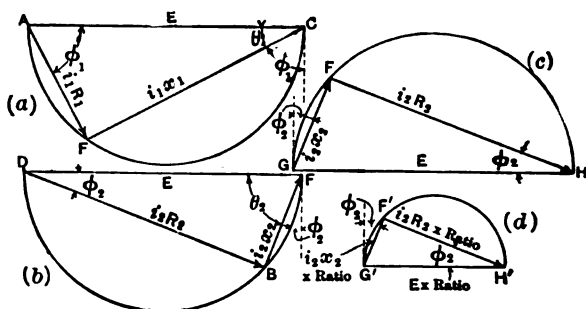


FIG. 298.

same scale. The first requirement is satisfied by inverting (b) into (c) of Fig. 298. The vectors  $DB$  and  $BF$  remain parallel and equal in their new positions  $FH$  and  $GF$  respectively, and in the new position (c)  $FH$  lags behind  $E$  in accordance with the lagging of  $DB$  behind  $E$  in (b). If  $H$  were now placed in coincidence with  $A$ , the vectors

$FH$  would be consecutive. The second requirement is obtained by changing the diameter of circle (c) as shown in Fig. 298 according to (24). In (a), (b), and (c) the lines  $AF$ ,  $FC$ ,  $DB$ ,  $BF$ ,  $GF$ ,  $FH$  are all measured on the same scale as the three equal diameters  $AC=DF=CH$ . When  $AF$  measures the current  $i_1$  in I then the scale of  $i_1$  is expressed in (20).

$$i_1 = \frac{E_{R_1}}{R_1},$$

$$\text{scale of } i_1 = \frac{\text{scale of } E_{R_1}}{R_1} = \frac{\text{scale of } E}{R_1}.$$

$DB$  measures the current  $i_2$  in II then the scale of  $i_2$  is expressed in (23).

$$i_2 = \frac{E_{R_2}}{R_2},$$

$$\text{scale of } i_2 = \frac{\text{scale of } E_{R_2}}{R_2} = \frac{\text{scale of } E}{R_2},$$

$$\therefore \frac{\text{scale of } i_1}{\text{scale of } i_2} = \frac{R_2}{R_1}.$$

is obtained from dividing (21) by (23) and states that the scales of the current lines in (a) compared with the scales of the current lines in (b) are inversely proportional to the constant resistances of the respective branches. In order to change the scale of  $i_2$  to correspond to the scale of  $i_1$  it is necessary to change the length of  $i_2$  by the ratio of  $\frac{R_2}{R_1}$ . This is accomplished most easily by changing the diameter of the circle in the ratio  $\frac{R_2}{R_1}$  as shown in (d) which has the same diameter as (c). Changing all lines of (d) in the same ratio. This is called the **diametric scale factor**.

$$\frac{FH}{F^1H} = \frac{GF}{G^1F^1} = \frac{GH}{G^1H} = \frac{R_2}{R_1}.$$

Instead of changing the diameter of circles (b) or (c) the diameter of circle (a) may be changed by multiplying it by a corresponding diametric scale factor  $\frac{R_1}{R_2}$ . Fig. 299 represents the circle diagrams for the two branch circuits I and II described above. The diametric scale factor  $\frac{R_1}{R_2}$  is applied to the circle diagram for branch I and therefore the length of  $CF$  equals  $\frac{R_1}{R_2}$  times the length of  $AB$ .

The vectors  $AC$  and  $CD$  representing  $i_1$  and  $i_2$ , are consecutive and constructed to the same scale, therefore

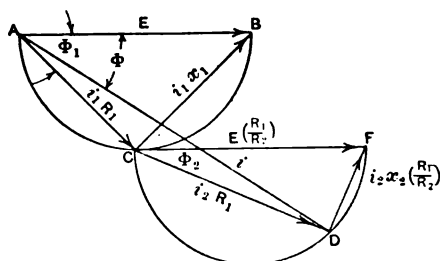


FIG. 299.

their resultant  $i$  is  $AD$  and is also measured with a like scale. Since the phase angles  $\phi_1$  and  $\phi_2$  for  $i_1$  and  $i_2$  respectively are measured from horizontal reference lines then the phase angle  $\phi$  for the resultant  $i$  is also measured from the horizontal reference line.

11. Another alternative method for the above problem is to use a single circle for both circuits I and II as shown in Fig. 300. Its principal disadvantage lies in the fact that more construction lines are necessary and there is no convenient method for applying the unequal scales to the two lines  $AC$  and  $AF$  in order to determine  $AP$  and respectively.

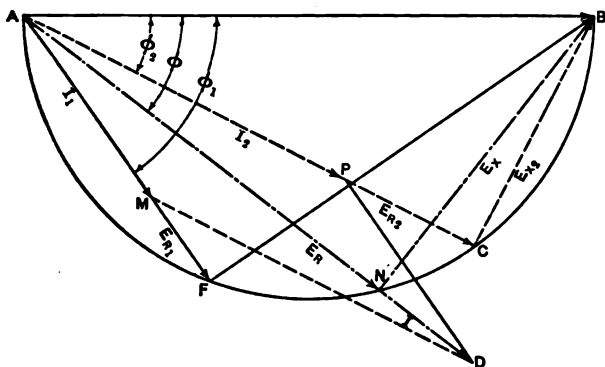


FIG. 300.

Another modification of the above problem is shown in Fig. 301. The second circle diagram  $CDB$  need not be

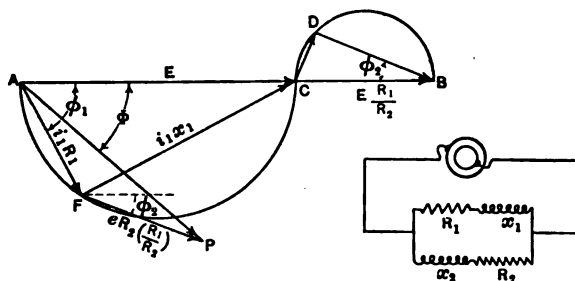


FIG. 301.

transposed if  $FP$  is drawn parallel and equal to  $DB$ . Then the resultant  $AP$  is the vector sum of  $AF$  and  $FP$ .

**12. Relabeling the Lines of a Circle Diagram.** Fig. 302 shows the circle diagram for a branch circuit. The circle for circuit I is drawn with a diameter  $CF=E$  and therefore the current  $i_1$  is proportional to  $CD$ . For circuit II the diameter  $AC=\frac{R_1}{R_2}E$  and therefore

$$BC=i_2R_2\frac{R_1}{R_2}=i_2R_1.$$

The three lines,  $CD$ ,  $BC$ , and  $BD$  are proportional to  $i_1$ ,  $i_2$ , and  $i$  and therefore  $CD = i_1 R_1$ ,  $BC = i_2 R_1$ , and  $BD = i R_1$ .

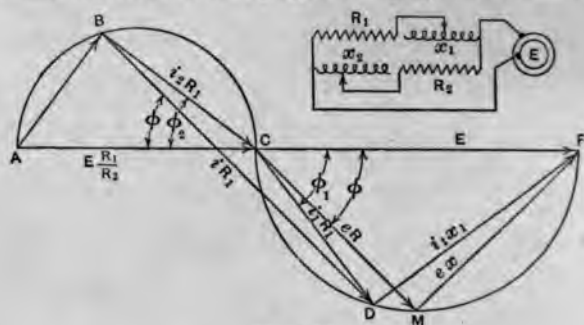


FIG. 302.

**Ex. 17.** In Fig. 302 show that  $AB = i_2 X_2 \frac{R_1}{R_2}$ .

**13. The Determination of the Equivalent Resistance and Equivalent Reactance.** Fig. 303 represents the circle

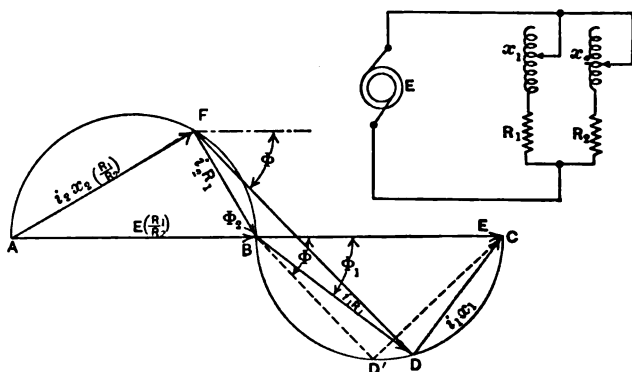


FIG. 303.

diagram for a parallel circuit having a constant resistance and a variable reactance in each branch. The total current  $i$  is represented by  $FD = iR_1$  which has the same scale as  $FR$  and  $BD$ . The total current lags  $\phi$  degrees

he E.M.F. and therefore  $BD'$  may be drawn parallel to  $FD$  so as to represent the total current in a circle whose diameter is  $E$ . If we designate the equivalent resistance and equivalent reactance of the whole circuit by  $R_a$  and  $x_a$ , then  $BD' = iR_a$ ,  $D'C = ix_a$ , and  $BC = E = iz_a$ .

$$26) \quad \frac{FD}{BD'} = \frac{iR_1}{iR_a} = \frac{R_1}{R_a},$$

$$27) \quad \therefore R_a = R_1 \times \frac{BD'}{FD},$$

$$28) \quad \frac{D'C}{BD'} = \frac{ix_a}{iR_a} = \frac{x_a}{R_a},$$

$$29) \quad x_a = R_a \times \frac{D'C}{BD'} = R_1 \times \frac{D'C}{FD}.$$

Interpret (27) and (29).

**Ex. 18.** Construct the circle diagram for a two-branch circuit which is supplied with a constant E.M.F.  $E$ . Branch I contains a constant resistance  $R_1 = 10\Omega$  and a variable reactance  $x_1$ , ranging from  $5\Omega$  to  $7.5\Omega$ . Branch II contains a resistance  $R_2$  equal to  $7.5\Omega$  and a variable reactance  $x_2$ , ranging from  $5\Omega$  to  $7.5\Omega$ . Designate the current lines, power lines, and power factor lines for each branch and for the entire circuit. Determine the equivalent resistance of the circuit and the equivalent reactance when  $x_1$  and  $x_2$  reach maximum and minimum values simultaneously. See Fig. 315.

**Ex. 19.** Construct the circle diagram for the circuit described in Ex. 18, excepting that the resistances are variable and the reactances are constant. Interchange the values for  $R$  and  $x$  given in Ex. 18. See Fig. 316.

**14. Parallel Circuits Whose Constant Elements are Unlike.** Fig. 304 represents a circuit with two parallel branches supplied with a constant E.M.F.  $E$ . Branch I contains a constant resistance  $R_1$  and a variable reactance  $x_1$ . Branch II contains a constant reactance  $X_2$  and a variable resistance  $r_2$ . In circuit I the current is proportional to  $E_1$ , whereas in circuit II the current is pro-

portional to  $E_{X_1}$ . The diameter of the circle diagram for circuit II will be obtained by multiplying  $E$  by  $\frac{R_1}{X_2}$  which is the diameter scale factor.

$$(30) \quad CA = i_2 X_2 \times \frac{R_1}{X_2} = i_2 R_1,$$

$$(31) \quad CF = i_2 r_2 \frac{R_1}{X_2}.$$

$$(32) \quad \frac{\text{The diameter of semicircle II}}{\text{The diameter of semicircle I}} = \frac{R_1}{X_2}.$$

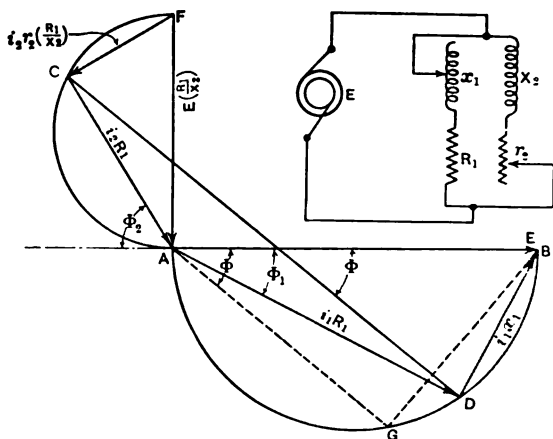


FIG. 304.

$CD$  is the vector sum of the currents  $i_1$  and  $i_2$  since the three lines  $CD = iR_1$ ,  $CA = i_2 R_1$ ,  $AD = i_1 R_1$ , are proportional to  $i$ ,  $i_1$  and  $i_2$ .  $AG$  is drawn parallel to  $CD$  and therefore  $AG$  is the drop across the equivalent simultaneous resistance and  $GE$  is the drop across the equivalent simultaneous reactance. Do Eqs. (27) and (29) apply to this circuit?

20. Construct and explain the diagram in Fig. 305 which is for a two-branch circuit supplied with a constant E.M.F.  $E$ . In branch I there is a constant reactance  $X_1$  and a variable resistor  $r_1$ .

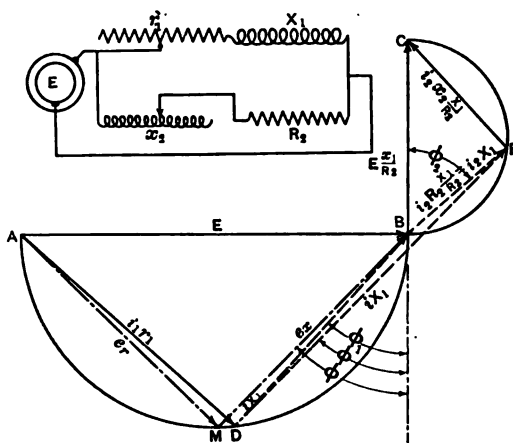


FIG. 305.

$r_1$ . In branch II there is a constant resistance  $R_2$  and a constant reactance  $x_2$ . Supply any necessary lines to show power factor and power in each branch and for the total circuit.

i. **The Circle Diagram for a Constant Current Circuit.** This circle diagram is applied to simple circuits supplied with a constant current. The diameter of the semicircle in

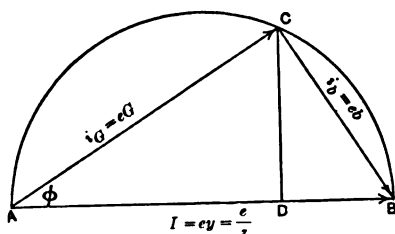


FIG. 306.

Fig. 306 represents the constant current  $I$  which is resolved into two orthogonal components  $i_Q$  and  $i_B$ , the locus of whose



$$(35) \quad i = \frac{e}{\sqrt{R^2 + x^2}} = \frac{Kx}{\sqrt{R^2 + x^2}},$$

$$(36) \quad \therefore i \propto \frac{x}{\sqrt{R^2 + x^2}}.$$

In Fig. 308  $AB$  represents the maximum value of  $x$  and  $AD$  the diameter of the semicircle  $AKD$  is constructed equal to  $R$ . Then  $DB$  represents the maximum impedance of the circuit. With a diminution of speed or frequency

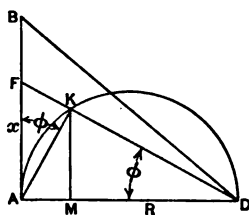


FIG. 308.

the value of  $x$  decreases so that when  $x = AF$ , then  $FD = \sqrt{R^2 + x^2}$ . Join  $A$  with  $K$  the intersection of  $FD$  with the semicircle. Then  $AKD$  is a right angle inscribed in a semicircle.

$$(37) \quad \frac{AF}{FD} = \frac{AK}{AD} = \frac{x}{\sqrt{R^2 + x^2}},$$

but

$$(38) \quad \frac{AK}{AD} = \sin \phi,$$

$$(39) \quad \therefore i \propto \sin \phi,$$

$$(40) \quad \therefore i \propto AK.$$

Therefore, the current line is represented by  $AK$ . What lines represent the power factor, the total volts generated, and the active power?

**Ex. 21.** Construct and explain the diagrams in Fig. 309 in which  $BF = E_c = ix_c$  and  $DF = E_L = ix_L$ .

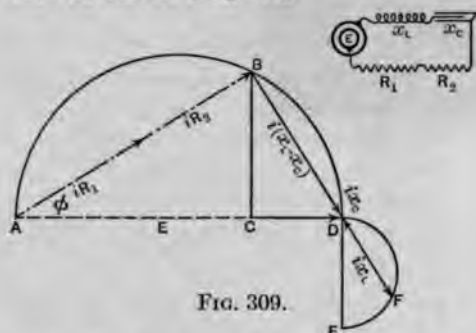


FIG. 309.

**Ex. 22.** Construct and explain the diagram in Fig. 310 which lines in the figure represent active and wattless power.

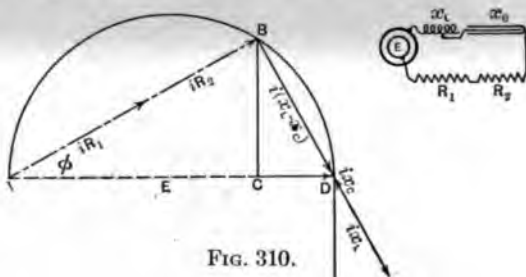


FIG. 310.

**Ex. 23.** Construct and explain the diagram in Fig. 311. Add a diagram of the circuit.

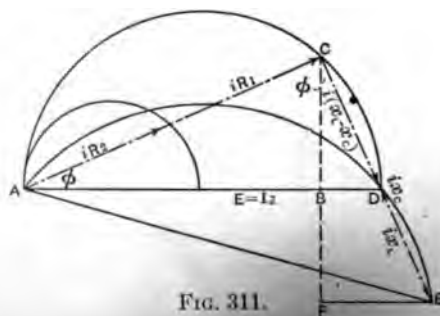


FIG. 311.

**17. Multiple Circuit Circle Diagrams.** Fig. 312 represents a multiple constant voltage circuit with three branches each containing a resistance and an inductive reactance.

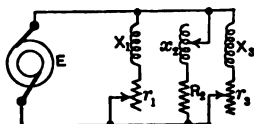


FIG. 312.

In circuits I and III the reactances are constant and the resistances are variable whereas in II the resistance is constant and the reactance is variable. Fig. 313 was

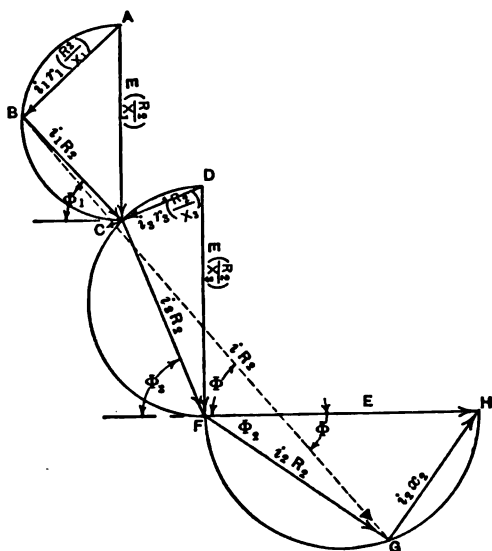


FIG. 313.

and by first representing the circle diagram  $FGH$  in which  $FG$  represents  $i_2$ . The diametric applied in constructing the circle diagram

*DCF* for circuit III in which *CF* represents  $i_3$ . The diametric scale factor  $\frac{R_2}{X_1}$  is applied in constructing the circle diagram

*ABC* for circuit I in which *BC* represents  $i_1$ . The phase angles for  $i_1$ ,  $i_2$ ,  $i_3$ , and  $i$  are measured from horizontal reference lines. The total current  $i = BG$  is the vector sum of *BC*, *CF*, and *FG*.

**Ex. 24.** Fig. 314 represents a multiple circuit in which the three branches I, II, and III each contain resistance, inductance,

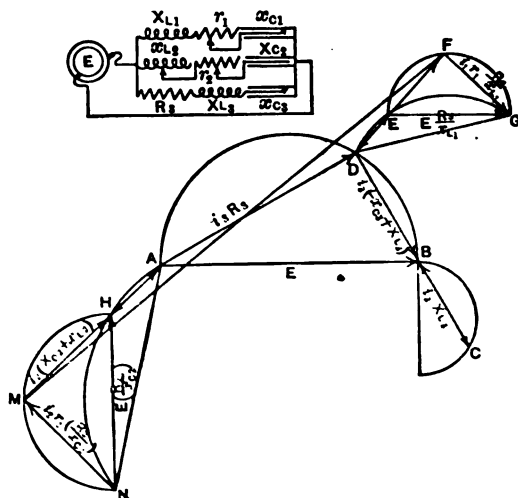


FIG. 314.

and capacity. In I both resistance  $r_1$  and capacity reactance  $x_{C1}$  are variable. In II the resistance  $r_2$  and the inductive reactance  $x_{L2}$  are variable. In circuit III the capacity reactance  $x_{C3}$  only is variable. Discuss the diagram for the possible conditions when one of the two variables in each circuit reduces to zero.

TABLE XXXVI.—RESISTANCE OF SOFT OR ANNEALED  
COPPER WIRE

B. & S. Gauge.	Diameter in Mils. d.	Area in Circular Mils. d <sup>2</sup> .	Ohms per 1000 ft. at 20° C. or 68° F.	B. & S. Gauge.	Diameter in Mils. d.	Area in Circular Mils. d <sup>2</sup> .	Ohms per 1000 ft. at 20° C. or 68° F.
0000	460.00	211,600	.04893	19	35.890	1288.1	8.038
000	409.64	167,810	.06170	20	31.961	1021.5	10.14
00	364.80	133,080	.07780	21	28.462	810.10	12.78
0	324.86	105,530	.09811	22	25.347	642.40	16.12
1	289.30	83,694	.1237	23	22.571	509.45	20.32
2	257.63	66,373	.1560	24	20.100	404.01	25.63
3	229.42	52,634	.1967	25	17.900	320.40	32.31
4	204.31	41,742	.2480	26	15.940	254.10	40.75
5	181.94	33,102	.3128	27	14.195	201.50	51.38
6	162.02	26,250	.3944	28	12.641	159.79	64.70
7	144.28	20,816	.4973	29	11.257	126.72	81.70
8	129.49	16,509	.6271	30	10.025	100.50	103.0
9	114.43	13,094	.7908	31	8.928	79.70	129.9
10	101.89	10,381	.9972	32	7.950	63.21	163.8
11	90.742	8,234.0	1.257	33	7.080	50.13	206.6
12	80.808	6,529.9	1.586	34	6.305	39.75	260.5
13	71.961	5,178.4	1.999	35	5.615	31.52	328.4
14	64.084	4,106.8	2.521	36	5.000	25.00	414.2
15	57.068	3,256.7	3.179	37	4.453	19.82	522.2
16	50.820	2,582.9	4.009	38	3.965	15.72	658.5
17	45.257	2,048.2	5.055	39	3.531	12.47	830.4
18	40.303	1,624.3	6.374	40	3.145	9.89	1047



## APPENDIX

### GREEK ALPHABET

Letters.	Name.	Letters.	Name.
A α	Alpha	N ν	Nu
B β	Bêta	Ξ ξ	Xi
Γ γ	Gamma	Ο ο	Omicron
Δ δ	Delta	Π, π	Pi
E ε	Epsilon	Ρ ρ	Rho
Z ζ	Zêta	Σ σ ς	Sigma
H η	Eta	Τ τ	Tau
Θ θ	Thêta	Υ υ	Upsilon
I ι	Iôta	Φ φ	Phi
K κ	Kappa	Χ χ	Chi
Λ λ	Lambda	Ψ ψ	Psi
M μ	Mu	Ω ω	Omega

#### INSTRUCTIONS FOR PREPARING AND REPORTING DAILY WORK

**1. The Work-book.** The student should provide himself with a **Mathematics Work-book**, which consists of a work-book cover intended to hold 300 sheets of 20-pound plain linen paper, measuring  $5\frac{1}{2} \times 8\frac{1}{2}$  ins. The work-book cover has a flexible leather back and is perforated with two pairs of eyelets. Attach 20 sheets of the plain linen paper to the rear cover for use. The front cover should be used

for finished work. Procure a good fountain pen and fountain-pen ink.

**2. Identity of Property.** Print a suitable label as directed and place it on the outside of the front cover of the work-book. Enter your name, class, home and boarding address on the inside of the front cover of both your text-book and work-book. Take sufficient precaution to mark every piece of your personal property so that it may be distinguished and identified easily.

**3. The Text.** The work-book is not to be used for notes as the text is deemed ample, but should any necessity arise for adding additional information, it may be entered in the text-book in the blank space at the end of a chapter. Both text-book and work-book should be carried daily to the class room unless instruction is given to the contrary.

**4. Preparation of Daily Work.** All work should represent individual effort and should be done neatly and accurately in **black writing ink**. Drawings and diagrams should be constructed lightly with a hard pencil and straight-edge and after the instructor has given his approval they should be inked with **India ink**. Title pages should be lettered in India ink on the work-book paper, and should precede each important division or chapter of the text. The main title page should be protected by a blank fly-leaf. Titles should be brief and lettered across the center of the page. Before entering the class room place your name on each page at the top between the holes. Place the date when the work is due at the upper right-hand corner one inch from the top of the paper. As the finished pages accumulate enter consecutive page numbers in red ink at the lower right-hand corner of the page. One inch from the upper left corner of the page write a fraction whose numerator is the text page number, and whose denominator is the exercise number.

All work should be written horizontally across the page and if the student experiences any difficulty fr

careless previous training he should procure or prepare a ruled sheet of paper with guide lines spaced one-quarter of an inch apart. Three vertical lines on the ruled sheet will prove of assistance. Headings should be carefully printed, centered and spaced. The work should be legible and carefully spaced, not more than one equation should appear on a line. Incidental calculations should appear on the paper near the right margin. Special care should be given to the formation of numerals, and symbols and more especially to the equality sign and the parenthesis. A five-minute daily practice making alphabetic characters and symbols will prove very profitable. Wherever convenient arrange the steps of your work so that equality symbols are placed in a vertical column. All proofs of analytic statements must bear an equation number placed in a parenthesis at the left of the line and also a mathematic authority placed at the right of the line. All equations must be numbered in sequence for each exercise.

All incorrect or rejected work must be presented again showing the corrections in red ink, at or before the beginning of the period immediately following its non-acceptance.

An excuse for the non-performance of any assigned work must be written upon the regular work paper dated and signed and presented in lieu of the work.

**5. Presentation of Work.** On the stroke of the bell at the beginning of the period all assigned work for the day, excuses and delinquent work, must be passed in each row so as to be received in alphabetic order by the end man in each row. The rear end man will collect all work and deposit it in a designated drawer of the instructor's desk. In order that this may be done quietly and without delay all work should be ready for collection when the student enters the class room, and he should make an immediate entry on the daily record sheet which has been distributed by the end man in his row.

**6. Reporting Daily Work.** A daily record sheet is attached to the front of the student's accepted work, and contains sufficient space for twenty daily entries under the following column headings: date, pages and exercises, title of matter assigned for preparation; a sub-divided time column for indicating daily the number of hours spent for each prepared subject in the course; two other columns headed "foreman" and "remarks" are reserved for the instructor's use.

When the time sheet is filled and the work described thereon has been accepted, then the work and the time sheet is placed in charge of a custodian, who stores, arranges and credits it upon a designated shelf in the class room closet, from which it may be removed only upon special permission from the instructor.

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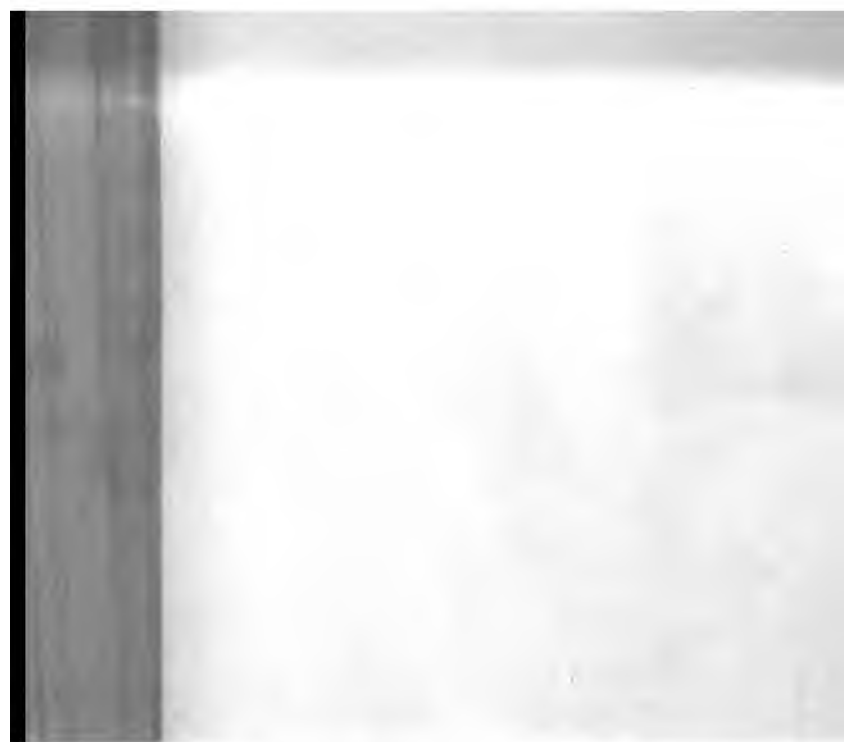
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